DORFMAN CONNECTIONS AND COURANT ALGEBROIDS

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Abstract. We define Dorfman connections, which are to Courant algebroids what connections are to Lie algebroids. Several examples illustrate this analogy.

A linear connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ on a vector bundle E over a smooth manifold M is tantamount to a linear splitting $TE\simeq T^{q_E}E\oplus H_\nabla$, where $T^{q_E}E$ is the set of vectors tangent to the fibres of E. Furthermore, the curvature of the connection measures the failure of the horizontal space H_∇ to be integrable. We show that linear horizontal complements to $T^{q_E}E\oplus(T^{q_E}E)^\circ$ in the Pontryagin bundle over the vector bundle E can be described in the same manner via a certain class of Dorfman connections $\Delta\colon \Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$. Similarly to the tangent bundle case, we find that, after the choice of a linear splitting, the standard Courant algebroid structure of $TE\oplus T^*E\to E$ can be completely described by properties of the Dorfman connection.

As an application, we study splittings of $TA\oplus T^*A$ over a Lie algebroid A and, following Gracia-Saz and Mehta, we compute the representations up to homotopy defined by any linear splitting of $TA\oplus T^*A$ and the linear Lie algebroid $TA\oplus T^*A\to TM\oplus A^*$. Further, we characterise VB- and LA-Dirac structures in $TA\oplus T^*A$ via Dorfman connections.

Contents

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1. Introduction

This paper introduces Dorfman connections, and studies in depth the standard Courant algebroid over a vector bundle. Let us begin with a simple observation. Take a subbundle $F \subseteq TM$ of the tangent bundle of a smooth manifold M. Then the $\mathbb{R}$-bilinear map

$$\tilde{\nabla}\colon \Gamma(F) \times \mathfrak{X}(M) \to \Gamma(TM/F), \qquad \tilde{\nabla}_X Y = \overline{[X,Y]}$$

measures the failure of vector fields on M to preserve F. The subbundle F is involutive if and only if $\tilde{\nabla}_X Y = 0$ for all $X, Y \in \Gamma(F)$. In this case, $\tilde{\nabla}$ induces a flat connection

$$\nabla\colon \Gamma(F) \times \Gamma(TM/F) \to \Gamma(TM/F), \qquad \nabla_X \bar{Y} = \overline{[X,Y]},$$

the **Bott connection** associated to F [2].

In the same manner, given a Courant algebroid $\mathsf{E} \to M$ with bracket $[\![\cdot\,,\cdot]\!]$, anchor ρ and pairing $\langle\cdot\,,\cdot\rangle$, and a subbundle $K \subseteq \mathsf{E}$, we define an $\mathbb{R}$-bilinear map

$$\tilde{\Delta}\colon \Gamma(K) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E}/K), \qquad \tilde{\Delta}_k e = \overline{[\![k, e]\!]}.$$

Again, we have $\tilde{\Delta}_k k' = 0$ for all $k, k' \in \Gamma(K)$ if and only if $\Gamma(K)$ is closed under the bracket on $\Gamma(\mathsf{E})$. If K is in addition isotropic, it is a Lie algebroid over M and the pairing on E induces a pairing $K \times_M (\mathsf{E}/K) \to \mathbb{R}$. The $\mathbb{R}$-bilinear map

$$\Delta\colon \Gamma(K) \times \Gamma(\mathsf{E}/K) \to \Gamma(\mathsf{E}/K), \qquad \tilde{\Delta}_k \bar{e} = \overline{[\![k, e]\!]}$$

that is induced by $\tilde{\Delta}$ is not a connection because it is not $C^\infty(M)$-homogeneous in the first argument, but the obstruction to this is, as we will see, measured by the pairing, the anchor of the Courant algebroid and the de Rham derivative on $C^\infty(M)$. This map is an example of what we call a Dorfman connection, namely the **Bott–Dorfman connection** associated to K in E. Dorfman connections appear naturally in several situations related to Courant algebroids and play a role similar to the one that connections play for tangent bundles and Lie algebroids. We illustrate this with a few examples.

Our main motivation for introducing this new concept is the following. It goes back to Dieudonné that a linear TM-connection ∇ on a vector bundle $q_E\colon E \to M$ corresponds to a splitting $TE \simeq T^{q_E}E \oplus H_\nabla$, where $T^{q_E}E \subseteq TE$ is the set of vectors tangent to the fibers of the vector bundle E, and H_∇ is a subbundle of $TE \to E$ that is also closed under the addition in $TE \to TM$. There exists then for each vector field $X \in \mathfrak{X}(M)$ a unique section $X^\nabla \in \Gamma(H_\nabla) \subseteq \mathfrak{X}(E)$ (a *horizontal* vector field) such that $Tq_E \circ X^\nabla = X \circ q_E$. The Lie bracket of two such vector fields $X^\nabla, Y^\nabla \in \Gamma(H_\nabla)$, for $X, Y \in \mathfrak{X}(M)$, is given by

$$\left[X^\nabla, Y^\nabla\right] = [X,Y]^\nabla - \widetilde{R_\nabla(X,Y)},$$

where $\widetilde{R_\nabla(X,Y)} \in \mathfrak{X}(E)$ is given by

$$\widetilde{R_\nabla(X,Y)}(e_m) = \left.\frac{d}{dt}\right|_{t=0} e_m + t \cdot R_\nabla(X,Y)(e_m)$$

for all $e_m \in E$, and so has values in the vertical space $T^{q_E}E$. Since $\Gamma(H_\nabla)$ is generated as a $C^\infty(E)$-module by the set of sections $\{X^\nabla \mid X \in \mathfrak{X}(M)\}$, this means that the failure of the horizontal space H_∇ to be involutive is measured by the curvature of the connection. The connection itself encodes the Lie bracket of horizontal and vertical vector fields. The space $\Gamma(T^{q_E}E)$ is indeed generated as a $C^\infty(E)$-module by the *vertical* vector fields $e^\uparrow$ with flow $\phi_t^{e^\uparrow}(e'_m) = e'_m + te(m)$ for $e \in \Gamma(E)$, and we have

$$\left[X^\nabla, e^\uparrow\right] = (\nabla_X e)^\uparrow$$

for all $X \in \mathfrak{X}(M)$.

This paper answers the following question: what can be said about linear[1] splittings

$$TE \oplus T^*E \simeq (T^{q_E}E \oplus (T^{q_E}E)^\circ) \oplus L$$

of the standard Courant algebroid over E?

Our first main result is a similar one-to-one correspondence of such linear splittings with $TM \oplus E^*$*–Dorfman connections* Δ *on* $E \oplus T^*M$. Then we prove that the bundle L_Δ is isotropic (and thus also Lagrangian) relative to the canonical pairing on $TE \oplus T^*E$ if and only if a bracket on sections of $TM \oplus E^*$, that is dual of the Dorfman connection (in the sense of connections), is skew-symmetric. Further, the set of sections of L_Δ is closed under the Courant-Dorfman bracket if and only if the curvature of the Dorfman connection vanishes. The Dorfman connection itself is the Courant-Dorfman bracket restricted to horizontal and vertical sections of $TE \oplus T^*E \to E$.

The direct sum $TE \oplus T^*E$ has the structure of a *double vector bundle* [29, 23] over the bases E and $TM \oplus E^*$. Double vector subbundles of $(TE \oplus T^*E; E, TM \oplus E^*, M)$ have a double vector bundle structure over subbundles of E and $TM \oplus E^*$. After proving the main results on splittings of $TE \oplus T^*E \to E$, we characterise the double vector subbundles of $TE \oplus T^*E$ over the sides E and a subbundle $U \subseteq TM \oplus E^*$. These double vector subbundles can be described by triples (U, K, Δ), where Δ is a Dorfman connection and K is a subbundle of $E \oplus T^*M$ (the *core* or *double kernel* of $TE \oplus T^*E$). We prove that both maximal isotropy and integrability of this type of double subbundle depend only on simple properties of the corresponding triple (U, K, Δ).

For instance, assume that the vector bundle E is endowed with a linear Poisson structure $\{\cdot\,,\cdot\}$ (see Appendix A.1). It might then be useful (for instance in geometric mechanics), to describe in terms of linear and vertical vector fields the image of the morphism of vector bundles

$$\sharp\colon T^*E \to TE, \quad \mathbf{d}F \mapsto \{F, \cdot\}, \quad F \in C^\infty(E)$$

associated to the Poisson bracket. The cotangent space T^*E is spanned by the exact sections $q_E^*\mathbf{d}f$ for $f \in C^\infty(M)$ and $\mathbf{d}\ell_\varepsilon$ for $\varepsilon \in \Gamma(E^*)$, where $\ell_\varepsilon \in C^\infty(E)$ is the linear function defined by $\varepsilon \in \Gamma(E^*)$. The images of $\sharp$ on these one-forms are easy to describe, but there is in general no way of finding a splitting of TE such that they can be described in terms of horizontal and vertical vector fields. It turns out that the *graph* of the vector bundle morphism $\sharp$, can be shown to have a vertical and a horizontal part in the space $TE \oplus T^*E$. This is described in Examples 4.3 and 4.3.

Note that $TE \oplus T^*E$ has the natural structure of a *VB-Courant algebroid* with sides E and $TM \oplus E^*$ and with core $E \oplus T^*M$. We show in [16] that the Dorfman connections that we find here define (after a skew-symmetrisation) the split Lie 2-algebroids which are equivalent to decompositions of the VB-Courant algebroid $TE \oplus T^*E$ [19].

If the vector bundle $E =: A$ has a Lie algebroid structure $(q_A\colon A \to M, \rho, [\cdot\,,\cdot])$, then the standard Courant algebroid $TA \oplus T^*A$ also has a naturally induced VB-algebroid structure over $TM \oplus A^*$. Given a $TM \oplus A^*$-Dorfman connection Δ on $A \oplus T^*M$, we compute the representation up to homotopy that corresponds to the linear splitting $TA \oplus T^*A \simeq (T^{q_A}A \oplus (T^{q_A}A)^\circ) \oplus L_\Delta$ and describes the VB-algebroid $TA \oplus T^*A \to TM \oplus A^*$ [12]. This representation up to homotopy is in general not the product of the two representations up to homotopy describing $TA \to TM$ and $T^*A \to A^*$. Furthermore, we describe the sub-representations up to homotopy defined by linear Dirac structures on A, that are at the same time Lie subalgebroids of $TA \oplus T^*A \to TM \oplus A^*$ over a base $U \subseteq TM \oplus A^*$. In that case, the Dirac structure has the induced structure of a double Lie algebroid [16], and is called an **LA-Dirac**

[1]The subbundle $L \subseteq TE \oplus T^*E$ over E is said to be linear if it is also closed under the addition of $TE \oplus T^*E$ as a vector bundle over $TM \oplus E^*$.

structure on A. We elaborate on this in [15] to describe infinitesimally Dirac groupoids, i.e. Lie groupoids with Dirac structures that are compatible with their multiplication.

Let (A, A^*) be a Lie bialgebroid [27] and let π_A be the linear Poisson bivector field defined on A by the Lie algebroid structure on A^*. The graph of $\pi_A^\sharp\colon T^*A \to TA$ is a known example of an LA-Dirac structure on A. The second most common example of an LA-Dirac structure is the graph of a linear presymplectic form $\sigma^*\omega_{\rm can} \in \Omega^2(A)$, for an IM–2–form $\sigma\colon A \to T^*M$ [6, 5]. A third example is $F_A \oplus F_A^\circ$, where $F_A \to A$ is an involutive subbundle that has at the same time a Lie algebroid structure over some subbundle $F_M \subseteq TM$. We describe the 2-term representations up to homotopy encoding linear splittings of the three examples above.

Outline of the paper. Some background on Courant algebroids and Dirac structures, connections, and double vector bundles is collected in the second section. In the third section, Dorfman connections and dull algebroids are defined, and some examples are discussed. In the fourth section, splittings of the standard Courant algebroid $TE \oplus T^*E$ over a vector bundle E are shown to be equivalent to a certain class of $TM \oplus E^*$-Dorfman connections on $E \oplus T^*M$. Linear Dirac structures on the vector bundle $E \to M$ are studied via Dorfman connections. In the fifth section, the geometric structures on the two sides of the standard LA-Courant algebroid $TA \oplus T^*A$ over a Lie algebroid $A \to M$ are expressed via splittings of $TA \oplus T^*A$, and LA-Dirac structures on A are classified via Dorfman connections and some adequate vector bundles over the units M.

Notation and conventions. Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the space of sections of E is written as $\Gamma(E)$. We write in general $q_E\colon E \to M$ for vector bundle projections, except for $p_M = q_{TM}\colon TM \to M$, $c_M = q_{T^*M}\colon T^*M \to M$ and $\pi_M = q_{TM\oplus T^*M}\colon TM \oplus T^*M \to M$.

The flow of a vector field $X \in \mathfrak{X}(M)$ is written as $\phi^X_\cdot$, unless specified otherwise. Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be ***f*-related** if $Tf \circ X = Y \circ f$. We then write $X \sim_f Y$.

Given a section ε of E^*, we always write $\ell_\varepsilon\colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m\rangle$ for all $e_m \in E$. We write $\phi^t\colon B^* \to A^*$ for the dual morphism to a morphism $\phi\colon A \to B$ of vector bundles over the identity, and we write $F^*\omega$ for the pullback of a form $\omega \in \Omega(N)$ under a smooth map $F\colon M \to N$ of manifolds.

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2. Preliminaries

First we recall some necessary background on Courant algebroids, on the double vector bundle structures on the tangent and cotangent spaces TE and T^*E of a vector bundle E, and on linear connections.

2.1. **Courant algebroids and Dirac structures.** A Courant algebroid [20, 30] over a manifold M is a vector bundle $\mathsf{E} \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$, a bilinear bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$, and an anchor $\rho\colon \mathsf{E} \to TM$, which satisfy the following conditions

(1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
(2) $\rho(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,

(3) $[\![e_1,e_2]\!]+[\![e_2,e_1]\!]=\mathcal{D}\langle e_1,e_2\rangle$

for all $e_1,e_2,e_3\in\Gamma(\mathsf{E})$. Here, we use the notation $\mathcal{D}:=\rho^t\circ\mathbf{d}\colon C^\infty(M)\to\Gamma(\mathsf{E})$, using $\langle\cdot\,,\cdot\rangle$ to identify E with E^*: $\langle\mathcal{D}f,e\rangle=\rho(e)(f)$ for all $f\in C^\infty(M)$ and $e\in\Gamma(E)$. The compatibility of the bracket with the anchor and the Leibniz identity

(4) $\rho([\![e_1,e_2]\!])=[\rho(e_1),\rho(e_2)]$,
(5) $[\![e_1,fe_2]\!]=f[\![e_1,e_2]\!]+(\rho(e_1)f)e_2$

are then also satisfied. They are often part of the definition in the literature, but [32] observed that they follow from (1)-(3).[2] For a nice overview of the history of Courant algebroids, consult [18].

Example 2.1. [[7]] The direct sum $TM\oplus T^*M$ endowed with the projection on TM as anchor map, $\rho=\mathrm{pr}_{TM}$, the symmetric bracket $\langle\cdot\,,\cdot\rangle$ given by

$$\langle(v_m,\theta_m),(w_m,\eta_m)\rangle=\theta_m(w_m)+\eta_m(v_m) \tag{2.1}$$

for all $m\in M$, $v_m,w_m\in T_mM$ and $\alpha_m,\beta_m\in T_m^*M$ and the **Courant-Dorfman bracket** given by

$$[\![(X,\theta),(Y,\eta)]\!]=([X,Y],\pounds_X\eta-\mathbf{i}_Y\mathbf{d}\theta) \tag{2.2}$$

for all $(X,\theta),(Y,\eta)\in\Gamma(TM\oplus T^*M)$, yield the standard example of a Courant algebroid, which is often called the **standard Courant algebroid over** M. The map $\mathcal{D}\colon C^\infty(M)\to\Gamma(TM\oplus T^*M)$ is given by $\mathcal{D}f=(0,\mathbf{d}f)$.

We are particularly interested in the standard Courant algebroids over vector bundles.

A **Dirac structure** $\mathsf{D}\subseteq\mathsf{E}$ is a subbundle satisfying

(1) $\mathsf{D}^\perp=\mathsf{D}$ relative to the pairing on E,
(2) $[\![\Gamma(\mathsf{D}),\Gamma(\mathsf{D})]\!]\subseteq\Gamma(\mathsf{D})$.

The rank of the Dirac bundle D is then half the rank of E, and the triple $(\mathsf{D}\to M,\rho|_{\mathsf{D}},[\![\cdot\,,\cdot]\!]|_{\Gamma(\mathsf{D})\times\Gamma(\mathsf{D})})$ is a Lie algebroid on M. Dirac structures appear naturally in several contexts in geometry and geometric mechanics (see for instance [3] for an introduction to the geometry and applications of Dirac structures).

2.2. **Basic facts about connections.** In this paper, connections will not be linear actions of Lie algebroids, but more generally of **dull algebroids**.

Definition 2.2. *A **dull algebroid** is a vector bundle $Q\to M$ endowed with an **anchor**, i.e. a vector bundle morphism $\rho_Q\colon Q\to TM$ over the identity on M and a bracket $[\cdot\,,\cdot]_Q$ on $\Gamma(Q)$ with*

$$\rho_Q[q_1,q_2]_Q=[\rho_Q(q_1),\rho_Q(q_2)]$$

for all $q,q'\in\Gamma(Q)$, and satisfying the Leibniz identity in both terms

$$[f_1q_1,f_2q_2]_Q=f_1f_2[q_1,q_2]_Q+f_1\rho_Q(q_1)(f_2)q_2-f_2\rho_Q(q_2)(f_1)q_1$$

for all $f_1,f_2\in C^\infty(M)$, $q_1,q_2\in\Gamma(Q)$.

In other words, a dull algebroid is a Lie algebroid if its bracket is in addition skew-symmetric and satisfies the Jacobi-identity.

Let $(Q\to M,\rho_Q,[\cdot,\cdot]_Q)$ be a dull algebroid and $B\to M$ a vector bundle. A Q-connection on B is a map

$$\nabla\colon\Gamma(Q)\times\Gamma(B)\to\Gamma(B),$$

[2] We quickly give here a simple manner to get (4)-(5) from (1)-(3). To get (5), replace e_2 by fe_2 in (2). Then replace e_2 by fe_2 in (1) in order to get (4).

with the usual properties. By the properties of a dull algebroid, one can still make sense of the curvature R_∇ of the connection, which is an element of $\Gamma(Q^*\otimes Q^*\otimes B^*\otimes B)$. The dual connection $\nabla^*\colon \Gamma(Q)\times\Gamma(B^*)\to\Gamma(B^*)$ to ∇ is defined by

$$\langle \nabla^*_q\beta, b\rangle = \rho_Q(q)\langle\beta, b\rangle - \langle\beta, \nabla_q b\rangle$$

for all $q\in\Gamma(Q)$, $b\in\Gamma(B)$ and $\beta\in\Gamma(B^*)$.

2.2.1. *The Bott connection associated to a subbundle $F\subseteq TM$.* Recall the definition of the Bott connection associated to an involutive subbundle of TM: Let $F\subseteq TM$ be a subbundle, then the Lie bracket on vector fields on M induces a map

$$\tilde\nabla^F\colon \Gamma(F)\times\Gamma(TM)\to\Gamma(TM/F), \qquad \tilde\nabla^F_X Y = \overline{[X,Y]}.$$

The subbundle F is involutive if and only if $\tilde\nabla^F_X X' = 0$ for all $X, X'\in\Gamma(F)$. In that case, the map $\tilde\nabla^F$ quotients to a flat connection

$$\nabla^F\colon \Gamma(F)\times\Gamma(TM/F)\to\Gamma(TM/F),$$

the **Bott connection**.

2.2.2. *The basic connections associated to a connection on a dull algebroid.* Consider here a dull algebroid $(Q,\rho_Q,[\cdot\,,\cdot]_Q)$ together with a connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(Q)\to\Gamma(Q)$. The induced **basic connections** are Q-connections on Q and TM that are defined as follows [8].

$$\nabla^{\rm bas}=\nabla^{{\rm bas},Q}\colon \Gamma(Q)\times\Gamma(Q)\to\Gamma(Q), \qquad \nabla^{\rm bas}_q q' = [q,q']_Q + \nabla_{\rho_Q(q')}q$$

and

$$\nabla^{\rm bas}=\nabla^{{\rm bas},TM}\colon \Gamma(Q)\times\mathfrak{X}(M)\to\mathfrak{X}(M), \qquad \nabla^{\rm bas}_q X = [\rho_Q(q),X] + \rho_Q(\nabla_X q).$$

The **basic curvature** is the map $R^{\rm bas}_\nabla\colon \Gamma(Q)\times\Gamma(Q)\times\mathfrak{X}(M)\to\Gamma(Q)$,

$$R^{\rm bas}_\nabla(q,q')(X) = -\nabla_X[q,q']_Q + [\nabla_X q, q']_Q + [q,\nabla_X q']_Q + \nabla_{\nabla^{\rm bas}_{q'}X}q - \nabla_{\nabla^{\rm bas}_q X}q'.$$

The basic curvature is tensorial and we have the identities

$$\nabla^{{\rm bas},TM}\circ\rho_Q = \rho_Q\circ\nabla^{{\rm bas},Q}, \qquad \rho_Q\circ R^{\rm bas}_\nabla = R_{\nabla^{{\rm bas},TM}} \quad\text{and}\quad R^{\rm bas}_\nabla\circ\rho_Q = R_{\nabla^{{\rm bas},Q}}.$$

If the bracket $[\cdot\,,\cdot]_Q$ on $\Gamma(Q)$ is skew-symmetric, then $R^{\rm bas}_\nabla$ is an element of $\Omega^2(Q,\operatorname{Hom}(TM,Q))$.

2.3. **Double vector bundles, VB-algebroids and representations up to homotopy.** We briefly recall the definitions of double vector bundles, of their **linear** and **core** sections, and of their **linear splittings** and **lifts**. We refer to [29, 23, 12] for more detailed treatments. A **double vector bundle** is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle\pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

of vector bundles such that

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4) \tag{2.3}$$

for $d_1,d_2,d_3,d_4\in D$ with $\pi_A(d_1)=\pi_A(d_2)$, $\pi_A(d_3)=\pi_A(d_4)$ and $\pi_B(d_1)=\pi_B(d_3)$, $\pi_B(d_2)=\pi_B(d_4)$. Here, $+_A$ and $+_B$ are the additions in $D\to A$ and $D\to B$, respectively. The vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and of π_B. From (2.3) follows easily the existence of a natural vector bundle structure on C over M. The inclusion $C\hookrightarrow D$ is denoted by $C_m\ni c\longmapsto \overline{c}\in\pi_A^{-1}(0^A_m)\cap\pi_B^{-1}(0^B_m)$.

The space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by two special classes of sections (see [24]), the **linear** and the **core sections** which we now describe. For a section $c\colon M \to C$, the corresponding **core section** $c^\dagger\colon B \to D$ is defined as $c^\dagger(b_m) = \tilde{0}_{b_m} +_A \overline{c(m)}$, $m \in M$, $b_m \in B_m$. We denote the corresponding core section $A \to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B is written as $\Gamma_B^c(D)$.

A section $\xi \in \Gamma_B(D)$ is called **linear** if $\xi\colon B \to D$ is a bundle morphism from $B \to M$ to $D \to A$ over a section $a \in \Gamma(A)$. The space of linear sections of D over B is denoted by $\Gamma_B^\ell(D)$. Given $\psi \in \Gamma(B^* \otimes C)$, there is a linear section $\tilde{\psi}\colon B \to D$ over the zero section $0^A\colon M \to A$ given by $\widetilde{\psi}(b_m) = \tilde{0}_{b_m} +_A \overline{\psi(b_m)}$. We call $\widetilde{\psi}$ a **core-linear section**.

Example 2.3. Let A, B, C be vector bundles over M and consider $D = A \times_M B \times_M C$. With the vector bundle structures $D = q_A^!(B \oplus C) \to A$ and $D = q_B^!(A \oplus C) \to B$, one finds that $(D; A, B; M)$ is a double vector bundle called the **decomposed double vector bundle with core** C. The core sections are given by

$$c^\dagger\colon b_m \mapsto (0_m^A, b_m, c(m)), \text{ where } m \in M,\ b_m \in B_m,\ c \in \Gamma(C),$$

and similarly for $c^\dagger\colon A \to D$. The space of linear sections $\Gamma_B^\ell(D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, \psi) : b_m \mapsto (a(m), b_m, \psi(b_m)), \text{ where } \psi \in \Gamma(B^* \otimes C),\ a \in \Gamma(A).$$

In particular, the fibered product $A \times_M B$ is a double vector bundle over the sides A and B and has core $M \times 0$.

A **linear splitting** of $(D; A, B; M)$ is an injective morphism of double vector bundles $\Sigma\colon A \times_M B \hookrightarrow D$ over the identity on the sides A and B. That every double vector bundle admits local linear splittings was proved by [10]. Local linear splittings are equivalent to double vector bundle charts. Pradines originally defined double vector bundles as topological spaces with an atlas of double vector bundle charts [28]. Using a partition of unity, he proved that (provided the double base is a smooth manifold) this implies the existence of a global double splitting [29]. Hence, any double vector bundle in the sense of our definition admits a (global) linear splitting.

A linear splitting Σ of D is also equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$-modules

$$0 \longrightarrow \Gamma(B^* \otimes C) \hookrightarrow \Gamma_B^\ell(D) \longrightarrow \Gamma(A) \longrightarrow 0, \tag{2.4}$$

where the third map is the map that sends a linear section (ξ, a) to its base section $a \in \Gamma(A)$. The splitting σ_A is called a **horizontal lift**. Given Σ, the horizontal lift $\sigma_A : \Gamma(A) \to \Gamma_B^\ell(D)$ is given by $\sigma_A(a)(b_m) = \Sigma(a(m), b_m)$ for all $a \in \Gamma(A)$ and $b_m \in B$. By the symmetry of a linear splitting, we find that a lift $\sigma_A\colon \Gamma(A) \to \Gamma_B^\ell(D)$ is equivalent to a lift $\sigma_B\colon \Gamma(B) \to \Gamma_A^\ell(D)$. Given a lift $\sigma_A\colon \Gamma(A) \to \Gamma_B^\ell(D)$, the corresponding lift $\sigma_B\colon \Gamma(B) \to \Gamma_A^\ell(D)$ is given by $\sigma_B(b)(a(m)) = \sigma_A(a)(b(m))$ for all $a \in \Gamma(A)$, $b \in \Gamma(B)$.

Example 2.4. Let $q_E\colon E \to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE \to TM$ are the derivatives of the structure maps of $E \to M$.

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

The space TE is a double vector bundle with core bundle $E \to M$. The map $^{-}\colon E \to p_E^{-1}(0^E) \cap (Tq_E)^{-1}(0^{TM})$ sends $e_m \in E_m$ to $\bar{e}_m = \frac{d}{dt}\big|_{t=0} te_m \in T_{0_m^E}E$. Hence the core vector field corresponding to $e \in \Gamma(E)$ is the vertical lift $e^\uparrow\colon E \to TE$, i.e. the vector field with flow $\phi^{e^\uparrow}\colon E \times \mathbb{R} \to E$, $\phi_t(e'_m) = e'_m + te(m)$. An element of $\Gamma^\ell_E(TE) = \mathfrak{X}^\ell(E)$ is called a **linear vector field**. It is well-known (see e.g. [23]) that a linear vector field $\xi \in \mathfrak{X}^l(E)$ covering $X \in \mathfrak{X}(M)$ corresponds to a derivation $D\colon \Gamma(E) \to \Gamma(E)$ over $X \in \mathfrak{X}(M)$. The precise correspondence is given by the following equations

$$\xi(\ell_\varepsilon) = \ell_{D^*(\varepsilon)} \quad \text{and} \quad \xi(q_E^* f) = q_E^*(X(f)) \tag{2.5}$$

for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$, where $D^*\colon \Gamma(E^*) \to \Gamma(E^*)$ is the dual derivation to D. We write $\widehat{D}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation D of $\Gamma(E)$. Given a derivation D over $X \in \mathfrak{X}(M)$, the explicit formula for $\widehat{D}$ is

$$\widehat{D}(e_m) = T_m e X(m) +_E \left.\frac{d}{dt}\right|_{t=0} (e_m - tD(e)(m)) \tag{2.6}$$

for $e_m \in E$ and any $e \in \Gamma(E)$ such that $e(m) = e_m$. The choice of a linear splitting Σ for $(TE; TM, E; M)$ is equivalent to the choice of a connection on E: Since a linear splitting gives us for each $X \in \mathfrak{X}(M)$ exactly one linear vector field $\sigma_{TM}(X) \in \mathfrak{X}^l(E)$ over X, we can define $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by $\sigma_{TM}(X) = \widehat{\nabla_X}$ for all $X \in \mathfrak{X}(M)$. Conversely, a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defines a lift $\sigma^\nabla_{TM}\colon \mathfrak{X}(M) \to \mathfrak{X}^l(E)$ and a linear splitting $\Sigma^\nabla\colon TM \times_M E \to TE$:

$$\Sigma^\nabla(v_m, e_m) = T_m e v_m +_E \left.\frac{d}{dt}\right|_{t=0} (e_m - t\nabla_{v_m} e)$$

for any $e \in \Gamma(E)$ such that $e(m) = e_m$. Note that the image of Σ^∇ is a subbundle $H_\nabla \subseteq TE$ that is linear, i.e. also closed under the addition in $TE \to TM$ and satisfies $TE \simeq H_\nabla \oplus T^{q_E}E$ as a vector bundle over E. Hence we have just described the correspondence of the two definitions of a connection; the first as the map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, the second as a linear splitting $TE \simeq T^{q_E}E \oplus H$. Given ∇ or Σ^∇ it is easy to see, using the equalities in (2.5), that

$$\begin{aligned} \left[\sigma^\nabla(X), \sigma^\nabla(Y)\right] &= \sigma^\nabla[X,Y| - \widetilde{R_\nabla(X,Y)}, \\ \left[\sigma^\nabla(X), e^\uparrow\right] &= (\nabla_X e)^\uparrow, \\ \left[e_1^\uparrow, e_2^\uparrow\right] &= 0 \end{aligned} \tag{2.7}$$

for all $X, Y \in \mathfrak{X}(M)$ and $e, e_1, e_2 \in \Gamma(E)$. That is, the Lie bracket of vector fields on E can be described using the connection. The connection itself can also be seen as a suitable quotient of the Bott connection ∇^{H_∇}:

$$\nabla^{H_\nabla}_{\sigma^\nabla_{TM}(X)} \overline{e^\uparrow} = \overline{(\nabla_X e)^\uparrow}$$

for all $e \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. That is, the Bott connection associated to H_∇ restricts well to linear (horizontal) and vertical sections.

Example 2.5. Dualizing TE over E, we get the double vector bundle

$$\begin{array}{ccc} T^*E & \xrightarrow{c_E} & E \\ {\scriptstyle r_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ E^* & \xrightarrow[q_{E^*}]{} & M \end{array} .$$

The map r_E is given as follows. For θ_{e_m}, $r_E(\theta_{e_m}) \in E^*_m$,

$$\langle r_E(\theta_{e_m}), e'_m \rangle = \left\langle \theta_{e_m}, \left.\frac{d}{dt}\right|_{t=0} e_m + te'_m \right\rangle$$

for all $e'_m \in E_m$. The addition in $T^*E \to E^*$ is defined as follows. If θ_{e_m} and $\omega_{e'_m}$ are such that $r_E(\theta_{e_m}) = r_E(\omega_{e'_m}) = \varepsilon_m \in E^*_m$, then the sum $\theta_{e_m} +_{r_E} \omega_{e'_m} \in T^*_{e_m+e'_m}E$ is given by

$$\langle \theta_{e_m} +_{E^*} \omega_{e'_m}, v_{e_m} +_{TM} v_{e'_m}\rangle = \langle \theta_{e_m}, v_{e_m}\rangle + \langle \omega_{e'_m}, v_{e'_m}\rangle$$

for all $v_{e_m} \in T_{e_m}E$, $v_{e'_m} \in T_{e'm}E$ such that $(q_E)_*(v_{e_m}) = (q_E)_*(v_{e'_m})$.

For $\varepsilon \in \Gamma(E^*)$, the one-form $\mathbf{d}\ell_\varepsilon$ is linear over ε, and for $\theta \in \Omega^1(M)$, the one-form $q_E^*\theta$ is a core section of $TE \to E$. We have $r_E(\mathbf{d}_{e_m}\ell_\varepsilon) = \varepsilon(m)$ and $r_E((q_E^*\theta)(e_m)) = 0_m^{E^*}$. The sum $\mathbf{d}_{e_m}\ell_\varepsilon +_{r_E} \mathbf{d}_{e'_m}\ell_\varepsilon$ equals $\mathbf{d}_{e_m+e'_m}\ell_\varepsilon$. The vector space $T^*_{e_m}E$ is spanned by $\mathbf{d}_{e_m}\ell_\varepsilon$ and $\mathbf{d}_{e_m}(q_E^*f)$ for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

Example 2.6. By taking the direct sum of the two double vector bundles in the two preceding examples, we get a double vector bundle

$$\begin{array}{ccc} TE \oplus T^*E & \xrightarrow{\pi_E} & E \\ {\scriptstyle \Phi_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM \oplus E^* & \xrightarrow[q_{TM\oplus E^*}]{} & M \end{array},$$

with $\Phi_E = (q_E)_* \oplus r_E$.

In the following, for any section (e, θ) of $E \oplus T^*M$, the vertical section $(e, \theta)^\uparrow \in \Gamma_E(T^{q_E}E \oplus (T^{q_E}E)^\circ)$ is the pair defined by

$$(e,\theta)^\uparrow(e'_m) = \left(\left.\frac{d}{dt}\right|_{t=0} e'_m + te(m), (T_{e'_m}q_E)^t\theta(m)\right) \tag{2.8}$$

for all $e'_m \in E$. Note that by construction the vertical sections $(e, \theta)^\uparrow$ are core sections of $TE \oplus T^*E$ as a vector bundle over E.

A subbundle L of $TE \oplus T^*E \to E$ is said to be **linear** if it projects to a subbundle $U \subseteq TM \oplus E^*$ under Φ_E and if it is also closed under the addition on $TE \oplus T^*E$ as a vector bundle over $TM \oplus E^*$. Such a linear subbundle defines a **sub double vector bundle** of $TE \oplus T^*E$.

A double vector bundle $(D; A, B; M)$ is a **VB-algebroid** ([22]; see also [12]) if there are Lie algebroid structures on $D \to B$ and $A \to M$, such that the anchor $\Theta\colon D \to TB$ is a morphism of double vector bundles over $\rho_A\colon A \to TM$ on one side and if the Lie bracket is linear:

$$[\Gamma^\ell_B(D), \Gamma^\ell_B(D)] \subset \Gamma^\ell_B(D), \qquad [\Gamma^\ell_B(D), \Gamma^c_B(D)] \subset \Gamma^c_B(D), \qquad [\Gamma^c_B(D), \Gamma^c_B(D)] = 0.$$

The vector bundle $A \to M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1, \xi_2 \in \Gamma^\ell_B(D)$ are linear over $a_1, a_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]$ is linear over $[a_1, a_2]$.

Now let $A \to M$ be a Lie algebroid and consider an A-connection ∇ on a vector bundle $E \to M$. Then the space $\Omega^\bullet(A, E)$ of E-valued Lie algebroid forms has an induced operator $\mathbf{d}_\nabla$ given by the Koszul formula:

$$\begin{aligned}\mathbf{d}_\nabla\omega(a_1, \ldots, a_{k+1}) = &\sum_{i<j}(-1)^{i+j}\omega([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{k+1}) \\ &+ \sum_i (-1)^{i+1}\nabla_{a_i}(\omega(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))\end{aligned}$$

for all $\omega \in \Omega^k(A, E)$ and $a_1, \ldots, a_{k+1} \in \Gamma(A)$.

Let e_0, e_1 be two vector bundles over the same base M as A. A **2-term representation up to homotopy of A on $E_0 \oplus E_1$** [1, 12] is the collection of

(1) a map $\partial\colon E_0 \to E_1$,

(2) two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial\circ\nabla^0=\nabla^1\circ\partial$,
(3) an element $R\in\Omega^2(A,\operatorname{Hom}(E_1,E_0))$ such that $R_{\nabla^0}=R\circ\partial$, $R_{\nabla^1}=\partial\circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}}R=0$, where ∇^{Hom} is the connection induced on $\operatorname{Hom}(E_1,E_0)$ by ∇^0 and ∇^1.

Note that Gracia-Saz and Mehta [12] defined this concept independently and called them "superrepresentations".

Consider again a VB-algebroid $(D\to B, A\to M)$ and choose a linear splitting $\Sigma\colon A\times_M B\to D$. Since the anchor Θ_B is linear, it sends a core section $c^\dagger$, $c\in\Gamma(C)$ to a vertical vector field on B. This defines the **core-anchor** $\partial_B\colon C\to B$ given by, $\Theta(c^\dagger)=(\partial_B c)^\uparrow$ for all $c\in\Gamma(C)$ and does not depend on the splitting (see [21]). Since the anchor Θ of a linear section is linear, for each $a\in\Gamma(A)$ the vector field $\Theta(\sigma_A(a))\in\mathfrak{X}^l(B)$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$. This defines a linear connection $\nabla^{AB}\colon\Gamma(A)\times\Gamma(B)\to\Gamma(B)$:

$$\Theta(\sigma_A(a))=\widehat{\nabla^{AB}_a}$$

for all $a\in\Gamma(A)$. Recall further that the anchor $\Theta(c^\dagger)$ of a core section $c^\dagger\in\Gamma^c_B(D)$ is given by $\Theta(c^\dagger)=(\partial_B c)^\uparrow$. Since the bracket of a linear section with a core section is again a core section, we find a linear connection $\nabla^{AC}\colon\Gamma(A)\times\Gamma(C)\to\Gamma(C)$ such that

$$[\sigma_A(a),c^\dagger]=(\nabla^{AC}_a c)^\dagger$$

for all $c\in\Gamma(C)$ and $a\in\Gamma(A)$. The difference $\sigma_A[a_1,a_2]-[\sigma_A(a_1),\sigma_A(a_2)]$ is a core-linear section for all $a_1,a_2\in\Gamma(A)$. This defines a vector valued form $R\in\Omega^2(A,\operatorname{Hom}(B,C))$ such that

$$[\sigma_A(a_1),\sigma_A(a_2)]=\sigma_A[a_1,a_2]-\widetilde{R(a_1,a_2)},$$

for all $a_1,a_2\in\Gamma(A)$. For more details on these constructions, see [12], where the following result is proved.

Theorem 2.7. *Let $(D\to B;A\to M)$ be a VB-algebroid and choose a linear splitting $\Sigma\colon A\times_M B\to D$. The triple $(\nabla^{AB},\nabla^{AC},R)$ defined as above is a 2-term representation up to homotopy of A on the complex $\partial_B\colon C\to B$.*

Conversely, let $(D;A,B;M)$ be a double vector bundle such that A has a Lie algebroid structure and choose a linear splitting $\Sigma\colon A\times_M B\to D$. Then if $(\nabla^{AB},\nabla^{AC},R)$ is a 2-term representation up to homotopy of A on a complex $\partial_B\colon C\to B$, then the equations above define a VB-algebroid structure on $(D\to B;A\to M)$.

Example 2.8. Let $E\to M$ be a vector bundle. The tangent double $(TE;E,TM;M)$ has a VB-algebroid structure $(TE\to E,TM\to M)$. Consider a linear splitting $\Sigma\colon E\times_M TM\to TE$ and the corresponding linear connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ as in Example 2.4. By (2.7), the representation up to homotopy corresponding to this splitting is given by $\partial_E=\mathrm{id}_E\colon E\to E$, (∇,∇,R_∇).

Example 2.9. Now assume that the vector bundle E is a Lie algebroid A. Then the tangent prolongation $(TA\to TM, A\to M)$ has a VB-algebroid structure; see Appendix C. The linear splitting corresponding to a linear connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ defines a horizontal lift $\sigma_A\colon\Gamma(A)\to\Gamma^l_{TM}(TA)$. The corresponding 2-term representation up to homotopy is given by $\partial_{TM}=\rho\colon A\to TM$, $(\nabla^{\mathrm{bas}},\nabla^{\mathrm{bas}},R^{\mathrm{bas}}_\nabla)$, where $\nabla^{\mathrm{bas}}\colon\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ and $\nabla^{\mathrm{bas}}\colon\Gamma(A)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ are the basic connections associated to ∇.

Example 2.10. Let $(A,\rho,[\cdot\,,\cdot])$ be a Lie algebroid over a smooth manifold M. Then $(T^*A\to A^*, A\to M)$ is naturally a VB-algebroid; see Appendix C. A linear splitting Σ^∇ of TA can be dualised to a linear splitting $\Sigma^\star_\nabla\colon A\times_M A^*\to T^*A$. In this splitting, the VB-algebroid structure is equivalent to the 2-representation of A on the complex $\rho^t\colon T^*M\to A^*$ that is defined by the connections

$$\nabla^{\mathrm{bas}^*}\colon\Gamma(A)\times\Gamma(A^*)\to\Gamma(A^*),\qquad \nabla^{\mathrm{bas}^*}\colon\Gamma(A)\times\Omega^1(M)\to\Omega^1(M),\tag{2.9}$$

and the curvature term

$$-R_{\nabla}^{\mathrm{bas}^t} \in \Omega^2(A, \mathrm{Hom}(A^*, T^*M)). \tag{2.10}$$

For more details, consult [11].

Example 2.11. The linear splittings of TA and T^*A described in the previous examples define a linear splitting of the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^* \to TM \oplus A^*, A \to M)$, the fibered product of $TA \to TM$ and $T^*A \to A^*$. The representations up to homotopy found in these two examples sum up to a representation up to homotopy of A on the complex $(\rho, \rho^t)\colon A \oplus T^*M \to TM \oplus A^*$, which describes the VB-algebroid in this linear splitting.

One application of our main results is a general description of linear splittings of $TA \oplus T^*A$, and explicit formulas for the corresponding representations up to homotopy (see Section 5).

3. Dorfman connections: definition and examples

Definition 3.1. *Let $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ be a dull algebroid. Let $B \to M$ be a vector bundle with a fiberwise pairing $\langle\cdot\,,\cdot\rangle\colon Q \times_M B \to \mathbb{R}$ and a map $\mathbf{d}_B\colon C^\infty(M) \to \Gamma(B)$ such that*

$$\langle q, \mathbf{d}_B f\rangle = \rho_Q(q)(f) \tag{3.11}$$

*for all $q \in \Gamma(Q)$ and $f \in C^\infty(M)$. Then $(B, \mathbf{d}_B, \langle\cdot\,,\cdot\rangle)$ is called a **pre-dual** of Q and Q and B are said to be **paired by** $\langle\cdot\,,\cdot\rangle$.*

Remark 3.2. Note that if the pairing is non-degenerate, then $(B \to M, \mathbf{d}_B, \langle\cdot\,,\cdot\rangle)$ is isomorphic to the **dual** of $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ and $\mathbf{d}_{Q^*}\colon C^\infty(M) \to \Gamma(Q^*)$ is *defined* by (3.11), namely $\mathbf{d}_{Q^*} f = \rho_Q^t \mathbf{d} f$.

The following is our main definition.

Definition 3.3. *Let $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ be a dull algebroid and $(B \to M, \mathbf{d}_B, \langle\cdot\,,\cdot\rangle)$ a pre-dual of Q.*

1. *A **Dorfman (Q-)connection on** B is an $\mathbb{R}$-bilinear map*
$$\Delta\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$$
such that
   (a) $\Delta_{fq} b = f\Delta_q b + \langle q, b\rangle \cdot \mathbf{d}_B f$,
   (b) $\Delta_q(fb) = f\Delta_q b + \rho_Q(q)(f) b$ *and*
   (c) $\Delta_q(\mathbf{d}_B f) = \mathbf{d}_B(\pounds_{\rho_Q(q)} f)$
for all $f \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$.
2. *The curvature of Δ is the map*
$$R_\Delta\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(B^* \otimes B),$$
defined on $q, q' \in \Gamma(Q)$ by $R_\Delta(q, q') := \Delta_q \Delta_{q'} - \Delta_{q'} \Delta_q - \Delta_{[q,q']_Q}$.

The failure of a Dorfman connection to be a connection is hence measured by the map $\mathbf{d}_B$ and the pairing of Q with B. We omit the proof of the following proposition.

Proposition 3.4. *Let $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ be a dull algebroid and $(B, \mathbf{d}_B, \langle\cdot\,,\cdot\rangle)$ a pre-dual of Q. Let Δ be a Dorfman Q-connection on B. Then:*

1. *For all $f \in C^\infty(M)$ and $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$, we have $R_\Delta(q, q')(f \cdot b) = f \cdot R_\Delta(q, q')$.*
2. *For all $q_1, q_2, q_3 \in \Gamma(Q)$ and $b \in \Gamma(B)$, we have*
$$\langle R_\Delta(q_1, q_2)(b), q_3\rangle = \langle [[q_1, q_2]_Q, q_3]_Q + [q_2, [q_1, q_3]_Q]_Q - [q_1, [q_2, q_3]_Q]_Q, b\rangle.$$

Note that this does not mean that the curvature of the Dorfman connection vanishes everywhere if Q is a Lie algebroid, since the pairing of Q and B can be degenerate. The following example is a trivial example for this phenomenon.

Example 3.5. Let $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ be a dull algebroid and $B \to M$ a vector bundle. Take the pairing $\langle\cdot\,,\cdot\rangle\colon Q \times_M B \to \mathbb{R}$ and the map $\mathbf{d}_B\colon C^\infty(M) \to \Gamma(B)$ to be trivial. Then any Q-connection on B is also a Dorfman connection.

Example 3.6. The easiest non-trivial example of a Dorfman connection is the map $\pounds : \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$,

$$\langle \pounds_q \tau, q'\rangle = \rho_Q(q)\langle q, \tau\rangle - \langle \tau, [q, q']_Q\rangle,$$

for a dull algebroid $(Q \to M, \rho_Q, [\cdot\,,\cdot]_Q)$ and its dual $(Q^*, \mathbf{d}_{Q^*})$, i.e. with the canonical pairing $Q \times_M Q^* \to \mathbb{R}$ and $\mathbf{d}_{Q^*} = \rho_Q^t \mathbf{d}\colon C^\infty(M) \to \Gamma(Q^*)$.

The third property of a Dorfman connection is immediate by definition of $\pounds$ and the first two properties are easily verified. The curvature vanishes if and only if $[\cdot\,,\cdot]_Q$ satisfies the Jacobi-identity in Leibniz form $[[q_1, q_2]_Q, q_3]_Q + [q_2, [q_1, q_3]_Q]_Q = [q_1, [q_2, q_3]_Q]_Q$ for all $q_1, q_2, q_3 \in \Gamma(Q)$.

The following proposition illustrates the general idea that Dorfman connections are to Courant algebroids what linear connections are to Lie algebroids. Our main result in Section 4 is a further example for this analogy.

Let $(\mathsf{E} \to M, \rho\colon \mathsf{E} \to TM, \langle\cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ be a Courant algebroid. If K is a subalgebroid of E, the (in general singular) distribution $S := \rho(K) \subseteq TM$ is algebraically involutive and we can define the "singular" Bott connection

$$\nabla^S\colon \Gamma(S) \times \frac{\mathfrak{X}(M)}{\Gamma(S)} \to \frac{\mathfrak{X}(M)}{\Gamma(S)}$$

by

$$\nabla^S_s \bar{X} = \overline{[s, X]}$$

for all $X \in \mathfrak{X}(M)$ and $s \in \Gamma(S)$. The anchor $\rho\colon \mathsf{E} \to TM$ induces a map $\bar\rho\colon \Gamma(\mathsf{E}/K) \to \mathfrak{X}(M)/\Gamma(S)$, $\bar\rho(\bar e) = \rho(e) + \Gamma(S)$.

Proposition 3.7. *Let $\mathsf{E} \to M$ be a Courant algebroid and $K \subseteq \mathsf{E}$ an isotropic subalgebroid. Then the map*

$$\Delta\colon \Gamma(K) \times \Gamma(\mathsf{E}/K) \to \Gamma(\mathsf{E}/K), \qquad \Delta_k \bar e = \overline{[\![k, e]\!]}$$

is a Dorfman connection. The dull algebroid structure on K is its induced Lie algebroid structure, the map $\mathbf{d}_{\mathsf{E}/K}$ is just $\mathcal{D} + \Gamma(K)$ and the pairing $\langle\cdot\,,\cdot\rangle\colon K \times_M (\mathsf{E}/K) \to \mathbb{R}$ is the natural pairing induced by the pairing on E.

We have

$$\bar\rho(\Delta_k \bar e) = \nabla^S_{\rho(k)} \bar\rho(\bar e)$$

for all $k \in \Gamma(K)$ and $\bar e \in \Gamma(\mathsf{E}/K)$.

Remark 3.8.
1. Because of the analogy of the Dorfman connection in the last proposition with the Bott connection defined by involutive subbundles of TM, we name this Dorfman connection the **Bott–Dorfman connection associated to** K.
2. Note that if K is a Dirac structure D in E, then $\mathsf{E}/\mathsf{D} \simeq \mathsf{D}^*$ and the Dorfman connection is just the Lie algebroid derivative of D on $\Gamma(\mathsf{D}^*)$.

4. Linear splittings of $TE \oplus T^*E$

Consider a vector bundle $q_E\colon E \to M$. Recall from Example 2.4 that an ordinary connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is equivalent to a linear splitting $\Sigma\colon E \times_M TM \to TE$. We show that a Dorfman connection $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ is the same as a linear splitting $\Sigma\colon (TM \oplus E^*) \times_M E \to TE \oplus T^*E$. Further, we show that the image L_Δ of Σ in $TE \oplus T^*E$ is maximally isotropic relatively to the canonical pairing if and only if the bracket $[\![\cdot\,,\cdot]\!]_\Delta$ dual[3] to the Dorfman connection is skew-symmetric, and we show how the failure of $\Gamma(L_\Delta)$ to be closed under the Dorfman bracket is measured by the curvature R_Δ.

Here, the vector bundle $TM\oplus E^*$ is always anchored by the projection $\operatorname{pr}_{TM}\colon TM\oplus E^*\to TM$ and the dual $E\oplus T^*M$ is always paired with $TM\oplus E^*$ via the canonical non-degenerate pairing. The map $\mathbf{d}_{E\oplus T^*M}\colon C^\infty(M)\to\Gamma(E\oplus T^*M)$ is consequently always

$$\mathbf{d}_{E\oplus T^*M}=\operatorname{pr}_{TM}^t\circ\mathbf{d},$$

i.e. $\mathbf{d}_{E\oplus T^*M}f=(0,\mathbf{d}f)$ for all $f\in C^\infty(M)$. A Dorfman connection Δ is here always a $TM\oplus E^*$-Dorfman connection on $E\oplus T^*M$, with dual $[\![\cdot\,,\cdot]\!]_\Delta$. Note that since the pairing is non-degenerate, the Dorfman connection is completely determined by its dual structure, the associated dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$ and vice-versa. Hence, we can say here that a Dorfman connection is equivalent to a dull algebroid $(TM\oplus E^*,\operatorname{pr}_{TM},[\![\cdot\,,\cdot]\!]_\Delta)$. It is easy to see, using Proposition 3.4, that the curvature R_Δ always vanishes on $(TM\oplus E^*)\otimes(TM\oplus E^*)\otimes(0\oplus T^*M)$ and so it can be identified with an element of $\Omega^2(TM\oplus E^*,\operatorname{Hom}(E,E\oplus T^*M))$.

4.1. **Dorfman connection associated to a linear splitting of $TE\oplus T^*E$.** Consider a linear splitting

$$\Sigma\colon E\times_M(TM\oplus E^*)\to TE\oplus T^*E$$

and the corresponding horizontal lift $\sigma_{TM\oplus E^*}\colon\Gamma(TM\oplus E^*)\to\Gamma^l_E(TE\oplus T^*E)$. Note that by the definition of the horizontal lift, we have $\sigma_{TM\oplus E^*}(f\cdot\nu)=q_E^*f\cdot\sigma_{TM\oplus E^*}(\nu)$ for all $f\in C^\infty(M)$ and $\nu\in\Gamma(TM\oplus E^*)$. Also by definition, $((q_E)_*,r_E)(\sigma_{TM\oplus E^*}(X,\varepsilon)(e(m)))=(X(m),\varepsilon(m))=((q_E)_*,r_E)(T_meX(m),\mathbf{d}_{e(m)}\ell_\varepsilon)$ for all $X\in\mathfrak{X}(M)$, $e\in\Gamma(E)$ and $\varepsilon\in\Gamma(E^*)$. Hence the difference

$$(T_meX(m),\mathbf{d}_{e(m)}\ell_\varepsilon)-\sigma_{TM\oplus E^*}(X,\varepsilon)(e(m))$$

is a core element, which can be written

$$(\delta_{(X,\varepsilon)}e)^\uparrow(e(m)),$$

defining a map[4] $\delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E)\to\Gamma(E\oplus T^*M)$.

Set $\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$, $\Delta_{(X,\varepsilon)}(e,\theta)=\delta_{(X,\varepsilon)}e+(0,\pounds_X\theta)$. We prove that Δ is a Dorfman connection. First

$$\begin{aligned}&(T_m(fe)X(m),\mathbf{d}_{f(m)e(m)}\ell_\varepsilon)\\&=(T_m(f(m)e)X(m)+X(f)(m)e^\uparrow(f(m)e(m)),\mathbf{d}_{f(m)e(m)}\ell_\varepsilon)\end{aligned}\tag{4.12}$$

and $\sigma_{TM\oplus E^*}(X,\varepsilon)((fe)(m))=\sigma_{TM\oplus E^*}(X,\varepsilon)(f(m)e(m))$ yield $\delta_{(X,\varepsilon)}(fe)=f\delta_{(X,\varepsilon)}e+X(f)(e,0)$. This implies

$$\Delta_{(X,\varepsilon)}(f(e,\theta))=f\Delta_{(X,\varepsilon)}(e,\theta)+X(f)(e,\theta)$$

for all $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$, $(e,\theta)\in\Gamma(E\oplus T^*M)$ and $f\in C^\infty(M)$. Then

$$(T_me(fX),\mathbf{d}_{e(m)}\ell_{f\varepsilon})=(T_me(f(m)X(m)),f(m)\mathbf{d}_{e(m)}\ell_\varepsilon+\langle\varepsilon,e\rangle(m)\mathbf{d}_{e(m)}(q^*f))\tag{4.13}$$

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[3]Since the Dorfman connection and the dual dull bracket corresponding to a linear splitting of $TE\oplus T^*E$ encode the Courant-Dorfman bracket on E, we write the dull brackets on $\Gamma(TM\oplus E^*)$ with double bars, as we write Courant algebroid brackets.

[4]To see that $\delta_{(X,\varepsilon)}e$ is a smooth section of $E\oplus T^*M$, it suffices to show that its pairing with each section of $TM\oplus E^*$ is smooth. For $(Y,\chi)\in\Gamma(TM\oplus E^*)$, we have $\langle\delta_{(X,\varepsilon)}e,(Y,\chi)\rangle(m)=\langle\delta_{(X,\varepsilon)}e^\uparrow(e(m)),(T_meY(m),\mathbf{d}_{e(m)}\ell_\chi)\rangle$ and so

$$\langle\delta_{(X,\varepsilon)}e,(Y,\chi)\rangle(m)=\langle(T_meX(m),\mathbf{d}_{e(m)}\ell_\varepsilon)-\sigma_{TM\oplus E^*}(X,\varepsilon)(e(m)),(T_meY(m),\mathbf{d}_{e(m)}\ell_\chi)\rangle,$$

which is

$$Y(m)\langle\varepsilon,e\rangle+X(m)\langle\chi,e\rangle-\langle\Sigma((X,\varepsilon)(m),e(m)),(T_meY(m),\mathbf{d}_{e(m)}\ell_\chi)\rangle.$$

This depends smoothly on m.

and $\sigma_{TM\oplus E^*}(f\cdot(X,\varepsilon))=q_E^*f\cdot\sigma_{TM\oplus E^*}(X,\varepsilon)$ yield $\delta_{f(X,\varepsilon)}e=f\delta_{(X,\varepsilon)}e+(0,\langle e,\varepsilon\rangle\mathbf{d}f)$. Since $\pounds_{fX}\theta=X(f)\theta+\langle X,\theta\rangle\mathbf{d}f$, we get

$$\Delta_{f(X,\varepsilon)}(e,\theta)=f\Delta_{(X,\varepsilon)}(e,\theta)+\langle(e,\theta),(X,\varepsilon)\rangle(0,\mathbf{d}f)$$

for all $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$, $(e,\theta)\in\Gamma(E\oplus T^*M)$ and $f\in C^\infty(M)$. The equality $\Delta_{(X,\varepsilon)}(0,\mathbf{d}f)=(0,\mathbf{d}\pounds_X f)$ is immediate.

Conversely let $E\to M$ be a vector bundle and consider a Dorfman connection $\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$. We want to define a linear splitting $\Sigma\colon(TM\oplus E^*)\times_M E\to TE\oplus T^*E$ by

$$\Sigma((v_m,\varepsilon_m),e_m)=(T_meX(m),\mathbf{d}\ell_\varepsilon(e_m))-\Delta_{(X,\varepsilon)}(e,0)^\uparrow(e_m) \tag{4.14}$$

for any sections $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $e\in\Gamma(E)$ such that $X(m)=v_m$, $\varepsilon(m)=\varepsilon_m$ and $e(m)=e_m$. For $X\in\mathfrak{X}(M)$, $\varepsilon\in\Gamma(E^*)$ and $e\in\Gamma(E)$ define the element

$$\Pi(X,\varepsilon,e)(m)=(T_meX(m),\mathbf{d}\ell_\varepsilon(e_m))-\Delta_{(X,\varepsilon)}(e,0)^\uparrow(e_m)$$

of $TE\oplus T^*E$. By (4.12) and the properties of the Dorfman connection we have $\Pi(X,\varepsilon,fe)(m)=f(m)\cdot_{TM\oplus E^*}\Pi(X,\varepsilon,e)(m)$ and by (4.13) we have $\Pi(fX,f\varepsilon,e)(m)=f(m)\cdot_E\Pi(X,\varepsilon,e)(m)$ and for all $f\in C^\infty(M)$, $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $e\in\Gamma(E)$. Using this, it is easy to show that the map in (4.14) is a well-defined, injective morphism of double vector bundles. Since it is the identity on the sides, it is a linear splitting of $TE\oplus T^*E$.

Hence, we have proved our main theorem:

Theorem 4.1. *Let $E\to M$ be a vector bundle. A linear splitting $\Sigma\colon(TM\oplus E^*)\times_M E\to TE\oplus T^*E$ defines a Dorfman connection $\Delta^\Sigma\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$ by*

$$\Sigma((X,\varepsilon)(m),e(m))=(T_meX(m),\mathbf{d}_{e_m}\ell_\varepsilon)-\Delta^\Sigma_{(X,\varepsilon)}(e,0)^\uparrow(e(m)) \tag{4.15}$$

and $\Delta_{(X,\varepsilon)}(0,\theta)=(0,\pounds_X\theta)$ for all $e\in\Gamma(E)$, $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $\theta\in\Omega^1(M)$. Conversely, each Dorfman connection $\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$ defines a linear splitting $\Sigma^\Delta\colon(TM\oplus E^*)\times_M E\to TE\oplus T^*E$ as in (4.15) *and the maps*

$$\Delta\mapsto\Sigma^\Delta,\qquad\qquad\Delta^\Sigma\leftarrow\!\shortmid\Sigma$$

are inverse to each other.

In short we have a bijection

$$\left\{\begin{array}{c}(TM\oplus E^*)\text{-Dorfman connections}\\ \Delta\text{ on }E\oplus T^*M\end{array}\right\}\leftrightarrow\left\{\begin{array}{c}\text{Linear splittings}\\ \Sigma\colon(TM\oplus E^*)\times_M E\to TE\oplus T^*E\end{array}\right\}.$$

Since a $(TM\oplus E^*)$-Dorfman connection Δ on $E\oplus T^*M$ is equivalent to a dull algebroid structure $(\mathrm{pr}_{TM},[\![\cdot\,,\cdot]\!]_\Delta)$ on $TM\oplus E^*$, we can reformulate this bijection as follows:

$$\left\{\begin{array}{c}\text{Dull algebroids}\\ (TM\oplus E^*,\mathrm{pr}_{TM},[\![\cdot\,,\cdot]\!])\end{array}\right\}\leftrightarrow\left\{\begin{array}{c}\text{Linear splittings}\\ \Sigma\colon(TM\oplus E^*)\times_M E\to TE\oplus T^*E\end{array}\right\}.$$

Example 4.2. Let $E\to M$ be a vector bundle with a linear connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$. Then the **standard Dorfman connection associated to** ∇ is the map

$$\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M),$$

$$\Delta_{(X,\varepsilon)}(e,\theta)=(\nabla_Xe,\pounds_X\theta+\langle\nabla^*_\cdot\varepsilon,e\rangle).$$

The dual bracket is in this case defined by

$$[\![(X,\varepsilon),(Y,\chi)]\!]_\Delta=([X,Y],\nabla^*_X\chi-\nabla^*_Y\varepsilon)$$

for all $(X,\varepsilon),(Y,\chi)\in\Gamma(TM\oplus E^*)$.

The curvature of the standard Dorfman connection Δ associated to ∇ is given by

$$R_\Delta((X,\varepsilon),(Y,\eta)) = (R_\nabla(X,Y), R_{\nabla^*}(\cdot,X)(\eta) - R_{\nabla^*}(\cdot,Y)(\varepsilon)).$$

As a consequence, we find easily that $(TM\oplus E^*, \mathrm{pr}_{TM}, [\![\cdot\,,\cdot]\!]_\Delta)$ is a Lie algebroid if and only if ∇ is flat.

For any section $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$, the horizontal lift is

$$\sigma^\Delta_{TM\oplus E^*}(X,\varepsilon)(e_m) = (T_meX(m), \mathbf{d}_{e_m}\ell_\varepsilon) - \left(\left.\frac{d}{dt}\right|_{t=0} e_m + t\nabla_X e, (T_{e_m}q_E)^t\langle\nabla^*_\cdot\varepsilon, e\rangle\right)$$

and the subbundle L_Δ spanned by these sections is equal to $H_\nabla\oplus H_\nabla^\circ$. Hence, the standard Dorfman connection associated to a connection ∇ is the same as the splitting

$$TE\oplus T^*E \cong (T^{q_E}E\oplus(T^{q_E}E)^\circ)\oplus(H_\nabla\oplus H_\nabla^\circ),$$

the sum of a (trivial) Dirac structure and an almost Dirac structure.

Note that $H_\nabla\oplus H_\nabla^\circ$ is a Dirac structure if and only if ∇ is flat, that is, if and only if $(TM\oplus E^*, \mathrm{pr}_{TM}, [\![\cdot\,,\cdot]\!]_\Delta)$ is a Lie algebroid. This is not a coincidence, but a special case of our next main result in Theorem 4.9.

Now we discuss more intricate examples of Dorfman connections $\Delta\colon \Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$. The geometric meaning of the corresponding linear splittings will be explained later.

Example 4.3. Consider a dull algebroid $(A,\rho,[\cdot\,,\cdot])$ with *skew-symmetric bracket*. We construct a $TM\oplus A$-Dorfman connection Δ on $A^*\oplus T^*M$, hence corresponding to a linear splitting $\Sigma\colon (TM\oplus A)\times_M A^*\to TA^*\oplus T^*A^*$ of the Pontryagin bundle over A^*. Take any connection $\nabla\colon \mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ and recall the definition of the basic connection $\nabla^{\mathrm{bas}}\colon \Gamma(A)\times\Gamma(A)\to\Gamma(A)$ associated to ∇ and the dull algebroid structure on A:

$$\nabla^{\mathrm{bas}}_a b = [a,b] + \nabla_{\rho(b)}a$$

for all $a,b\in\Gamma(A)$. The Dorfman connection

$$\Delta\colon \Gamma(TM\oplus A)\times\Gamma(A^*\oplus T^*M)\to\Gamma(A^*\oplus T^*M)$$

is defined by

$$\Delta_{(X,a)}(\alpha,\theta) = \left(\langle\alpha,\nabla^{\mathrm{bas}}_\cdot a\rangle + \nabla^*_X\alpha - \rho^t\langle\nabla_\cdot a,\alpha\rangle, \pounds_X\theta + \langle\nabla_\cdot a,\alpha\rangle\right).$$

The bracket $[\![\cdot\,,\cdot]\!]_\Delta$ on sections of $TM\oplus A$ is then given by

$$[\![(X,a),(Y,b)]\!]_\Delta = \left([X,Y], \nabla_X b - \nabla_Y a + \nabla_{\rho(b)}a - \nabla_{\rho(a)}b + [a,b]\right).$$

Since it is skew-symmetric, the image L_Δ of Σ is in this case maximally isotropic. The projection pr_{TM} obviously intertwines this bracket with the Lie bracket of vector fields. The curvature of this Dorfman connection is given by

(4.16)

$$\begin{aligned}
&\langle R_\Delta((X,a),(Y,b))(\alpha,\theta),(Z,c)\rangle = -\langle[\![(X,a),[\![(Y,b),(Z,c)]\!]_\Delta]\!]_\Delta + \text{c.p.},(\alpha,\theta)\rangle\\
&= -\left\langle\left(R_\nabla(X,Y)c - R_\nabla(\rho(a),Y)c\right) + \text{c.p.},\alpha\right\rangle - \left\langle\left(R_\nabla(\rho(a),\rho(b))c - R_\nabla(X,\rho(b))c\right) + \text{c.p.},\alpha\right\rangle\\
&\quad - \left\langle\left(R^{\mathrm{bas}}_\nabla(a,b)Z - R^{\mathrm{bas}}_\nabla(a,b)\rho(c)\right) + \text{c.p.},\alpha\right\rangle - \langle[a,[b,c]] + [b,[c,a]] + [c,[a,b]],\alpha\rangle.
\end{aligned}$$

The proof of this formula is a rather long, but straightforward computation and we omit it here.

The next subsection explains the signification of this example in terms of the linear almost Poisson structure defined on A^* by the skew-symmetric dull algebroid structure.

Example 4.4. Consider a vector bundle $E \to M$ endowed with a vector bundle morphism $\sigma\colon E \to T^*M$ over the identity and a connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Define the Dorfman connection

$$\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$$

by

$$\Delta_{(X,\varepsilon)}(e,\theta) = (\nabla_X e, \pounds_X(\theta - \sigma(e)) + \langle \nabla^*_\cdot(\sigma^t X + \varepsilon), e\rangle + \sigma(\nabla_X e)).$$

The bracket $[\![\cdot\,,\cdot]\!]_\Delta$ on sections of $TM \oplus E^*$ is here given by

$$[\![(X,\varepsilon),(Y,\eta)]\!]_\Delta = ([X,Y], \nabla^*_X(\eta + \sigma^t Y) - \nabla^*_Y(\varepsilon + \sigma^t X) - \sigma^t[X,Y]).$$

In this case also, the image L_Δ of Σ^Δ is maximally isotropic.

Here also, we give the curvature of the Dorfman connection in terms of the Jacobiator of the associated bracket:

$$[\![(X,\varepsilon),[\![(Y,\eta),(Z,\gamma)]\!]_\Delta]\!]_\Delta + \text{c.p.} = \Big(0, R_{\nabla^*}(X,Y)(\gamma + \sigma^t Z) + \text{c.p.}\Big). \tag{4.17}$$

Example 4.22 shows how this Dorfman connection is related to the 2-form $\sigma^*\omega_{\text{can}} \in \Omega^2(E)$, where ω_{can} is the canonical symplectic form on T^*M.

4.2. **The canonical pairing, the anchor and the Courant-Dorfman bracket on $TE \oplus T^*E$.** This section shows that the image of a linear splitting $\Sigma\colon (TM \oplus E^*) \times_M E \to TE \oplus T^*E$ is maximally isotropic if and only if the corresponding dull bracket $[\![\cdot\,,\cdot]\!]_\Sigma$ is skew-symmetric, and its set of sections is closed under the Courant-Dorfman bracket if and only if the curvature of Δ^Σ vanishes.

Here and later, we need the following notation. Let $E \to M$ be a vector bundle and $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ a Dorfman connection. We call $\text{Skew}_\Delta \in \Gamma((TM \oplus E^*) \otimes (TM \oplus E^*) \otimes E^*)$ the tensor defined by

$$\text{Skew}_\Delta(\nu_1,\nu_2) = \text{pr}_{E^*}([\![\nu_1,\nu_2]\!]_\Delta + [\![\nu_2,\nu_1]\!]_\Delta)$$

for all $\nu_1,\nu_2 \in \Gamma(TM \oplus E^*)$. By the Leibniz identity, this is indeed $C^\infty(M)$-linear in both arguments. Note that the TM-part of $[\![\nu_1,\nu_2]\!]_\Delta + [\![\nu_2,\nu_1]\!]_\Delta$ always vanishes since the Lie bracket of vector fields is skew-symmetric.

In this subsection, given a Dorfman connection $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$, we always write σ^Δ for the induced horizontal lift $\sigma^\Delta_{TM\oplus E^*}\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$.

Theorem 4.5. *Let $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ be a Dorfman connection and choose $\nu,\nu_1,\nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau,\tau_1,\tau_2 \in \Gamma(E \oplus T^*M)$. Then*

(1) $\left\langle \sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)\right\rangle = \ell_{\text{Skew}_\Delta(\nu_1,\nu_2)}$,
(2) $\left\langle \sigma^\Delta(\nu), \tau^\uparrow\right\rangle = q_E^*\langle \nu,\tau\rangle$,
(3) $\left\langle \tau_1^\uparrow, \tau_2^\uparrow\right\rangle = 0$.

Proof. Since the second and third equalities are immediate by (4.14), we prove only the first one. We write $\nu_1 = (X,\varepsilon)$, $\nu_2 = (Y,\eta)$ and compute for any section $e \in \Gamma(E)$:

$$\begin{aligned}
&\big\langle (T_m eX(m), \mathbf{d}\ell_\varepsilon(e_m)) - \Delta_{(X,\varepsilon)}(e,0)^\uparrow(e_m), (T_m eY(m), \mathbf{d}\ell_\eta(e_m)) - \Delta_{(Y,\eta)}(e,0)^\uparrow(e_m)\big\rangle \\
&= X(m)\langle \eta, e\rangle - \langle \text{pr}_{T^*M}\Delta_{(Y,\eta)}(e,0), X(m)\rangle - \langle \eta(m), \text{pr}_E\Delta_{(X,\varepsilon)}(e,0)\rangle \\
&\quad + Y(m)\langle \varepsilon, e\rangle - \langle \text{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e,0), Y(m)\rangle - \langle \varepsilon(m), \text{pr}_E\Delta_{(Y,\eta)}(e,0)\rangle \\
&= \big(X\langle \eta, e\rangle - \langle \Delta_{(Y,\eta)}(e,0), (X,\varepsilon)\rangle + Y\langle \varepsilon, e\rangle - \langle \Delta_{(X,\varepsilon)}(e,0), (Y,\eta)\rangle\big)(m) \\
&= \langle (e,0), [\![\nu_2,\nu_1]\!]_\Delta + [\![\nu_1,\nu_2]\!]_\Delta\rangle.
\end{aligned}$$

□

Corollary 4.6. *The dull bracket* $[\![\cdot\,,\cdot]\!]_\Delta$ *associated to a Dorfman connection* Δ *is skew-symmetric if and only if the image of* Σ^Δ *is maximally isotropic in* $TE \oplus T^*E$. *The corresponding splitting*

$$TE \oplus T^*E \cong (T^{q_E}E \oplus (T^{q_E}E)^\circ) \oplus L_\Delta$$

is then the direct sum of the Dirac structure $T^{q_E}E \oplus (T^{q_E}E)^\circ$ *and the linear almost Dirac structure* $L_\Delta = \Sigma^\Delta((TM \oplus E^*) \times_M E)$.

Proof. Since the rank of L_Δ as a vector bundle over E is equal to the dimension of E as a manifold, we have only to show that L_Δ is isotropic if and only if $[\![\cdot\,,\cdot]\!]_\Delta$ is skew-symmetric. But this is immediate by the preceding theorem. □

Next we describe the anchor of the Courant algebroid $TE \oplus T^*E \to E$ in terms of linear splittings and the corresponding Dorfman connections. We begin with a proposition, the proof of which is left to the reader.

Proposition 4.7. *Let* $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ *be a Dorfman connection. Then the map*

$$\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E), \qquad \nabla_\nu e = \mathrm{pr}_E(\Delta_\nu(e, 0))$$

is a linear connection.

This linear connection encodes in the following manner the anchor $\mathrm{pr}_{TE}\colon TE \oplus T^*E \to TE$.

Theorem 4.8. *Let* $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ *be a Dorfman connection and choose* $\nu \in \Gamma(TM \oplus E^*)$ *and* $\tau \in \Gamma(E \oplus T^*M)$. *Then* $\mathrm{pr}_{TE}\left(\sigma^\Delta_{TM\oplus E^*}(\nu)\right) = \widehat{\nabla_\nu}$ *and* $\mathrm{pr}_{TE}(\tau^\uparrow) = (\mathrm{pr}_E \tau)^\uparrow$.

Proof. The second claim is immediate by the definition of $\tau^\uparrow$ in (2.8). For the first equality, note that by definition of ∇ and $\sigma^\Delta_{TM\oplus E^*}(\nu)$,

$$\mathrm{pr}_{TE}\left(\sigma^\Delta_{TM\oplus E^*}(\nu)\right)(e(m)) = T_m e(\mathrm{pr}_{TM}\nu)(m) +_E \left.\frac{d}{dt}\right|_{t=0} e(m) - t\nabla_\nu e(m)$$

for all $e \in \Gamma(E)$ and $m \in M$. By (2.6), this proves the claim. □

Finally, we show how the Dorfman connection encodes the Courant-Dorfman bracket on linear and core sections. The next theorem shows how the integrability of L_Δ is related to the curvature R_Δ of the Dorfman connection.

Theorem 4.9. *Let* $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ *be a Dorfman connection and choose* $\nu, \nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ *and* $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. *Then*

(1) $\left[\!\!\left[\tau_1^\uparrow, \tau_2^\uparrow\right]\!\!\right] = 0$,
(2) $[\![\sigma^\Delta(\nu), \tau^\uparrow]\!] = (\Delta_\nu \tau)^\uparrow$,
(3) $[\![\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)]\!] = \sigma^\Delta([\![\nu_1, \nu_2]\!]_\Delta) - R_\Delta\widetilde{(\nu_1, \nu_2)}(\cdot, 0)$.

The proof of this theorem is relatively long and technical, it can be found in Appendix B.

Remark 4.10. (1) If the Courant-Dorfman bracket is twisted by a linear closed 3-form H over a map $\bar H\colon TM \wedge TM \to E^*$ [4], then the bracket $[\![\tilde\nu_1, \tilde\nu_2]\!]$ is linear over $[\![\nu_1, \nu_2]\!]_{\bar H,\Delta} = [\![\nu_1, \nu_2]\!]_\Delta + (0, \bar H(X_1, X_2))$. Note that the Dorfman connection dual to this bracket is $\Delta^{\bar H}_v \sigma = \Delta_v \sigma + (0, \langle \bar H(X, \cdot), e\rangle)$. A more careful study of general exact Courant algebroids [30] over vector bundles and of the corresponding twisting of the Dorfman connections and dull algebroids corresponding to splittings of $TE \oplus T^*E$ will be done somewhere else.

(2) The **Courant bracket**, i.e. the skew-symmetric counterpart of the Courant-Dorfman bracket, is given by
   (a) $\left[\!\left[\tau_1^\uparrow, \tau_2^\uparrow\right]\!\right]_C = 0$,
   (b) $[\![\sigma^\Delta(\nu), \tau^\uparrow]\!]_C = [\![\sigma^\Delta(\nu), \tau^\uparrow]\!] - (0, \frac{1}{2} q_E^* \mathbf{d}\langle \nu, \tau\rangle) = \left(\Delta_\nu \tau - (0, \frac{1}{2}\mathbf{d}\langle \nu, \tau\rangle)\right)^\uparrow$,
   (c) $[\![\widetilde{\nu_1}, \widetilde{\nu_2}]\!]_C = [\![\nu_1, \nu_2]\!]_\Delta - R_\Delta(\nu_1, \nu_2)(\cdot, 0)^\uparrow - (0, \frac{1}{2}\mathbf{d}\ell_{\mathrm{Skew}_\Delta(\nu_1,\nu_2)})$.
   We chose to work with the Courant Dorfman bracket because it is described naturally by Dorfman connections, as in Proposition 4.1. This is why we chose to call the Dorfman connections after I. Dorfman. Since Dorfman connections are equivalent to linear splittings of the standard Courant algebroid over a vector bundle, this suggests that the Courant Dorfman bracket is more natural than the Courant bracket.

The following corollary of Corollary 4.6 and Theorem 4.9 is immediate.

Corollary 4.11. *Let $E \to M$ be a vector bundle and consider a linear splitting $TE \oplus T^*E = (T^{q_E}E \oplus (T^{q_E}E)^\circ) \oplus L$. Then the horizontal space L is a Dirac structure if and only if the corresponding dull algebroid $(TM \oplus E^*, \mathrm{pr}_{TM}, [\![\cdot\,,\cdot]\!]_L)$ is a Lie algebroid.*

In the next section we study more general (non-horizontal) Dirac structures on E.

4.3. **VB-Dirac structures and Dorfman connections.** We consider linear subbundles

$$\begin{array}{ccc} D & \longrightarrow & U \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \qquad \text{of} \qquad \begin{array}{ccc} TE \oplus T^*E & \longrightarrow & TM \oplus E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

The intersection of such a sub- double vector bundle D with the vertical space $T^{q_E}E \oplus (T^{q_E}E)^\circ$ always has constant rank on E and there is a subbundle $K \subseteq E \oplus T^*M$ such that $D \cap (T^{q_E}E \oplus (T^{q_E}E)^\circ)$ is spanned over E by the sections $k^\uparrow$ for all $k \in \Gamma(K)$. In other words K is the core of D. The following proposition follows from this observation.

Proposition 4.12. *Let E be a vector bundle endowed with a linear subbundle $D \subseteq TE \oplus T^*E$ over $U \subseteq TM \oplus E^*$ and with core $K \subseteq E \oplus T^*M$. Then there exists a Dorfman connection Δ such that D is spanned by the sections $k^\uparrow$ for all $k \in \Gamma(K)$ and $\sigma^\Delta(u)$ for all $u \in \Gamma(U)$.*

The Dorfman connection Δ is then said to be **adapted** to D. Conversely, given a Dorfman connection and two subbundles $U \subseteq TM \oplus E^*$ and $K \subseteq E \oplus T^*M$, we call $D_{U,K,\Delta}$ the linear subbundle of $TE \oplus T^*E \to E$ that is spanned by $k^\uparrow$, for all $k \in \Gamma(K)$ and $\sigma^\Delta(u)$ for all $u \in \Gamma(U)$.

Proof. To see that such a splitting exist, we work with decompositions. Since D and $TE \oplus T^*E$ are both double vector bundles, there exist two decompositions $\mathbb{I}_D\colon E \times_M U \times_M K \to D$ and $\mathbb{I}\colon E \times_M (TM \oplus E^*) \times_M (E \oplus T^*M) \to TE \oplus T^*E$. Let $\iota\colon D \to TE \oplus T^*E$ be the double vector bundle inclusion, over $\iota_U\colon U \to TM \oplus E^*$ and the identity on E, and with core $\iota_K\colon K \to E \oplus T^*M$. Then there exists $\phi \in \Gamma(E^* \otimes U^* \otimes E \oplus T^*M)$ such that the map $\mathbb{I}^{-1} \circ \iota \circ \mathbb{I}_D\colon E \times_M U \times_M K \to E \times_M (TM \oplus E^*) \times_M (E \oplus T^*M)$ sends (e_m, u_m, k_m) to $(e_m, \iota_U(u_m), \iota_K(k_m) + \phi(e_m, u_m))$. Using local basis sections of $TM \oplus E^*$ adapted to U and a partition of unity on M, extend ϕ to $\hat\phi \in \Gamma(E^* \otimes (TM \oplus E^*) \otimes (E \oplus T^*M))$. Then define a new decomposition $\tilde{\mathbb{I}}^{-1}\colon TE \oplus T^*E \to E \times_M (TM \oplus E^*) \times_M (E \oplus T^*M)$ by $\tilde{\mathbb{I}}^{-1}(\xi) = \mathbb{I}^{-1}(e) +_E (e_m, 0_m, -\hat\phi(e_m, \nu_m)) = \mathbb{I}^{-1}(e) +_{TM\oplus E^*} (0_m, \nu_m, -\hat\phi(e_m, \nu_m))$ for $\xi \in Te_mE \times T^*_{e_m}E$ with $\Phi_E(\xi) = \nu_m$. Then $(\tilde{\mathbb{I}} \circ \iota \circ \mathbb{I}_D)(e_m, u_m, k_m) = (e_m, \iota_U(u_m), \iota_K(k_m))$ for all $(e_m, u_m, k_m) \in E \times_M U \times_M K$. The corresponding linear splitting $\tilde\Sigma\colon E \times_M (TM \oplus E^*) \to TE \oplus T^*E$, $\tilde\Sigma(e_m, \nu_m) = \tilde{\mathbb{I}}(e_m, \nu_m, 0_m)$ sends $(e_m, \iota_U(u_m))$ to $\iota(\mathbb{I}_D(e_m, u_m, 0_m)) \in \iota(D)$. $\square$

Next we ask how many linear splittings are adapted to D, and how two linear splittings that are adapted to D are related.

Definition 4.13. *Two Dorfman connections Δ, Δ' are said to be (U,K)-**equivalent** if $(\Delta - \Delta')(\Gamma(U) \times \Gamma(E \oplus 0)) \subseteq \Gamma(K)$.*

The following proposition shows that this defines an equivalence relation on the set of Dorfman connections. We write $[\Delta]_{U,K}$, or simply $[\Delta]$, for the (U,K)-class of the Dorfman connection Δ. By the next proposition, triples $(U, K, [\Delta])$ are in one-one correspondence with linear subbundles of $TE \oplus T^*E \to E$.

Proposition 4.14. *Choose two Dorfman connections Δ, Δ' and assume that Δ is adapted to D. Then Δ' is adapted to D if and only if Δ and Δ' are (U,K)-equivalent.*

Proof. Assume that Δ is adapted to D. Then D is spanned by the sections $\sigma^\Delta(u)$ and $k^\uparrow$ for all $k \in \Gamma(K)$ and $u \in \Gamma(U)$. If Δ and Δ' are (U,K)-equivalent, we have $\sigma_\Delta(u) - \sigma_{\Delta'}(u) = k^\uparrow$ for some $k \in \Gamma(K)$. By (4.14), this implies immediately that Δ' is adapted to D. The converse implication can be proved in a similar manner. □

The following theorem follows from the results in the preceding subsection.

Theorem 4.15. *Let D be a linear subbundle of $TE \oplus T^*E \to E$ over $U \subseteq TM \oplus E^*$ and with core $K \subseteq E \oplus T^*M$, and choose a Dorfman connection Δ that is adapted to D. Then*
1. *D is isotropic if and only if $\operatorname{Skew}_\Delta|_{U \otimes U} = 0$ and $K \subseteq U^\circ$.*
2. *D is maximally isotropic if and only if $\operatorname{Skew}_\Delta|_{U \otimes U} = 0$ and $K = U^\circ$.*
3. *$\Gamma(D)$ is closed under the Courant-Dorfman bracket if and only if*
   (a) *$\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$,*
   (b) *$[\![\Gamma(U), \Gamma(U)]\!]_\Delta \subseteq \Gamma(U)$,*
   (c) *$R_\Delta\Big(U \otimes U \otimes (E \oplus T^*M)\Big) \subseteq K$.*

Proof. This is an immediate corollary of the results in the preceding subsection, using $R_\Delta\Big((TM \oplus E^*) \otimes (TM \oplus E^*) \otimes (0 \oplus T^*M)\Big) = 0$. To see this use (2) of Proposition 3.4, bearing in mind that the anchor is pr_{TM}. □

Corollary 4.16. *Let D be a linear subbundle of $TE \oplus T^*E \to E$ over $U \subseteq TM \oplus E^*$ and with core $K \subseteq E \oplus T^*M$, and choose a Dorfman connection Δ that is adapted to D. Then*
1. *D is an isotropic subalgebroid of $TE \oplus T^*E \to E$ if and only if*
   (a) *$U \subseteq K^\circ$,*
   (b) *$\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$,*
   (c) *$(U, \operatorname{pr}_{TM}|_U, [\![\cdot\,,\cdot]\!]_\Delta|_{\Gamma(U)\times\Gamma(U)})$ is a skew-symmetric dull algebroid.*
   (d) *the induced Dorfman connection*
   $$\bar\Delta\colon \Gamma(U) \times \Gamma((E \oplus T^*M)/K) \to \Gamma((E \oplus T^*M)/K)$$
   is flat.
2. *D is a Dirac structure if and only if $U = K^\circ$ and $(U, \operatorname{pr}_{TM}|_U, [\![\cdot\,,\cdot]\!]_\Delta|_{\Gamma(U)\times\Gamma(U)})$ is a Lie algebroid.*

Note that in the second situation, the induced Dorfman connection $\bar\Delta$ is just the Lie derivative
$$\pounds = \bar\Delta\colon \Gamma(U) \times \Gamma(U^*) \to \Gamma(U^*),$$
which flatness is equivalent to the restriction of $[\![\cdot\,,\cdot]\!]_\Delta$ to $\Gamma(U)$ satisfying the Jacobi-identity. The Dorfman connection $\bar\Delta$ depends only on the class $[\Delta]$ of the connection Δ. Conversely, a Dorfman connection $\bar\Delta\colon \Gamma(U) \times \Gamma((E \oplus T^*M)/K) \to \Gamma((E \oplus T^*M)/K)$, can be extended to a Dorfman connection $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ (by extending in a dull manner the corresponding Lie algebroid bracket on U). Two such extensions of $\bar\Delta$ are automatically (U,K)-equivalent.

Proof of Corollary 4.16. The proof is immediate. For (2), note only that $K = U^\circ$ and $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$ imply together that the dull bracket restricts to a bracket on $\Gamma(U)$, and vice-versa. □

Remark 4.17.
(1) Using the following Proposition 4.18, one can see that if the conditions in (2) of Corollary 4.16 are satisfied for Δ, then they are also satisfied for any Δ' that is (U,K)-equivalent to Δ.
(2) We say that $(U, K, [\Delta])$ is a Dirac triple if the corresponding linear subbundle $D_{(U,K,[\Delta])}$ is a Dirac structure on E. By the considerations above, we find that linear Dirac structures in $TE \oplus T^*E \to E$ are in one-one correspondence with Dirac triples.

Proposition 4.18. *Let $E \to M$ be a vector bundle and choose a triple $(U, K, [\Delta]_{U,K})$ such that $U = K^\circ$. Then for any two representatives $\Delta, \Delta' \in [\Delta]_{U,K}$, we have*
$$[\![u_1, u_2]\!]_\Delta = [\![u_1, u_2]\!]_{\Delta'}$$
for all $u_1, u_2 \in \Gamma(U)$.

Proof. Since $\mathrm{pr}_{TM}[\![u_1, u_2]\!]_\Delta = [\mathrm{pr}_{TM} u_1, \mathrm{pr}_{TM} u_2] = \mathrm{pr}_{TM}[\![u_1, u_2]\!]_{\Delta'}$, we need only to check that
$$\langle [\![u_1, u_2]\!]_\Delta, (e, 0)\rangle = \langle [\![u_1, u_2]\!]_{\Delta'}, (e, 0)\rangle$$
for all $e \in \Gamma(E)$. But this is immediate by the hypothesis, the duality of Δ and $[\![\cdot\,,\cdot]\!]_\Delta$ and the definition of (U,K)-equivalence. □

Since a linear Dirac structure D in $TE \oplus T^*E$ over the base $U \subseteq TM \oplus E^*$ is a VB-algebroid $(D \to E, U \to M)$, we get the following corollary from Theorem 4.15, Corollary 4.16 and Proposition 4.18.

Corollary 4.19. *Let $(D; E, U; M)$ be a linear Dirac structure in $(TE \oplus T^*E; E, TM \oplus E^*; M)$. A linear splitting Σ^Δ of $TE \oplus T^*E$ that is adapted to D defines a linear splitting Σ of D. Then $(U, \mathrm{pr}_{TM}|_U, [\![\cdot\,,\cdot]\!]_\Delta|_{\Gamma(U)\times\Gamma(U)})$ is a Lie algebroid (that does not depend on the splitting), the restriction $\tilde\Delta$ of Δ to $\Gamma(U) \times \Gamma(U^\circ) \to \Gamma(U^\circ)$ is a linear connection, the linear connection ∇ restricts to $\tilde\nabla\colon \Gamma(U) \times \Gamma(E) \to \Gamma(E)$ and the vector-valued 2-form R_Δ restricts to $\tilde R_\Delta \in \Omega^2(U, \mathrm{Hom}(E, U^\circ))$.*

The triple $(\tilde\Delta, \tilde\nabla, \tilde R_\nabla)$ is a 2-term representation up to homotopy of U on $\mathrm{pr}_E|_{U^\circ}\colon U^\circ \to E$, that describes the VB-algebroid structure on D in the linear splitting Σ.

We conclude with a series of examples.

Example 4.20. In the situation of Example 4.2, choose two subbundles $F_M \subseteq TM$ and $C \subseteq E$. Set $U := F_M \oplus C^\circ$ and $K := C \oplus F_M^\circ = U^\circ$. The linear subbundle $D_{U,K,\Delta}$ corresponding to U, K and the standard Dorfman connection associated to ∇ is then the direct sum of a subbundle $F_E \subseteq TE$, with $C_E \subseteq T^*E$. Since $U = K^\circ$, we get immediately that $C_E = F_E^\circ$ and $D_{U,K,[\Delta]}$ is the trivial almost Dirac structure $F_E \oplus F_E^\circ$. An application of Corollary 4.16 to this situation yields that $F_E \oplus F_E^\circ$ is a Dirac structure if and only if
(1) F_M is involutive,
(2) $\nabla_X c \in \Gamma(C)$ for all $X \in \Gamma(F_M)$ and $c \in \Gamma(C)$ and
(3) the induced connection $\tilde\nabla\colon \Gamma(F_M) \times \Gamma(E/C) \to \Gamma(E/C)$ is flat.

Since $F_E \oplus F_E^\circ$ is Dirac if and only if $F_E \subseteq TE$ is involutive, we have recovered a result in [17], see also [9].

Example 4.21. In the situation of Example 4.3, consider $U = \mathrm{graph}(\rho\colon A \to TM) \subseteq TM \oplus A^*$ and $K = \mathrm{graph}(-\rho^t\colon T^*M \to A^*) = U^\circ$. A straightforward computation shows that
$$\Delta_{(\rho(a),a)}(-\rho^t(\theta), \theta) = \left(-\rho^t\left(\nabla_a^{\mathrm{bas}^*}\theta\right), \nabla_a^{\mathrm{bas}^*}\theta\right) \in \Gamma(K)$$

for all $a \in \Gamma(A)$ and $\theta \in \Omega^1(M)$. Furthermore, we have

$$[\![(\rho(a_1), a_1), (\rho(a_2), a_2)]\!]_\Delta = (\rho([a_1, a_2]), [a_1, a_2])$$

for all $a_1, a_2 \in \Gamma(A)$, which shows that $(U, \mathrm{pr}_{TM}, [\![\cdot\,,\cdot]\!]_\Delta)$ is a Lie algebroid if and only if A is a Lie algebroid. We have:

$$\begin{aligned}\bar{\Delta}_{(\rho(a),a)}\overline{(\alpha, 0)} &= \overline{\left(\langle \alpha, \nabla^{\mathrm{bas}}_{\cdot} a\rangle + \nabla^*_{\rho(a)}\alpha - \rho^t\langle \nabla_{\cdot} a, \alpha\rangle, \langle \nabla_{\cdot} a, \alpha\rangle\right)} \\ &= \overline{\left(\langle \alpha, \nabla^{\mathrm{bas}}_{\cdot} a\rangle + \nabla^*_{\rho(a)}\alpha, 0\right)} = \overline{(\pounds_a\alpha, 0)}.\end{aligned}$$

Finally, the right-hand side of (4.16) vanishes for $(\rho(a), a), (\rho(b), b), (\rho(c), c) \in \Gamma(U)$ and arbitrary $(\alpha, \theta) \in \Gamma(A^* \oplus T^*M)$ if and only if A is a Lie algebroid.

Hence, we find that the linear subbundle D of $TA^* \oplus T^*A^* \to A^*$ associated to U, K and Δ is an almost Dirac structure on A^*, and that is is a Dirac structure if and only if A is a Lie algebroid. The vector bundle $D \to A^*$ is the graph of the vector bundle morphism

$$\pi_A^\sharp \colon T^*A^* \to TA^*$$

associated to the linear almost Poisson structure defined on A^* by the skew-symmetric dull algebroid structure on A. More precisely, D is spanned by the sections $k^\uparrow$ for $k \in \Gamma(K)$ and $\sigma_\Delta(u)$ for $u \in \Gamma(U)$, or, equivalently, by the sections

$$(-\rho^t\theta^\uparrow, q_{A^*}^*\theta) \qquad \text{and} \qquad \left(\widehat{\pounds_a}, \mathbf{d}\ell_a\right)$$

for $\theta \in \Omega^1(M)$ and for $a \in \Gamma(A)$. By Appendix A.1, these are exactly the sections $(\pi_A^\sharp(q_{A^*}^*\theta), q_{A^*}^*\theta)$ and $(\pi_A^\sharp(\mathbf{d}\ell_a), \mathbf{d}\ell_a)$.

Example 4.22. Consider, in the situation of Example 4.4, $U := \mathrm{graph}(-\sigma^t \colon TM \to E^*)$ and $K := \mathrm{graph}(\sigma \colon E \to T^*M)$. Then $U = K^\circ$ by definition and since

$$\Delta_{(X, -\sigma^t X)}(e, \sigma(e)) = (\nabla_X e, \sigma(\nabla_X e)),$$

we find that $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$ and $k \in \Gamma(K)$. Furthermore, we have

$$[\![(X, -\sigma^t X), (Y, -\sigma^t Y)]\!]_\Delta = ([X, Y], -\sigma^t[X, Y])$$

for all $X, Y \in \mathfrak{X}(M)$ and U is a Lie algebroid (isomorphic to TM with the Lie bracket of vector fields). Alternatively, the Jacobiator in (4.17) is easily seen to vanish on sections of U. This shows that the double vector subbundle $D \subseteq TE \oplus T^*E$ defined by U, K and Δ is a Dirac structure.

By the considerations in Appendix A.2, D is the graph of the vector bundle morphism $TE \to T^*E$ defined by the closed 2-form $\sigma^*\omega_{\mathrm{can}}$.

Example 4.23. In this example, we consider the vector bundle $E = TM$, for a smooth manifold M. Consider a Dirac structure D on M and the Bott-Dorfman connection

$$\Delta^D \colon \Gamma(D) \times \Gamma(TM \oplus T^*M/D) \to \Gamma(TM \oplus T^*M/D)$$

defined by D (see Proposition 3.7). Choose an extension $\Delta \colon \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ of Δ^D, i.e. a dull extension of the restriction to $\Gamma(D)$ of the Courant-Dorfman bracket.

It is easy to check that the triple $(D, D, [\Delta]) = (D, D, \Delta^D)$ is a Dirac triple. Later we will see the meaning of the Dirac structure on TM associated to it.

Example 4.24. We now combine Examples 4.20 and 4.22 to recover an example in [14].

We consider the vector bundle $T^*M \to M$ endowed with a TM-connection ∇ and the Dorfman connection

$$\begin{aligned}&\Delta \colon \Gamma(TM \oplus TM) \times \Gamma(T^*M \oplus T^*M) \to \Gamma(T^*M \oplus T^*M), \\ &\Delta_{(X,Y)}(\theta, \omega) = (\nabla_X\theta, \pounds_X(\omega - \theta) + \langle \nabla^*_{\cdot}(X + Y), \omega\rangle + \nabla_X\theta).\end{aligned}$$

Consider a subbundle $F \subseteq TM$ and $U := \{(v, -v) \mid x \in F\} \subseteq TM \oplus TM$. The annihilator $K = U^\circ$ is then given by $K = \{(\theta, \omega) \in T^*M \oplus T^*M \mid \theta - \omega \in F^\circ\}$.

Note that by Example 4.4, the dull bracket on $TM \oplus TM$ is skew-symmetric. It is easy to see that its restriction to U is just the Lie bracket of vector fields

$$[\![(X, -X), (Y, -Y)]\!]_\Delta = ([X, Y], -[X, Y])$$

for all $X, Y \in \Gamma(F)$. Hence, we know already that the linear subbundle $D_{(U,K,[\Delta])}$ is an almost Dirac structure on T^*M. An easy computation using Appendix A.2 yields that

$$D_{(U,K,[\Delta])}(\theta) = \{(v_\theta, \omega_{\text{can}}^\flat(v_\theta) + \eta_\theta) \mid v_\theta \in \mathcal{F}(\theta), \eta_\theta \in \mathcal{F}^\circ(\theta)\}$$

for all $\theta \in T^*M$, where $\mathcal{F} = (Tc_M)^{-1}(F)$. Assume that M is the configuration space of a nonholonomic mechanical system and F the constraints distribution. If L is the Lagrangian of the system, then the pullback to the contraints submanifold $\mathbb{F}L(F) \subseteq T^*M$ of the Dirac structure $D_{(U,K,[\Delta])}$ is one of the frameworks proposed in [14] for the study of the nonholonomic system.

5. Application: the prolongation $TA \oplus T^*A \to TM \oplus A^*$ of a Lie algebroid A

We consider a Lie algebroid $(A \to M, \rho, [\cdot\,,\cdot])$ and a Dorfman connection

$$\Delta\colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$$

with corresponding dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$ and anchor pr_{TM} on $TM \oplus A^*$. By Theorem 4.1, this Dorfman connection corresponds to a linear splitting of $TA \oplus T^*A$. Our goal is to compute the representation up to homotopy defined by this linear splitting of the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$. Note that until now, only the representations up to homotopy defined by standard Dorfman connections were known (Example 2.11). The results in this section are used in [15] to describe infinitesimally Dirac groupoids.

We define a map $\Omega\colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M)$ by

$$\Omega_{(X,\alpha)}a = \Delta_{(X,\alpha)}(a, 0) - (0, \mathbf{d}\langle\alpha, a\rangle).$$

Ω satisfies $\Omega_{f(X,\alpha)}a = f\Omega_{(X,\alpha)}a$ and $\Omega_{(X,\alpha)}(fa) = f\Omega_{(X,\alpha)}a + X(f)(a, 0) - \langle\alpha, a\rangle(0, \mathbf{d}f)$ for all $f \in C^\infty(M)$, $a \in \Gamma(A)$ and $(X, \alpha) \in \Gamma(TM \oplus A^*)$. For each $a \in \Gamma(A)$, we have two derivations over $\rho(a) \in \mathfrak{X}(M)$:

$$\pounds_a\colon \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M), \qquad \pounds_a(a', \theta) = ([a, a'], \pounds_{\rho(a)}\theta)$$

and

$$\pounds_a\colon \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*), \qquad \pounds_a(X, \alpha) = ([\rho(a), X], \pounds_a\alpha).$$

Note that $\pounds_{fa}(a', \theta) = f\pounds_a(a', \theta) + (-\rho(a')(f)a, \langle\theta, \rho(a)\rangle\mathbf{d}f)$.

5.1. The basic connections associated to Δ.

Proposition 5.1. *The two maps*

$$\nabla^{\text{bas}}\colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*), \quad \nabla^{\text{bas}}_a(X, \alpha) = (\rho, \rho^t)(\Omega_{(X,\alpha)}a) + \pounds_a(X, \alpha)$$
$$\nabla^{\text{bas}}\colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M), \quad \nabla^{\text{bas}}_a(a', \theta) = \Omega_{(\rho,\rho^t)(a',\theta)}a + \pounds_a(a', \theta)$$

are ordinary linear connections.

Proof. The proof is straightforward and left to the reader. □

The following proposition is easily checked, and shows that the connections are dual to each other if and only if the dull bracket on $\Gamma(TM \oplus A^*)$ is skew-symmetric.

Proposition 5.2. *We have*

$$\langle \nabla_a^{\mathrm{bas}}\nu, \tau\rangle + \langle \nu, \nabla_a^{\mathrm{bas}}\tau\rangle = \rho(a)\langle \nu, \tau\rangle - \langle \mathrm{Skew}_\Delta(\nu, (\rho, \rho^t)\tau), a\rangle \tag{5.18}$$

$$\nabla_a^{\mathrm{bas}}(\rho, \rho^t)\tau = (\rho, \rho^t)\nabla_a^{\mathrm{bas}}\tau \tag{5.19}$$

for all $a \in \Gamma(A)$, $\nu \in \Gamma(TM \oplus A^*)$ and $\tau \in \Gamma(A \oplus T^*M)$.

Definition 5.3. *The connections in Proposition 5.1 is called the **basic connections** associated to Δ.*

Proposition 5.4. *The map*

$$R_\Delta^{\mathrm{bas}}\colon \Gamma(A) \times \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(A \oplus T^*M)$$

given by

$$R_\Delta^{\mathrm{bas}}(a, b)(X, \alpha) = -\Omega_{(X,\alpha)}[a, b] + \pounds_a\left(\Omega_{(X,\alpha)}b\right) - \pounds_b\left(\Omega_{(X,\alpha)}a\right) + \Omega_{\nabla_b^{\mathrm{bas}}(X,\alpha)}a - \Omega_{\nabla_a^{\mathrm{bas}}(X,\alpha)}b.$$

is tensorial, i.e. it defines $R_\Delta^{\mathrm{bas}} \in \Omega^2(A, \mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$.

Proof. This proof also is a straightforward computation. □

Definition 5.5. *We call the tensor R_Δ^{bas} the **basic curvature** associated to Δ.*

Proposition 5.6. *The basic curvature satisfies $R_{\nabla^{\mathrm{bas}}} = R_\Delta^{\mathrm{bas}} \circ (\rho, \rho^t)$ and $R_{\nabla^{\mathrm{bas}}} = (\rho, \rho^t) \circ R_\Delta^{\mathrm{bas}}$.*

Proof. For $\tau \in \Gamma(A \oplus T^*M)$ and $a, b \in \Gamma(A)$, we have

$$\begin{aligned} R_\Delta(a,b)((\rho,\rho^t)\tau) &= -\Omega_{(\rho,\rho^t)\tau}[a,b] + \pounds_a\left(\Omega_{(\rho,\rho^t)\tau}b\right) - \pounds_b\left(\Omega_{(\rho,\rho^t)\tau}a\right) \\ &\quad + \Omega_{\nabla_b^{\mathrm{bas}}(\rho,\rho^t)\tau}a - \Omega_{\nabla_a^{\mathrm{bas}}(\rho,\rho^t)\tau}b \\ &= -\Omega_{(\rho,\rho^t)\tau}[a,b] - \pounds_{[a,b]}\tau + \pounds_a\left(\Omega_{(\rho,\rho^t)\tau}b + \pounds_b\tau\right) - \pounds_b\left(\Omega_{(\rho,\rho^t)\tau}a + \pounds_a\tau\right) \\ &\quad + \Omega_{\nabla_b^{\mathrm{bas}}(\rho,\rho^t)\tau}a - \Omega_{\nabla_a^{\mathrm{bas}}(\rho,\rho^t)\tau}b \\ &= -\nabla_{[a,b]}^{\mathrm{bas}}\tau + \nabla_a^{\mathrm{bas}}\nabla_b^{\mathrm{bas}}\tau - \nabla_b^{\mathrm{bas}}\nabla_a^{\mathrm{bas}}\tau = R_{\nabla^{\mathrm{bas}}}(a,b)\tau. \end{aligned}$$

Note that in the second equality, we use $\pounds_a\pounds_b - \pounds_b\pounds_a - \pounds_{[a,b]} = 0$. The second equality is shown in a similar manner. □

5.2. **The Lie algebroid structure on $TA \oplus T^*A \to TM \oplus A^*$.** Consider a Lie algebroid A and a Dorfman connection $\Delta\colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$. Then, for any section $a \in \Gamma(A)$, the horizontal lift $\sigma_A(a) \in \Gamma_{TM\oplus A^*}(TA \oplus T^*A)$ is given by

$$\sigma_A^\Delta(a)(v_m, \alpha_m) = (T_m a v_m, \mathbf{d}_{a_m}\ell_\alpha) - \Delta_{(X,\alpha)}(a, 0)^\uparrow(a_m)$$

for any choice of section $(X, \alpha) \in \Gamma(TM \oplus A^*)$ such that $(X, \alpha)(m) = (v_m, \alpha_m)$. That is, we have

$$\sigma_A^\Delta(a) = (Ta, R(\mathbf{d}\ell_a)) - \widetilde{\Omega.a} = a^l - \widetilde{\Omega.a}$$

for all $a \in \Gamma(A)$ (using the notation of Appendix C). For simplicity, we write σ_A for σ_A^Δ.

Theorem 5.7. *The Lie algebroid structure on $TA \oplus T^*A \to TM \oplus A^*$ with anchor $\Theta\colon TA \oplus T^*A \to T(TM \oplus A^*)$ is described as follows:*

(1) $[\sigma_A(a_1)\sigma_A(a_2)] = \sigma_A([a_1, a_2]) - \widetilde{R_\Delta^{\mathrm{bas}}(a_1, a_2)}$,
(2) $[\sigma_A(a), \tau^\dagger] = (\nabla_a^{\mathrm{bas}}\tau)^\dagger$,
(3) $[\tau_1^\dagger, \tau_2^\dagger] = 0$,
(4) $\Theta(\sigma_A(a)) = \widehat{\nabla_a^{\mathrm{bas}}} \in \mathfrak{X}(TM \oplus A^*)$,
(5) $\Theta(\tau^\dagger) = ((\rho, \rho^t)\tau)^\uparrow \in \mathfrak{X}(TM \oplus A^*)$.

Corollary 5.8. *In other words,* $(\rho,\rho^t)\colon A\oplus T^*M\to TM\oplus A^*$*, the basic connections* $\nabla^{\rm bas}$ *and the basic curvature* $R_\Delta^{\rm bas}$ *define the representation up to homotopy describing the VB-Lie algebroid structure on* $TA\oplus T^*A\to TM\oplus A^*$ *in the linear splitting given by* Δ*.*

Proof of Theorem 5.7. The proof of this theorem consists in checking the formulas, using the description of the Lie algebroid structure on $TA\oplus T^*A\to TM\oplus A^*$ in Appendix C. We begin with the Lie algebroid brackets. Choose $a_1,a_2\in\Gamma(A)$ and $\tau\in\Gamma(A\oplus T^*M)$. Using Proposition C.1, we find

$$\begin{aligned}[]
[\sigma_A(a_1),\sigma_A(a_2)] &= \left[a_1^l-\widetilde{\Omega.a_1},\, a_2^l-\widetilde{\Omega.a_2}\right]\\
&= [a_1,a_2]^l-\widetilde{\pounds_{a_1}\Omega.a_2}+\widetilde{\pounds_{a_2}\Omega.a_1}+\Omega.a_2\circ\widetilde{(\rho,\rho^t)}\circ\Omega.a_1-\Omega.a_1\circ\widetilde{(\rho,\rho^t)}\circ\Omega.a_2\\
&= \sigma_A[a_1,a_2]-\widetilde{R_\Delta^{\rm bas}(a_1,a_2)}.
\end{aligned}$$

We have used

$$\begin{aligned}
&-(\pounds_{a_1}\Omega.a_2)(v)+(\pounds_{a_2}\Omega.a_1)(v)+(\Omega.a_2\circ(\rho,\rho^t)\circ\Omega.a_1)(v)-(\Omega.a_1\circ(\rho,\rho^t)\circ\Omega.a_2)(v)\\
&=-\pounds_{a_1}\Omega_v a_2+\Omega_{\pounds_{a_1}v}a_2+\pounds_{a_2}\Omega_v a_1-\Omega_{\pounds_{a_2}v}a_1+\Omega_{(\rho,\rho^t)\Omega_v a_1}a_2-\Omega_{(\rho,\rho^t)\Omega_v a_2}a_1\\
&=-\pounds_{a_1}\Omega_v a_2+\pounds_{a_2}\Omega_v a_1+\Omega_{\nabla^{\rm bas}_{a_1}v}a_2-\Omega_{\nabla^{\rm bas}_{a_2}v}a_1=-R_\Delta^{\rm bas}(a_1,a_2)v-\Omega_v[a_1,a_2]
\end{aligned}$$

for all $v\in\Gamma(TM\oplus A^*)$. Next, we find $[\sigma_A(a),\tau^\dagger]=(\pounds_a\tau)^\dagger+\Omega_{(\rho,\rho^t)\tau}a^\dagger=(\nabla_a^{\rm bas}\tau)^\dagger$. For the anchor map, we compute $\Theta(\sigma_A(a))(\ell_\tau)=\ell_{\pounds_a\tau-(\Omega.a)^t((\rho,\rho^t)\tau)}$, which yields the desired equality since

$$\begin{aligned}
\langle(\Omega.a)^t((\rho,\rho^t)\tau),v\rangle &= \langle(\rho,\rho^t)\Omega_v a,\tau\rangle=\langle\nabla_a^{\rm bas}v-\pounds_a v,\tau\rangle\\
&=\rho(a)\langle v,\tau\rangle-\langle v,\nabla_a^{{\rm bas}^*}\tau\rangle-\langle\pounds_a v,\tau\rangle
\end{aligned}$$

and consequently $\langle v,\pounds_a\tau-(\Omega.a)^t((\rho,\rho^t)\tau)\rangle=\langle v,\nabla_a^{{\rm bas}^*}\tau\rangle$. The remaining equalities follow from Proposition C.1. □

Theorem 5.9. *Consider a Lie algebroid* A *and a Dorfman connection* $\Delta\colon\Gamma(TM\oplus A^*)\times\Gamma(A\oplus T^*M)\to\Gamma(A\oplus T^*M)$*. Let* $U\subseteq TM\oplus A^*$ *and* $K\subseteq A\oplus T^*M$ *be subbundles. Then the linear subbundle* $D_{(U,K,[\Delta])}$ *is a subalgebroid of* $TA\oplus T^*A\to TM\oplus A^*$ *over* U *if and only if:*

1. $(\rho,\rho^t)(K)\subseteq U$*,*
2. $\nabla_a^{\rm bas}k\in\Gamma(K)$ *for all* $a\in\Gamma(A)$ *and* $k\in\Gamma(K)$*,*
3. $\nabla_a^{\rm bas}u\in\Gamma(U)$ *for all* $a\in\Gamma(A)$ *and* $u\in\Gamma(U)$*,*
4. $R_\Delta^{\rm bas}(a_1,a_2)u\in\Gamma(K)$ *for all* $u\in\Gamma(U)$*,* $a_1,a_2\in\Gamma(A)$*.*

Proof. Assume that $D_{(U,K,[\Delta])}\to U$ is a subalgebroid of $TA\oplus T^*A\to TM\oplus A^*$. Then we have $((\rho,\rho^t)k)^\uparrow|_U=\Theta(k^\dagger|_U)\in\mathfrak{X}(U)$ and $\widehat{\nabla_a^{\rm bas}}|_U=\Theta(\sigma_A(a)|_U)\in\mathfrak{X}(U)$ for all $a\in\Gamma(A)$ and $k\in\Gamma(K)$. This is the case if and only if $((\rho,\rho^t)k)^\uparrow(\ell_\tau)|_U=0$ and $\widehat{\nabla_a^{\rm bas}}(\ell_\tau)|_U=0$ for all $\tau\in\Gamma(U^\circ)$. Since $((\rho,\rho^t)k)^\uparrow(\ell_\tau)=\pi^*\langle(\rho,\rho^t)k,\tau\rangle$ and $\widehat{\nabla_a^{\rm bas}}(\ell_\tau)=\ell_{\nabla_a^{{\rm bas}^*}\tau}$, we find that $(\rho,\rho^t)k$ must be a section of U and $\nabla_a^{{\rm bas}^*}\tau\in\Gamma(U^\circ)$ for all $\tau\in\Gamma(U^\circ)$. But the latter is equivalent to $\nabla_a^{\rm bas}u\in\Gamma(U)$ for all $u\in\Gamma(U)$. We have in the same manner $(\nabla_a^{\rm bas}k)^\dagger|_U=[\sigma_A(a),k^\dagger]|_U\in\Gamma(D_{(U,K,[\Delta])})$ and $\big(\sigma_A[a_1,a_2]-\widetilde{R_\Delta^{\rm bas}(a_1,a_2)}\big)|_U=[\sigma_A(a_1),\sigma_A(a_2)]|_U\in\Gamma(D_{(U,K,[\Delta])})$ for all $a_1,a_2\in\Gamma(A)$ and $k\in\Gamma(K)$. But this is only the case if $\nabla_a^{\rm bas}k\in\Gamma(K)$ and, since $\sigma_A[a_1,a_2]|_U\in\Gamma(D_{(U,K,[\Delta])})$, if $R_\Delta^{\rm bas}(a_1,a_2)^\dagger|_U\in\Gamma(D_{(U,K,[\Delta])})$. This holds only if $R_\Delta^{\rm bas}(a_1,a_2)u\in\Gamma(K)$ for all $u\in\Gamma(U)$.

The converse implication is shown in a similar manner. □

5.3. **LA-Dirac structures in** $TA\oplus T^*A$**.** Our last result is a description of the triples $(U,K,[\Delta]_{U,K})$ associated to Dirac structures on A that are at the same time Lie subalgebroids of $TA\oplus T^*A\to TM\oplus A^*$. We call such a Dirac structure D_A an **LA-Dirac structure** on A, and we call the pair (A,D_A) a **Dirac algebroid**.

Theorem 5.10. *Consider a Lie algebroid A and a Dorfman connection $\Delta\colon \Gamma(TM\oplus A^*)\times \Gamma(A\oplus T^*M)\to\Gamma(A\oplus T^*M)$. Let $U\subseteq TM\oplus A^*$ and $K\subseteq A\oplus T^*M$ be subbundles. Then $D_{(U,K,[\Delta])}$ is a Dirac structure in $TA\oplus T^*A\to A$ and a subalgebroid of $TA\oplus T^*A\to TM\oplus A^*$ over U if and only if:*

1. $K=U^\circ$
2. $(\rho,\rho^t)(K)\subseteq U$,
3. $(U,\mathrm{pr}_{TM},[\![\cdot\,,\cdot]\!]_\Delta)$ *is a Lie algebroid,*
4. $\nabla^{\rm bas}_a k\in\Gamma(K)$ *for all* $a\in\Gamma(A)$ *and* $k\in\Gamma(K)$,
5. $R^{\rm bas}_\Delta(a_1,a_2)u\in\Gamma(K)$ *for all* $u\in\Gamma(U)$, $a_1,a_2\in\Gamma(A)$.

Proof. This theorem follows from (2) in Corollary 4.16, together with Theorem 5.9. Note that if $U=K^\circ$, $(\rho,\rho^t)K\subseteq U$ and $(U,\mathrm{pr}_{TM},[\![\cdot\,,\cdot]\!]_\Delta)$ is a Lie algebroid, then $\nabla^{\rm bas}_a$ preserves $\Gamma(U)$ if and only if $\nabla^{\rm bas}_a$ preserves $\Gamma(K)$. So (2) and (3) in Theorem 5.9 become one single condition. □

Corollary 5.11. *We have found a one-one correspondence between triples $(U,K,[\Delta]_{U,K})$ satisfying (1)–(5) in Theorem 5.9 and LA-Dirac structures on the Lie algebroid A.*

Remark 5.12. If D is an LA-Dirac structure over $U\subseteq TM\oplus A^*$ in $TA\oplus T^*A$, then $(D\to U, A\to M)$ and $(D\to A, U\to M)$ are VB-algebroids. A linear splitting Σ^Δ that is adapted to D defines a linear splitting Σ of D. The 2-term representation up to homotopy $(\tilde\Delta,\tilde\nabla,\tilde R_\Delta)$ of U on $pr_E\colon U^\circ\to A$ describes $(D\to A, U\to M)$ in this splitting (see Corollary 4.19). Theorem 5.9 shows that the representation up to homotopy $(\nabla^{\rm bas},\nabla^{\rm bas},R^{\rm bas}_\Delta)$ of A on $(\rho,\rho^t)\colon A\oplus T^*M\to TM\oplus A^*$ restricts to a representation up to homotopy $(\widetilde{\nabla^{\rm bas}},\widetilde{\nabla^{\rm bas}},\widetilde{R^{\rm bas}_\Delta})$ of A on $(\rho,\rho^t)|_{U^\circ}\colon U^\circ\to U$. One can check that these two 2-term representations up to homotopy form a matched pair, which implies that D is a double Lie algebroid [11]. The computation is very similar to the one for the double Lie algebroid TA in [11, Section 3.3].

Finally we discuss our previous examples. To avoid confusions, we write ∇^A for the A-basic connections induced on A and TM by the Lie algebroid structure on A and the connection ∇, and R^A_∇ for the basic curvature associated to it (§2.2.2).

Example 5.13. In the situation of Examples 4.2 and 4.20, assume that the vector bundle E is a Lie algebroid A. We show that the conditions in Theorem 5.10 define in this case an *infinitesimal ideal system* [17], see also [13]. Condition (1) is trivially satisfied by construction and Condition (3) is the involutivity of F_M and the quotient of ∇ to a flat connection $\tilde\nabla\colon\Gamma(F_M)\times\Gamma(A/C)\to\Gamma(A/C)$ (Example 4.20). Condition (2) is $\rho(C)\subseteq F_M$, Condition (4) is $\nabla^A_a c\in\Gamma(C)$ for all $c\in\Gamma(C)$ and $\nabla^A_a X\in\Gamma(F_M)$ for all $X\in\Gamma(F_M)$. To see this, note that $\nabla^{\rm bas}\colon\Gamma(A)\times\Gamma(A\oplus T^*M)\to\Gamma(A\oplus T^*M)$ is here just $\nabla^{\rm bas}_a(a',\theta)=(\nabla^A_a a',\nabla^{A^*}_a\theta)$. Finally, an easy computation shows $R^{\rm bas}_\Delta(a_1,a_2)(X,\alpha)=(R^A_\nabla(a_1,a_2)X,-R^A_\nabla(a_1,a_2)^*\alpha)$, which implies the equivalence of Condition (5) with $R^A_\nabla(a_1,a_2)X\in\Gamma(C^\circ)$ for all $X\in\Gamma(F_M)$ and all $a_1,a_2\in\Gamma(A)$. Therefore, following [9, §5.2] we find that the conditions of Theorem 5.10 are satisfied if and only if $(F_M,C,\tilde\nabla)$ is an infinitesimal ideal system in A.

Example 5.14. Consider again Examples 4.3 and 4.21. Assume that A^* has itself also a Lie algebroid structure with anchor ρ_* and bracket $[\cdot\,,\cdot]_*$. For simplicity, we switch the roles of A and A^* in Examples 4.3 and 4.21. We show that U, K, Δ satisfy the conditions of Theorem 5.10 if and only if (A,A^*) is a Lie bialgebroid. Recall that we have already found that (1) and (3) are equivalent to A^* being a Lie algebroid. Then, (2) in Theorem 5.10 is equivalent to

$$\rho_*\circ\rho^t=-\rho\circ\rho^t_*. \tag{5.20}$$

We assume in the following that this condition is satisfied. We also have:

$$\Omega_{(\rho_*(\alpha),\alpha)}a=\left(\pounds_\alpha a-\rho_*^t\langle\nabla^*_\cdot\alpha,a\rangle,\langle\nabla^*_\cdot\alpha,a\rangle\right)-(0,\mathbf{d}\langle\alpha,a\rangle)=\left(\mathbf{i}_\alpha\mathbf{d}_A a+\rho^t_*\langle\alpha,\nabla_\cdot a\rangle,-\langle\alpha,\nabla_\cdot a\rangle\right),$$

for all $\alpha \in \Gamma(A^*)$ and $a \in \Gamma(A)$ and so

$$\Omega_{(\rho,\rho^t)(-\rho_*^t\theta,\theta)}a = \Omega_{(\rho_*(\rho^t\theta),\rho^t\theta)}a = \left(\mathbf{i}_{\rho^t\theta}\mathbf{d}_A a + \rho_*^t\langle\rho^t\theta, \nabla_\cdot a\rangle, -\langle\rho^t\theta, \nabla_\cdot a\rangle\right).$$

for all $\theta \in \Omega^1(M)$. In particular, if $\theta = \mathbf{d}f$ for some $f \in C^\infty(M)$, we get:

$$\begin{aligned}\nabla_a^{\rm bas}(-\rho_*^t\mathbf{d}f, \mathbf{d}f) &= \Omega_{(\rho,\rho^t)(-\rho_*^t\mathbf{d}f,\mathbf{d}f)}a + \pounds_a(-\rho_*^t\mathbf{d}f, \mathbf{d}f)\\ &= \left(\mathbf{i}_{\mathbf{d}_{A^*}f}\mathbf{d}_A a + \rho_*^t\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle - [a, \mathbf{d}_A f], -\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle + \mathbf{d}(\rho(a)(f))\right).\end{aligned}$$

Thus, using (1) in Theorem 5.10, $\nabla_a^{\rm bas}(-\rho_*^t\mathbf{d}f, \mathbf{d}f) \in \Gamma(K)$ if and only if

$$\langle\left(\mathbf{i}_{\mathbf{d}_{A^*}f}\mathbf{d}_A a + \rho_*^t\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle - [a, \mathbf{d}_A f], -\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle + \mathbf{d}(\rho(a)(f))\right), (\rho_*\alpha, \alpha)\rangle = 0$$

for all $\alpha \in \Gamma(A^*)$. But this pairing equals

$$\langle\left(\mathbf{i}_{\mathbf{d}_{A^*}f}\mathbf{d}_A a + \rho_*^t\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle - [a, \mathbf{d}_A f], -\langle\mathbf{d}_{A^*}f, \nabla_\cdot a\rangle + \mathbf{d}(\rho(a)(f))\right), (\rho_*\alpha, \alpha)\rangle,$$

which is easily shown to be the same as $\left([\rho_*(\alpha), \rho(a)] + \rho_*(\pounds_a\alpha) - \rho(\pounds_\alpha a) + \rho(\mathbf{d}_A\langle\alpha, a\rangle)\right)(f)$. Since f was arbitrary, we have shown that the fourth condition is satisfied if and only if

$$[\rho(a), \rho_*(\alpha)] - \rho_*(\pounds_a\alpha) + \rho(\pounds_\alpha a) = \rho(\mathbf{d}_A\langle\alpha, a\rangle) \tag{5.21}$$

for all $a \in \Gamma(A)$ and $\alpha \in \Gamma(A^*)$. Thus, we have found until here (5.20) and (5.21), which are properties of Lie bialgebroids (see [23]).

Using these equations, we study Condition (5), on the basic curvature. Since $\Omega_{(\rho_*(\alpha),\alpha)}a = (\mathbf{i}_\alpha\mathbf{d}_A a, 0) - (-\rho_*^t\langle\alpha, \nabla_\cdot a\rangle, \langle\alpha, \nabla_\cdot a\rangle)$, we find $\langle\Omega_{(\rho_*(\alpha),\alpha)}a, (\rho_*\alpha', \alpha')\rangle = (\mathbf{d}_A a)(\alpha, \alpha')$ for all $a \in \Gamma(A)$, $\alpha, \alpha' \in \Gamma(A^*)$. The fourth condition together with (5.18) and the first and third conditions imply that $\nabla_a^{\rm bas}u \in \Gamma(A)$ for all $u \in \Gamma(U)$. Hence,

$$\begin{aligned}\nabla_a^{\rm bas}(\rho_*(\alpha), \alpha) &= (\rho, \rho^t)\left(\mathbf{i}_\alpha\mathbf{d}_A a + \rho_*^t\langle\alpha, \nabla_\cdot a\rangle, -\langle\alpha, \nabla_\cdot a\rangle\right) + \pounds_a(\rho_*(\alpha), \alpha)\\ &= \left(\rho_*(-\rho^t\langle\alpha, \nabla_\cdot a\rangle + \pounds_a\alpha), -\rho^t\langle\alpha, \nabla_\cdot a\rangle + \pounds_a\alpha\right)\end{aligned}$$

and

$$\langle\Omega_{\nabla_a^{\rm bas}(\rho_*\alpha,\alpha)}a', (\rho_*\alpha', \alpha')\rangle = (\mathbf{d}_A a')(-\rho^t\langle\alpha, \nabla_\cdot a\rangle + \pounds_a\alpha, \alpha')$$

for all $a, a' \in \Gamma(A)$, $\alpha, \alpha' \in \Gamma(A^*)$. Then a computation yields

$$\begin{aligned}\langle R_\Delta^{\rm bas}(a, a')(\rho_*\alpha, \alpha), (\rho_*\alpha', \alpha')\rangle &= (\mathbf{d}_A[a, a'] - [a, \mathbf{d}_A a'] + [a', \mathbf{d}_A a])(\alpha, \alpha')\\ &+ \langle\alpha, \nabla_{\rho_*(\pounds_a\alpha')}a'\rangle - \langle\alpha, \nabla_{[\rho(a),\rho_*\alpha']}a'\rangle - \langle\alpha, \nabla_{\rho_*(\pounds_{a'}\alpha')}a\rangle + \langle\alpha, \nabla_{[\rho(a'),\rho_*\alpha']}a\rangle\\ &+ \langle\alpha, \nabla_{\rho(\mathbf{d}_A\langle a,\alpha'\rangle)}a'\rangle - \langle\alpha, \nabla_{\rho(\pounds_{\alpha'}a)}a'\rangle - \langle\alpha, \nabla_{\rho(\mathbf{d}_A\langle a',\alpha'\rangle)}a\rangle + \langle\alpha, \nabla_{\rho(\pounds_{\alpha'}a')}a\rangle.\end{aligned}$$

By (5.21), the second and the third lines vanish. We find hence that the last condition is satisfied if and only if (A, A^*) is a Lie bialgebroid. Hence, $(U, K, [\Delta])$ is an LA-Dirac triple if and only if (A, A^*) is a Lie bialgebroid, and so the graph of π_A is a subalgebroid and Dirac if and only if (A, A^*) is a Lie bialgebroid. This was already found in [27].

Example 5.15. In the situation of Examples 4.4 and 4.22, assume furthermore that $E =: A$ is a Lie algebroid. Condition (2) in Theorem 5.10 reads here $(\rho, \rho^t)(a, \sigma(a)) = (\rho(a), -\sigma^t\rho(a))$ for all $a \in \Gamma(A)$, that is, $\rho^t \circ \sigma = -\sigma^t \circ \rho$. This is equivalent to the first axiom defining an IM–2–form $\sigma\colon A \to T^*M$ [6, 5], namely $\langle\sigma(a_1), \rho(a_2)\rangle = -\langle\rho(a_1), \sigma(a_2)\rangle$ for all $a_1, a_2 \in \Gamma(A)$. Next we compute $\nabla_a^{\rm bas}(a', \sigma(a'))$. We have

$$\Omega_{(X,-\sigma^tX)}a = (\nabla_X a, -\pounds_X\sigma(a) + \sigma(\nabla_X a)) + (0, \mathbf{d}\langle\sigma(a), X\rangle) = (\nabla_X a, -\mathbf{i}_X\mathbf{d}\sigma(a) + \sigma(\nabla_X a))$$

and as a consequence

$$\begin{aligned}\nabla_a^{\rm bas}(a, \sigma(a')) &= \Omega_{(\rho,\rho^t)(a',\sigma(a'))}a + \pounds_a(a', \sigma(a'))\\ &= (\nabla_{\rho(a')}a + [a, a'], \pounds_{\rho(a)}\sigma(a') - \mathbf{i}_{\rho(a')}\mathbf{d}\sigma(a) + \sigma(\nabla_{\rho(a')}a)).\end{aligned}$$

Hence we find that $\nabla_a^{\rm bas}(a', \sigma(a')) \in \Gamma(K)$ if and only if $([a, a'], \pounds_{\rho(a)}\sigma(a') - \mathbf{i}_{\rho(a')}\mathbf{d}\sigma(a)) \in \Gamma(K)$, i.e. if and only if $\sigma([a, a']) = \pounds_{\rho(a)}\sigma(a') - \mathbf{i}_{\rho(a')}\mathbf{d}\sigma(a)$. Since this is the second axiom in the definition of an IM–2–form, we find that the graph of $(\sigma^*\omega_{\rm can})^\flat\colon TA\to T^*A$ is a subalgebroid of $TA\oplus T^*A\to TM\oplus A^*$ over $U=\operatorname{graph}(-\sigma^t)$ only if $\sigma\colon A\to T^*M$ is an IM–2–form. To recover the equivalence [5], we show that in this example, Condition (5) follows from the four previous conditions. We have also for $a,a'\in\Gamma(A)$ and $X\in\mathfrak{X}(M)$:

$$\begin{aligned}\pounds_{a'}\Omega_{(X,-\sigma^tX)}a&=-(0,\pounds_{\rho(a')}\mathbf{i}_X\mathbf{d}\sigma(a))+([a',\nabla_Xa],\pounds_{\rho(a')}\sigma(\nabla_Xa))\\&=-(0,\mathbf{i}_{[\rho(a'),X]}\mathbf{d}\sigma(a)+\mathbf{i}_X\pounds_{\rho(a')}\mathbf{d}\sigma(a))+([a',\nabla_Xa],\sigma([a',\nabla_Xa])+\mathbf{i}_{\rho(\nabla_Xa)}\mathbf{d}\sigma(a'))\end{aligned}$$

and

$$\nabla^{\rm bas}_a(X,-\sigma^tX)=-(\rho,\rho^t)(0,\mathbf{i}_X\mathbf{d}\sigma(a))+(\rho,\rho^t)(\nabla_Xa,\sigma(\nabla_Xa))+\pounds_a(X,-\sigma^tX)$$

which equals $(\nabla^A_aX,-\sigma^t(\nabla^A_aX))$. Then we easily get

$$R^{\rm bas}_\Delta(a,a')(X,-\sigma^tX)=\big(R^A_\nabla(a,a')(X),\sigma(R^A_\nabla(a,a')(X))\big)\in\Gamma(K).$$

Example 5.16. We are here in the situation of Example 4.23. Recall that $TM\to M$ with the Lie bracket of vector fields and the anchor Id_{TM} is the standard example of a Lie algebroid. We check here that the Dirac triple (D,D,Δ^D) satisfies the conditions of Theorem 5.10. First, we obviously have $(\mathrm{Id}_{TM},\mathrm{Id}^t_{TM})(D)\subseteq D$. Then, note that for all $X,Y\in\mathfrak{X}(M)$ and $\theta\in\Omega^1(M)$, we have

$$\begin{aligned}\nabla^{\rm bas}_X(Y,\theta)&=\pounds_X(Y,\theta)+\Omega_{(Y,\theta)}X=[\![(X,0),(Y,\theta)]\!]+\Delta_{(Y,\theta)}(X,0)-(0,\mathbf{d}\langle\theta,X\rangle)\\&=\Delta_{(Y,\theta)}(X,0)-[\![(Y,\theta),(X,0)]\!].\end{aligned}$$

Thus, we can compute for $X\in\mathfrak{X}(M)$ and $d_1,d_2\in\Gamma(D)$:

$$\begin{aligned}\langle\nabla^{\rm bas}_Xd_1,d_2\rangle&=\langle\Delta_{d_1}(X,0)-[\![d_1,(X,0)]\!],d_2\rangle=\langle\Delta^D_{d_1}\overline{(X,0)},d_2\rangle-\langle[\![d_1,(X,0)]\!],d_2\rangle\\&=\langle[\![d_1,(X,0)]\!],d_2\rangle-\langle[\![d_1,(X,0)]\!],d_2\rangle=0.\end{aligned}$$

This shows that $\nabla^{\rm bas}_Xd\in\Gamma(D)$ for all $d\in\Gamma(D)$. Finally we check Condition (5), involving the basic curvature. For this, note first that an easy computation using $\langle\Delta_d(X,0),d'\rangle=\langle\Delta^D\overline{(X,0)},d'\rangle=\langle[\![d,(X,0)]\!],d'\rangle$ yields $\langle\Omega_dX,d'\rangle=-\langle\pounds_Xd,d'\rangle$ for all $X\in\mathfrak{X}(M)$ and $d,d'\in\Gamma(D)$. We get

$$\begin{aligned}\langle R^{\rm bas}_\Delta(X_1,X_2)d,d'\rangle&=\langle-\Omega_d[X_1,X_2]+\pounds_{X_1}\Omega_dX_2-\pounds_{X_2}\Omega_dX_1+\Omega_{\nabla^{\rm bas}_{X_2}d}X_1-\Omega_{\nabla^{\rm bas}_{X_1}d}X_2,d'\rangle\\&=\langle\pounds_{[X_1,X_2]}d+\pounds_{X_1}\Omega_dX_2-\pounds_{X_2}\Omega_dX_1-\pounds_{X_1}\nabla^{\rm bas}_{X_2}d+\pounds_{X_2}\nabla^{\rm bas}_{X_1}d,d'\rangle,\end{aligned}$$

since we have found above that $\nabla^{\rm bas}_{X_2}d$, $\nabla^{\rm bas}_{X_1}d\in\Gamma(D)$. But since $\pounds_{X_1}\Omega_dX_2-\pounds_{X_1}\nabla^{\rm bas}_{X_2}d=-\pounds_{X_1}\pounds_{X_2}d$, we find $\langle R^{\rm bas}_\Delta(X_1,X_2)d,d'\rangle=\langle\pounds_{[X_1,X_2]}d-\pounds_{X_1}\pounds_{X_2}d+\pounds_{X_2}\pounds_{X_1}d,d'\rangle=0$.

There is a canonical isomorphism from the Courant algebroid over TM

$$\begin{array}{ccc}TTM\oplus T^*TM&\longrightarrow&TM\oplus T^*M\\\downarrow&&\downarrow\\TM&\longrightarrow&M\end{array}\quad\longrightarrow\quad\begin{array}{ccc}T(TM)\oplus T(T^*M)&\longrightarrow&TM\oplus T^*M\\\downarrow&&\downarrow\\TM&\longrightarrow&M\end{array}$$

to the double tangent of the vector bundle $TM\oplus T^*M$ [31], see also [23]. One can check in a straightforward manner (using for instance [23]) that this isomorphism is nothing else than the anchor of the VB-Lie algebroid $(T(TM)\oplus T^*(TM),TM\oplus T^*M;TM,M)$.

$D_{(D,D,\Delta^D)}$ is spanned as a vector bundle over D by the sections $\sigma^\Delta_{TM}(X)|_D$ for all $X\in\mathfrak{X}(M)$ and $d^\dagger|_D$ for all $d\in\Gamma(D)$. By Theorem 5.7, the image of $\sigma^\Delta_{TM}(X)$ under the anchor Θ is $\widehat{\nabla^{\rm bas}_X}$ and the image of $d^\dagger$ is $d^\uparrow$. Hence, since $\nabla^{\rm bas}$ restricts to a TM-connection on D, we get that the linear subbundle $(D_{(D,D,\Delta^D)},D;TM,M)$ is sent via this isomorphism to $(TD,D;TM,M)$, the tangent Dirac structure in [7].

Appendix A. Linear almost Poisson structures and the canonical symplectic form on T^*E

A.1. **Linear almost Poisson structures.** Consider here a skew-symmetric dull algebroid $(A,\rho,[\cdot\,,\cdot])$. This is equivalent to a linear almost Poisson bracket on the vector bundle $A^*\to M$, i.e. a skew-symmetric bracket $\{\cdot\,,\cdot\}\colon C^\infty(A^*)\times C^\infty(A^*)\to C^\infty(A^*)$ such that

(1) $\{\cdot\,,\cdot\}$ satisfies the Leibniz identity,
(2) $\{\ell_a,\ell_b\}=\ell_{[a,b]}$ is again linear for two sections $a,b\in\Gamma(A)$ and
(3) $\{\ell_a,q_{A^*}^*f\}=q_{A^*}^*(\rho(a)(f))$ is again a pullback for all $a\in\Gamma(A)$ and $f\in C^\infty(M)$.

Let $\pi_A\in\mathfrak{X}^2(A^*)$ be the bivector field associated to this almost Poisson structure. We describe the vector bundle morphism $\pi_A^\sharp\colon T^*A^*\to TA^*$, $\mathbf{d}F\mapsto\{F,\cdot\}$, $F\in C^\infty(A^*)$, associated to it.

We compute the vector fields $\pi_A^\sharp(\mathbf{d}\ell_a)$ and $\pi_A^\sharp(q_{A*}^*\theta)$ for all $a\in\Gamma(A)$ and $\theta\in\Omega^1(M)$. Since $\pi_A^\sharp(q_{A^*}^*\mathbf{d}f)(q_{A^*}^*g)=\pi_A^\sharp(\mathbf{d}q_{A^*}^*f)(q_{A^*}^*g)=0$ and $\pi_A^\sharp(\mathbf{d}q_{A^*}^*f)(\ell_a)=-q_{A^*}^*(\rho(a)(f))$ for all $f,\psi\in C^\infty(M)$ and $a\in\Gamma(A)$, we find $\pi_A^\sharp(q_{A^*}^*\mathbf{d}f)=-(\rho^t(\mathbf{d}f))^\uparrow$ for all $f\in C^\infty(M)$ and consequently $\pi_A^\sharp(q_{A^*}^*\theta)=-(\rho^t\theta)^\uparrow$ for all $\theta\in\Omega^1(M)$. In the same manner, we have

$$\pi_A^\sharp(\mathbf{d}\ell_a)(\ell_b)=\ell_{[a,b]}\qquad\text{and}\qquad\pi_A^\sharp(\mathbf{d}\ell_a)(q_{A^*}^*f)=q_{A^*}^*(\rho(a)(f))$$

for $a,b\in\Gamma(A)$ and $f\in C^\infty(M)$. Recall that the vector field $\widehat{\mathcal{L}_a}\in\mathfrak{X}(A^*)$ satisfies $\widehat{\mathcal{L}_a}(\alpha_m)(\ell_b)=\ell_{[a,b]}(\alpha_m)$ and $\widehat{\mathcal{L}_a}(\alpha_m)(q_{A^*}^*f)=\rho(a(m))(f)$ for $\alpha_m\in A^*$. This shows the equality $\pi_A^\sharp(\mathbf{d}\ell_a)=\widehat{\mathcal{L}_a}$.

A.2. **The canonical symplectic form on T^*E.** Now let M be a smooth manifold and $c_M\colon T^*M\to M$ its cotangent bundle. Recall that there is a canonical 1-form $\theta_{\rm can}\in\Omega^1(T^*M)$, given by

$$\langle\theta_{\rm can}(\eta_m),v_{\eta_m}\rangle=\langle\eta_m,T_{\eta_m}c_M(v_{\eta_m})\rangle$$

for all $\eta_m\in T^*M$ and $v_{\eta_m}\in T_{\eta_m}(T^*M)$. The canonical symplectic form $\omega_{\rm can}\in\Omega^2(T^*M)$ is defined by $\omega_{\rm can}=-\mathbf{d}\theta_{\rm can}$.

Consider a vector bundle $E\to M$ endowed with a vector bundle morphism $\sigma\colon E\to T^*M$ over the identity, and a connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$. The one-form $\sigma^*\theta_{\rm can}\in\Omega^1(E)$ can be described as follows

$$\begin{aligned}\langle(\sigma^*\theta_{\rm can})(e'_m),\sigma_{TM}^\nabla(X)(e'_m)\rangle&=\langle\theta_{\rm can}(\sigma(e'_m)),T_{e'_m}\sigma(\sigma_{TM}^\nabla(X)(e'_m))\rangle=\langle\sigma(e'_m),X(m)\rangle\\ \langle(\sigma^*\theta_{\rm can})(e'_m),e^\uparrow(e'_m)\rangle&=\langle\theta_{\rm can}(\sigma(e'_m)),\sigma(e)^\uparrow(e'_m)\rangle=0\end{aligned}$$

for all $e'_m\in E$, $X\in\mathfrak{X}(M)$ and $e\in\Gamma(E)$. This shows in particular the equality $\langle\sigma^*\theta_{\rm can},\sigma_{TM}^\nabla(X)\rangle=\ell_{\sigma^t(X)}$. As a consequence, we get for all $e,e_1,e_2\in\Omega^1(M)$, $X,Y\in\mathfrak{X}(M)$:

$$\begin{aligned}\sigma^*\omega_{\rm can}(\sigma_{TM}^\nabla(X),\sigma_{TM}^\nabla(Y))&=\sigma_{TM}^\nabla(X)((\sigma^*\theta_{\rm can})(\sigma_{TM}^\nabla(Y)))-\sigma_{TM}^\nabla(Y)((\sigma^*\theta_{\rm can})(\sigma_{TM}^\nabla(X)))\\&\qquad-(\sigma^*\theta_{\rm can})\left(\sigma_{TM}^\nabla[X,Y]-\widetilde{R_\nabla(X,Y)}\right)\\&=\ell_{\nabla_X^*(\sigma^tY)}-\ell_{\nabla_Y^*(\sigma^tX)}-\ell_{\sigma^t[X,Y]}=\ell_{\nabla_X^*(\sigma^tY)-\nabla_Y^*(\sigma^tX)-\sigma^t[X,Y]}\\ \sigma^*\omega_{\rm can}(\sigma_{TM}^\nabla(X),e^\uparrow)&=\sigma_{TM}^\nabla(X)(0)-e^\uparrow(\ell_{\sigma^tX})-\sigma^*\theta_{\rm can}\left(\left[\sigma_{TM}^\nabla(X),e^\uparrow\right]\right)\\&=-q_E^*\langle\sigma(e),X\rangle\end{aligned}$$

and $\sigma^*\omega_{\rm can}(e_1^\uparrow,e_2^\uparrow)=0$. Hence, the one-forms $(\sigma^*\omega_{\rm can})^\flat(\sigma_{TM}^\nabla(X))$ and $(\sigma^*\omega_{\rm can})^\flat(e^\uparrow)\in\Omega^1(E)$ are given by

$$(\sigma^*\omega_{\rm can})^\flat(\sigma_{TM}^\nabla(X))=\mathbf{d}\ell_{-\sigma^tX}+\widetilde{\sigma(\nabla_X\cdot)-\mathcal{L}_X(\sigma(\cdot))},$$

where $\sigma(\nabla_X\cdot)-\mathcal{L}_X(\sigma(\cdot))$ is seen as a section of $\operatorname{Hom}(E,T^*M)$, and $(\sigma^*\omega_{\rm can})^\flat(e^\uparrow)=q_E^*(\sigma(e))$.

Appendix B. Proof of Theorem 4.9

In this section we prove Theorem 4.9. For simplicity, given a Dorfman connection

$$\Delta\colon \Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M),$$

we write $\tilde X=\operatorname{pr}_{TE}\left(\sigma^{\Delta}_{TM\oplus E^*}(X,\varepsilon)\right)$ and $\tilde\varepsilon=\operatorname{pr}_{T^*E}\left(\sigma^{\Delta}_{TM\oplus E^*}(X,\varepsilon)\right)$. The reader should bear in mind that both $\tilde X$ and $\tilde\varepsilon$ depend on X and ε. More precisely, $\tilde X$ is the linear vector field $\widehat{\nabla_{(X,\xi)}}$, with the connection $\nabla\colon\Gamma(TM\oplus E^*)\times\Gamma(E)\to\Gamma(E)$ in Proposition 4.7. Further, recall that by construction, the Dorfman connection can be written

$$\Delta_{(X,\varepsilon)}(e,\theta)=\Delta_{(X,\varepsilon)}(e,0)+(0,\pounds_X\theta) \tag{2.22}$$

for all $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $(e,\theta)\in\Gamma(E\oplus T^*M)$. We begin by proving three lemmas.

Lemma B.1. *Choose $(X_1,\varepsilon_1),(X_2,\varepsilon_2)\in\Gamma(TM\oplus E^*)$ and $e\in\Gamma(E)$. Then*

$$\nabla_{(X_2,\varepsilon_2)}\nabla_{(X_1,\varepsilon_1)}e=\operatorname{pr}_E\left(\Delta_{(X_2,\varepsilon_2)}\Delta_{(X_1,\varepsilon_1)}(e,0)\right).$$

Proof. Choose $\varepsilon_3\in\Gamma(E^*)$. Then

$$\begin{aligned}
&\langle\varepsilon_3,\operatorname{pr}_E\Delta_{(X_2,\varepsilon_2)}\left(\operatorname{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0),0\right)\rangle=\langle(0,\varepsilon_3),\Delta_{(X_2,\varepsilon_2)}\left(\operatorname{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0),0\right)\rangle\\
&=\ X_2\langle(0,\varepsilon_3),\left(\operatorname{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0),0\right)\rangle-\langle[\![(X_2,\varepsilon_2),(0,\varepsilon_3)]\!]_\Delta,\left(\operatorname{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0),0\right)\rangle\\
&=\ X_2\langle(0,\varepsilon_3),\Delta_{(X_1,\varepsilon_1)}(e,0)\rangle-\langle[\![(X_2,\varepsilon_2),(0,\varepsilon_3)]\!]_\Delta,\Delta_{(X_1,\varepsilon_1)}(e,0)\rangle\\
&=\ \langle(0,\varepsilon_3),\Delta_{(X_2,\varepsilon_2)}\Delta_{(X_1,\varepsilon_1)}(e,0)\rangle=\langle\varepsilon_3,\operatorname{pr}_E\Delta_{(X_2,\varepsilon_2)}\Delta_{(X_1,\varepsilon_1)}(e,0)\rangle.
\end{aligned}$$

In the third equality, we have used $\operatorname{pr}_{TM}[\![(X_2,\varepsilon_2),(0,\varepsilon_3)]\!]_\Delta=[X_2,0]=0$. □

Lemma B.2. *Choose $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $e\in\Gamma(E)$. Then*

1. $\langle\tilde\varepsilon,e^\uparrow\rangle=q_E^*\langle\varepsilon,e\rangle$.
2. $\pounds_{e^\uparrow}\tilde\varepsilon=q_E^*\left(\mathbf{d}\langle\varepsilon,e\rangle-\operatorname{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e,0)\right)$.
3. $\left[\tilde X,e^\uparrow\right]=(\nabla_{(X,\varepsilon)}e)^\uparrow$.

Proof. The first claim is immediate by the definition of $\tilde\varepsilon$. For any $e'\in\Gamma(E)$, we have

$$\langle\pounds_{e^\uparrow}\tilde\varepsilon,e'^\uparrow\rangle=e^\uparrow(\langle\tilde\varepsilon,e'^\uparrow\rangle)-\langle\tilde\varepsilon,[e^\uparrow,e'^\uparrow]\rangle=e^\uparrow(q_E^*\langle\varepsilon,e'\rangle)-\langle\tilde\varepsilon,0\rangle=0.$$

This shows that $\pounds_{e^\uparrow}\tilde\varepsilon$ is vertical, i.e. the pullback under q_E of a 1-form on M. Thus, we just need to compute $\langle(\pounds_{e^\uparrow}\tilde\varepsilon)(e'(m)),T_me'v_m\rangle$ for $e'\in\Gamma(E)$ and $v_m\in TM$. But we have

$$\begin{aligned}
\langle(\pounds_{e^\uparrow}\tilde\varepsilon)(e'(m)),T_me'v_m\rangle&=\left.\frac{d}{dt}\right|_{t=0}\langle\tilde\varepsilon(e'(m)+te(m)),T_{e'(m)}\phi_t^{e^\uparrow}(T_me'v_m)\rangle\\
&=\left.\frac{d}{dt}\right|_{t=0}\langle\tilde\varepsilon(e'(m)+te(m)),T_m(e'+te)v_m\rangle\\
&=\left.\frac{d}{dt}\right|_{t=0}v_m\langle\varepsilon,e'+te\rangle-\langle\operatorname{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e'+te,0),v_m\rangle\\
&=v_m\langle\varepsilon,e\rangle-\langle\operatorname{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e,0),v_m\rangle.
\end{aligned}$$

For the third equality we just need to compute $[\tilde X,e^\uparrow](\ell_{\varepsilon'})$ for sections $\varepsilon'\in\Gamma(E^*)$ and $[\tilde X,e^\uparrow](q_E^*f)$ for functions $f\in C^\infty(M)$. We have

$$\begin{aligned}
[\tilde X,e^\uparrow](\ell_{\varepsilon'})&=\tilde X(e^\uparrow(\ell_{\varepsilon'}))-e^\uparrow(\tilde X(\ell_{\varepsilon'}))=\tilde X(q_E^*\langle\varepsilon',e\rangle)-e^\uparrow(\ell_{\nabla^*_{(X,\varepsilon)}\varepsilon'})\\
&=q_E^*\left(X\langle\varepsilon',e\rangle-\langle\nabla^*_{(X,\varepsilon)}\varepsilon',e\rangle\right)=(\nabla_{(X,\xi)}e)^\uparrow(\ell_{\varepsilon'}),
\end{aligned}$$

and $[\tilde X,e^\uparrow](q_E^*f)=0=(\nabla_{(X,\xi)}e)^\uparrow(q_E^*f)$ since $e^\uparrow\sim_{q_E}0$ and $\tilde X\sim_{q_E}X$. □

Next note that since $\tilde{X}$ is linear over X, the flow $\phi_t^{\tilde{X}}$ of $\tilde{X}$ is a vector bundle morphism $E \to E$ over $\phi_t^X\colon M \to M$, for any $t \in \mathbb{R}$ where this is defined. Hence, for any section $e \in \Gamma(E)$, we can define a new section $\psi_t^{(X,\varepsilon)}(e) \in \Gamma(E)$ by

$$\psi_t^{(X,\varepsilon)}(e) = \phi_{-t}^{\tilde{X}} \circ e \circ \phi_t^X.$$

Lemma B.3. *The time derivative of $\psi_t^{(X,\varepsilon)}$ satisfies*

$$\left.\frac{d}{dt}\right|_{t=0} \psi_t^{(X,\varepsilon)}(e) = \nabla_{(X,\varepsilon)} e.$$

Proof. The curve $c\colon t \mapsto \psi_t^X(e)(m)$ is a curve in E with $c(0) = e_m$ and satisfying $q_E \circ c = m$. Hence, the derivative $\dot{c}(0)$ is a vertical vector over e_m. Since $\phi_t^{\tilde{X}}$ is linear, we have

$$((\phi_t^{\tilde{X}})^* e^\uparrow)(e'_m) = \left.\frac{d}{ds}\right|_{s=0} \phi_{-t}^{\tilde{X}}(\phi_t^{\tilde{X}}(e'_m) + se(\phi_t^X(m))) = \left.\frac{d}{ds}\right|_{s=0} e'_m + s\psi_t^{(X,\varepsilon)}(e)(m)$$

for $e'_m \in E$. Thus, we get for any $\varepsilon \in \Gamma(E^*)$:

$$\begin{aligned}
[\tilde{X}, e^\uparrow](\ell_\varepsilon)(e'_m) &= \left.\frac{d}{dt}\right|_{t=0} \langle \mathbf{d}_{e'_m}\ell_\varepsilon, ((\phi_t^{\tilde{X}})^* e^\uparrow)\rangle = \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} \langle \varepsilon(m), e'_m + s\psi_t^{(X,\varepsilon)}(e)(m)\rangle \\
&= \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \langle \varepsilon(m), e'_m + s{\psi_t}^X(e)(m)\rangle \\
&= \left.\frac{d}{ds}\right|_{s=0} s\left\langle \varepsilon(m), \left.\frac{d}{dt}\right|_{t=0} \psi_t^X(e)(m)\right\rangle = \left\langle \varepsilon(m), \left.\frac{d}{dt}\right|_{t=0} \psi_t^X(e)(m)\right\rangle.
\end{aligned}$$

This shows that

$$[\tilde{X}, e^\uparrow] = \left(\left.\frac{d}{dt}\right|_{t=0} \psi_t^X(e)\right)^\uparrow.$$

By (3) of Lemma B.2, we are done. □

Now we can prove Theorem 4.9. We write $\tau = (e,\theta)$, $\tau_i = (e_i,\theta_i)$ and $\nu = (X,\varepsilon)$, $\nu_i = (X_i,\varepsilon_i)$ for $i = 1,2$.

Proof of Theorem 4.9. The first equality is easy to check: for the tangent part, we use the commutativity of the flows of the vertical vector fields. For the cotangent part, note that since $e_i^\uparrow \sim_{q_E} 0$ for $i = 1,2$, we get immediately $\pounds_{e_1^\uparrow} q_E^*\theta_2 - \mathbf{i}_{e_2^\uparrow}\mathbf{d}q_E^*\theta_1 = q_E^*(\pounds_0\theta_2 - \mathbf{i}_0\mathbf{d}\theta_1) = 0$.

For the second equality, we know by Lemma B.2 that $[\tilde{X}, e^\uparrow] = (\operatorname{pr}_E \Delta_{(X,\varepsilon)}(e,0))^\uparrow$. We compute the cotangent part of the Courant-Dorfman bracket. Using Lemma B.2, we have

$$\begin{aligned}
\pounds_{\tilde{X}} q_E^*\theta - \mathbf{i}_{e^\uparrow}\mathbf{d}\tilde{\varepsilon} &= \pounds_{\tilde{X}} q_E^*\theta - \pounds_{e^\uparrow}\tilde{\varepsilon} + \mathbf{d}\langle\tilde{\varepsilon}, e^\uparrow\rangle \\
&= \pounds_{\tilde{X}} q_E^*\theta - q_E^*\left(\mathbf{d}\langle\varepsilon, e\rangle - \operatorname{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e,0)\right) + \mathbf{d}q_E^*\langle\varepsilon, e\rangle \\
&= q_E^*(\pounds_X\theta + \operatorname{pr}_{T^*M}\Delta_{(X,\varepsilon)}(e,0)) \overset{(2.22)}{=} q_E^*\operatorname{pr}_{T^*M}\Delta_\nu\tau.
\end{aligned}$$

This leads to our claim $[\![\sigma_{TM\oplus E^*}^\Delta(\nu), \tau^\uparrow]\!] = \Delta_\nu\tau^\uparrow$.

For the last equality choose a section $\tau = (e,\theta)$ of $E \oplus T^*M$. Then:

$$\left\langle \pounds_{\tilde{X}_1}\tilde{\varepsilon}_2 - \mathbf{i}_{\tilde{X}_2}\mathbf{d}\tilde{\varepsilon}_1, e^\uparrow\right\rangle = \tilde{X}_1\langle\tilde{\varepsilon}_2, e^\uparrow\rangle - \left\langle\tilde{\varepsilon}_2, \left[\tilde{X}_1, e^\uparrow\right]\right\rangle - \tilde{X}_2\langle\tilde{\varepsilon}_1, e^\uparrow\rangle + e^\uparrow\langle\tilde{\varepsilon}_1, \tilde{X}_2\rangle + \left\langle\tilde{\varepsilon}_1, \left[\tilde{X}_2, e^\uparrow\right]\right\rangle$$

First we have $\langle\tilde{\varepsilon}_2, e^\uparrow\rangle = q_E^*\langle\varepsilon_2, e\rangle$ by Lemma B.2, and consequently $\tilde{X}_1\langle\tilde{\varepsilon}_2, e^\uparrow\rangle = q_E^*(X_1\langle\varepsilon_2, e\rangle)$. Then, we get $\left\langle\tilde{\varepsilon}_2, \left[\tilde{X}_1, e^\uparrow\right]\right\rangle = q_E^*\langle\varepsilon_2, \nabla_{(X_1,\varepsilon_1)}e\rangle$ by (3) of Lemma B.2, and $\langle\tilde{\varepsilon}_1, \tilde{X}_2\rangle(e_m) =$

$X_2(m)\langle\varepsilon_1, e\rangle - \langle\varepsilon_1, \nabla_{\nu_2}e\rangle(m) - \langle X_2, \mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0)\rangle(m)$, which defines a linear function on E. This yields $e^\uparrow\langle\tilde\varepsilon_1, \tilde X_2\rangle = q_E^*\left(X_2\langle\varepsilon_1, e\rangle - \langle\varepsilon_1, \nabla_{\nu_2}e\rangle - \langle X_2, \mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0)\rangle\right)$. Thus, we get

$$\begin{aligned}\left\langle \pounds_{\tilde X_1}\tilde\varepsilon_2 - \mathbf{i}_{\tilde X_2}\mathbf{d}\tilde\varepsilon_1, e^\uparrow\right\rangle &= q_E^*\big(X_1\langle\varepsilon_2, e\rangle - \langle\varepsilon_2, \nabla_{\nu_1}e\rangle - \cancel{X_2\langle\varepsilon_1, e\rangle} + \cancel{X_2\langle\varepsilon_1, e\rangle} - \cancel{\langle\varepsilon_1, \nabla_{\nu_2}e\rangle}\\ &\qquad -\langle X_2, \mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0)\rangle + \cancel{\langle\varepsilon_1, \nabla_{\nu_2}e\rangle}\big)\\ &= q_E^*\left(X_1\langle\varepsilon_2, e\rangle - \langle(X_2,\varepsilon_2), \Delta_{\nu_1}(e,0)\rangle\right) = q_E^*\langle[\![\nu_1,\nu_2]\!]_\Delta, (e,0)\rangle.\end{aligned}$$

This leads to $\left\langle \left[\!\left[\sigma^\Delta_{TM\oplus E^*}(\nu_1), \sigma^\Delta_{TM\oplus E^*}(\nu_2)\right]\!\right], (e,\theta)^\uparrow\right\rangle = q_E^*\langle[\![\nu_1,\nu_2]\!]_\Delta, (e,\theta)\rangle$, which shows that

$$\left[\!\left[\sigma^\Delta_{TM\oplus E^*}(\nu_1), \sigma^\Delta_{TM\oplus E^*}(\nu_2)\right]\!\right](e_m) = (T_m e[X_1, X_2](m), \mathbf{d}_{e_m}\ell_{\mathrm{pr}_{E^*}[\![\nu_1,\nu_2]\!]_\Delta}) + \tau^\uparrow(e_m),$$

for some $\tau\in\Gamma(E\oplus T^*M)$.

Hence we know that for any $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$, we have

$$\begin{aligned}\langle\tau, (X,\varepsilon)\rangle(m) &= \left\langle \left[\!\left[\sigma^\Delta_{TM\oplus E^*}(\nu_1), \sigma^\Delta_{TM\oplus E^*}(\nu_2)\right]\!\right](e_m), (T_m eX(m), \mathbf{d}_{e_m}\ell_\varepsilon)\right\rangle\\ &\quad - X(m)\langle[\![\nu_1,\nu_2]\!]_\Delta, (e,0)\rangle - [X_1, X_2]\langle\varepsilon, e\rangle.\end{aligned}\tag{2.23}$$

First we find $\left[\tilde X_1, \tilde X_2\right](\ell_\varepsilon) = \ell_{\nabla^*_{\nu_1}\nabla^*_{\nu_2}\varepsilon - \nabla^*_{\nu_2}\nabla^*_{\nu_1}\varepsilon}$. Next, we compute $\langle\pounds_{\tilde X_1}\tilde\varepsilon_2, T_m eX(m)\rangle$. Using Lemma B.3 and the identity $\phi_t^{\tilde X_1}(e_m) = \psi_{-t}^{X_1}(e)(\phi_t^{X_1}(m))$, we find

$$\begin{aligned}&\langle\pounds_{\tilde X_1}\tilde\varepsilon_2(e_m), T_m eX(m)\rangle = \left.\frac{d}{dt}\right|_{t=0}\left\langle\tilde\varepsilon_2(\phi_t^{\tilde X_1}(e_m)), (T_{e_m}\phi_t^{\tilde X_1}\circ T_m e)X(m)\right\rangle\\ &= \left.\frac{d}{dt}\right|_{t=0}\langle\tilde\varepsilon_2(\psi_{-t}^{X_1}(e)(\phi_t^{X_1}(m))), T_{\phi_t^{X_1}(m)}\psi_{-t}^{X_1}(e)((\phi_{-t}^{X_1})^*(X)(\phi_t^{X_1}(m)))\rangle\\ &= \left.\frac{d}{dt}\right|_{t=0}((\phi_{-t}^{X_1})^*(X))\langle\varepsilon_2, \psi_{-t}^{X_1}(e)\rangle(\phi_t^{X_1}(m))\\ &\qquad - \left.\frac{d}{dt}\right|_{t=0}\langle\mathrm{pr}_{T^*M}\Delta_{(X_2,\varepsilon_2)}(\psi_{-t}^{X_1}(e),0), (\phi_{-t}^{X_1})^*(X)\rangle(\phi_t^{X_1}(m))\\ &= \big(-[X_1,X]\langle\varepsilon_2, e\rangle + X_1X\langle\varepsilon_2, e\rangle - X\langle\varepsilon_2, \mathrm{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0)\rangle - X_1\langle\mathrm{pr}_{T^*M}\Delta_{(X_2,\varepsilon_2)}(e,0), X\rangle\\ &\qquad + \langle\mathrm{pr}_{T^*M}\Delta_{(X_2,\varepsilon_2)}(e,0), [X_1,X]\rangle + \langle\mathrm{pr}_{T^*M}\Delta_{(X_2,\varepsilon_2)}(\mathrm{pr}_E\Delta_{(X_1,\varepsilon_1)}(e,0),0), X\rangle\big)(m).\end{aligned}$$

We also have

$$\begin{aligned}\langle\mathbf{d}_{e_m}\langle\tilde\varepsilon_1, \tilde X_2\rangle, T_m eX(m)\rangle &= X\left(\langle\tilde\varepsilon_1, \tilde X_2\rangle\circ e\right)\\ &= X\left(X_2\langle\varepsilon_1, e\rangle - \langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0), X_2\rangle - \langle\varepsilon_1, \nabla_{\nu_2}e\rangle\right),\end{aligned}$$

which leads to

$$\begin{aligned}&\langle\pounds_{\tilde X_1}\tilde\varepsilon_2 - \mathbf{i}_{\tilde X_2}\mathbf{d}\tilde\varepsilon_1, T_m eX(m)\rangle = \langle\pounds_{\tilde X_1}\tilde\varepsilon_2 - \pounds_{\tilde X_2}\tilde\varepsilon_1 + \mathbf{d}\langle\tilde\varepsilon_1, \tilde X_2\rangle, T_m eX(m)\rangle\\ &= (XX_1\langle\varepsilon_2, e\rangle - X\langle\varepsilon_2, \nabla_{\nu_1}e\rangle - X_1\langle\mathrm{pr}_{T^*M}\Delta_{\nu_2}(e,0), X\rangle + \langle\mathrm{pr}_{T^*M}\Delta_{\nu_2}(e,0), [X_1,X]\rangle\\ &\qquad + \langle\mathrm{pr}_{T^*M}\Delta_{\nu_2}(\nabla_{\nu_1}e,0), X\rangle - \cancel{XX_2\langle\varepsilon_1, e\rangle} + \cancel{X\langle\varepsilon_1, \nabla_{\nu_2}e\rangle}\\ &\qquad + X_2\langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0), X\rangle - \langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0), [X_2,X]\rangle - \langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(\nabla_{\nu_2}e,0), X\rangle\\ &\qquad + \cancel{XX_2\langle\varepsilon_1, e\rangle} - X\langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0), X_2\rangle - \cancel{X\langle\varepsilon_1, \nabla_{\nu_2}e\rangle})(m)\end{aligned}$$

The first, second and last remaining terms add up to $X(X_1\langle[\![\nu_1,\nu_2]\!]_\Delta, (e,0)\rangle - \langle\Delta_{\nu_1}(e,0), \nu_2\rangle = X\langle[\![\nu_1,\nu_2]\!]_\Delta, (e,0)\rangle$. The fifth remaining term can be written $\langle\Delta_{\nu_2}(\mathrm{pr}_E\Delta_{\nu_1}(e,0),0), (X,0)\rangle$. But this equals

$$\begin{aligned}&X_2\langle(\mathrm{pr}_E\Delta_{\nu_1}(e,0),0), (X,0)\rangle - \langle(\mathrm{pr}_E\Delta_{\nu_1}(e,0),0), [\![\nu_2, (X,0)]\!]_\Delta\rangle\\ &= 0 - \langle\Delta_{\nu_1}(e,0), [\![\nu_2, (X,0)]\!]_\Delta\rangle + \langle\mathrm{pr}_{T^*M}\Delta_{\nu_1}(e,0), [X_2, X]\rangle,\end{aligned}$$

which, together with the seventh remaining term, add up to $-\langle \Delta_{\nu_1}(e,0), [\![\nu_2,(X,0)]\!]_\Delta\rangle$. This and the sixth remaining term add up to $\langle \Delta_{\nu_2}\Delta_{\nu_1}(e,0),(X,0)\rangle$. Similarly, the eighth, third and fourth remaining terms add up to $-\langle \Delta_{\nu_1}\Delta_{\nu_2}(e,0),(X,0)\rangle$. This leads to

$$\begin{aligned}&\langle \pounds_{\tilde X_1}\tilde\varepsilon_2 - \mathbf{i}_{\tilde X_2}\mathbf{d}\tilde\varepsilon_1, T_m eX(m)\rangle\\ &= \left(X\langle [\![\nu_1,\nu_2]\!]_\Delta,(e,0)\rangle - \langle \Delta_{\nu_1}\Delta_{\nu_2}(e,0)-\Delta_{\nu_2}\Delta_{\nu_1}(e,0),(X,0)\rangle\right)(m).\end{aligned}$$

Now we find that (2.23) reads

$$\begin{aligned}\langle \tau,(X,\varepsilon)\rangle &= \langle \nabla^*_{\nu_1}\nabla^*_{\nu_2}\varepsilon - \nabla^*_{\nu_2}\nabla^*_{\nu_1}\varepsilon, e\rangle - [X_1,X_2]\langle\varepsilon,e\rangle - \langle \Delta_{\nu_1}\Delta_{\nu_2}(e,0)-\Delta_{\nu_2}\Delta_{\nu_1}(e,0),(X,0)\rangle\\ &= \langle R_{\nabla^*}(\nu_1,\nu_2)\varepsilon + \nabla^*_{[\![\nu_1,\nu_2]\!]_\Delta}\varepsilon, e\rangle - [X_1,X_2]\langle\varepsilon,e\rangle - \langle R_\Delta(\nu_1,\nu_2)(e,0) + \Delta_{[\![\nu_1,\nu_2]\!]_\Delta}(e,0),(X,0)\rangle\\ &= \langle -R_\nabla(\nu_1,\nu_2)e - \nabla_{[\![\nu_1,\nu_2]\!]_\Delta}e,\varepsilon\rangle - \langle R_\Delta(\nu_1,\nu_2)(e,0) + \Delta_{[\![\nu_1,\nu_2]\!]_\Delta}(e,0),(X,0)\rangle\\ &= \langle -R_\Delta(\nu_1,\nu_2)(e,0) - \Delta_{[\![\nu_1,\nu_2]\!]_\Delta}(e,0),(X,\varepsilon)\rangle.\end{aligned}$$

In the last equality, we have used Lemma B.1. This shows

$$\tau = -R_\Delta(\nu_1,\nu_2)(e,0) - \Delta_{[\![\nu_1,\nu_2]\!]_\Delta}(e,0). \qquad \square$$

Appendix C. The Lie algebroid structure on $TA\oplus T^*A\to TM\oplus A^*$

Let $(q_A\colon A\to M,\rho,[\cdot\,,\cdot])$ be a Lie algebroid. We describe here the Lie algebroid structures on $TA\to TM$, $T^*A\to A^*$ and $TA\oplus T^*A\to TM\oplus A^*$. For simplicity, we write $q:=q_A\colon A\to M$ and $q_*:=q_{A^*}\colon A^*\to M$ for the vector bundle maps.

The Lie algebroid $TA\to TM$. Recall that for $a\in\Gamma(A)$, we have two particular types of sections of $TA\to TM$: the *linear* sections $Ta\colon TM\to TA$, which are vector bundle morphisms over $a\colon M\to A$, and the *core* sections $a^\dagger\colon TM\to TA$, $a^\dagger(v_m) = T_m0^A v_m +_{p_A} \frac{d}{dt}\big\an{t=0} t\cdot a(m)$. The Lie algebroid bracket on sections of $TA\to TM$ is given by

$$[Ta_1,Ta_2]=T[a_1,a_2], \qquad [Ta_1,a_2^\dagger]=[a_1,a_2]^\dagger, \qquad [a_1^\dagger,a_2^\dagger]=0$$

and the anchor is given by $\rho_{TA}(Ta)=\widehat{[\rho(a),\cdot]}\in\mathfrak{X}(TM)$, $\rho_{TA}(a^\dagger)=(\rho(a))^\uparrow\in\mathfrak{X}(TM)$ (see [25]).

The Lie algebroid $T^*A\to A^*$. There is an isomorphism of double vector bundles

$$\begin{array}{ccc} T^*A^* & \xrightarrow{r_{A^*}} & A \\ {\scriptstyle c_{A^*}}\downarrow & & \downarrow \\ A^* & \longrightarrow & M \end{array} \quad \xrightarrow{R} \quad \begin{array}{ccc} T^*A & \xrightarrow{c_A} & A \\ {\scriptstyle r_A}\downarrow & & \downarrow \\ A^* & \longrightarrow & M \end{array}$$

over the identity on the sides, and $-\operatorname{id}_{T^*M}$ on the core T^*M. The map R is given as follows: for $\theta\in\Omega^1(M)$, we have

$$R(q_*^*\theta(\alpha_m)) = \mathbf{d}_{0^A_m}\ell_\alpha - q^*\theta(0^A_m)$$

and for $\alpha\in\Gamma(A^*)$ and $a\in\Gamma(A)$, we have

$$R(\mathbf{d}_{\alpha(m)}\ell_a) = \mathbf{d}_{a(m)}(\ell_\alpha - q^*\langle\alpha,a\rangle)$$

for all $m\in M$. Hence, we find that for $\theta\in\Omega^1(M)$, the core section $\theta^\dagger\in\Gamma_{A^*}(T^*A)$ is $\alpha_m\mapsto R(-q_*^*\theta(\alpha_m))$. For $a\in\Gamma(A)$, we write $a^R\in\Gamma_{A^*}(T^*A)$ for the section $\alpha_m\mapsto R(\mathbf{d}_{\alpha(m)}\ell_a)$.

Recall that since A is a Lie algebroid, its dual A^* is endowed with a linear Poisson structure given by

$$\{\ell_{a_1},\ell_{a_2}\}=\ell_{[a,b]}, \qquad \{\ell_a,q_*^*f\}=q_*^*(\rho(a)(f)), \qquad \{q_*^*f,q_*^*g\}=0$$

for all $a_1,a_2\in\Gamma(A)$ and $f,g\in C^\infty(M)$. Hence, there is a Lie algebroid structure on $T^*A^*\to A^*$ associated to this Poisson structure, and the Lie algebroid structure on $T^*A\to A^*$ is exactly such that the isomorphism $R\colon T^*A^*\to T^*A$ is an isomorphism of Lie algebroids [25, 26].

Therefore, we first give the Lie brackets and images under the anchor map $\rho_{T^*A^*}$ of the sections $\mathbf{d}\ell_a$ and $q_*^*\theta \in \Omega^1(A^*) = \Gamma_{A^*}(T^*A^*)$, for $\theta \in \Omega^1(M)$ and $a \in \Gamma(A)$. By the definition of the Lie algebroid structure $T^*A^* \to A^*$ associated to the linear Poisson structure on A^*, one finds easily that the Lie algebroid structure on $T^*A^* \to A^*$ is given by the following identities:

$$[\mathbf{d}\ell_a, \mathbf{d}\ell_b] = \mathbf{d}\ell_{[a,b]}, \qquad [\mathbf{d}\ell_a, q_*^*\theta] = q_*^*(\pounds_{\rho(a)}\theta), \qquad [q_*^*\theta, q_*^*\theta] = 0,$$
$$\rho_{T^*A^*}(\mathbf{d}\ell_a) = \widehat{\pounds_a} \in \mathfrak{X}(A^*), \qquad \rho_{T^*A^*}(q_*^*\theta) = (-\rho^t\theta)^\uparrow \in \mathfrak{X}(A^*)$$

for $a, b \in \Gamma(A)$ and $\theta, \theta \in \Omega^1(M)$. As a consequence, we find that the Lie algebroid structure on $T^*A \to A^*$ is given by

$$[a_1^R, a_2^R] = [a_1, a_2]^R, \qquad [a_1^R, \theta^\dagger] = (\pounds_{\rho(a_1)}\theta)^\dagger, \qquad [\theta_1^\dagger, \theta_2^\dagger] = 0,$$
$$\rho_{T^*A}(a^R) = \widehat{\pounds_a} \in \mathfrak{X}(A^*), \qquad \rho_{T^*A}(\theta^\dagger) = (\rho^t\theta)^\uparrow \in \mathfrak{X}(A^*)$$

for $a_1, a_2 \in \Gamma(A)$ and $\theta_1, \theta_2 \in \Omega^1(M)$.

The fibered product $TA \times_A T^*A \to TM \times_M A^*$. The Lie algebroid $TA \oplus T^*A \to TM \oplus A^*$ is defined as the pullback to the diagonals $\Delta_A \to \Delta_M$ of the Lie algebroid $TA \times T^*A \to TM \times A^*$. We have the special sections

$$a^l := (Ta, a^R) \colon TM \oplus A^* \to TA \oplus T^*A$$

for $a \in \Gamma(A)$ and

$$(a, \theta)^\dagger := (a^\dagger, \theta^\dagger) \colon TM \oplus A^* \to TA \oplus T^*A$$

for $(a, \theta) \in \Gamma(A \oplus T^*M)$. The set of sections of $TA \oplus T^*A \to TM \oplus A^*$ is spanned as a $C^\infty(TM \oplus A^*)$-module by these two types of sections. We write $\pi \colon TM \oplus A^* \to M$ for the projection and $\Theta \colon TA \oplus T^*A \to T(TM \oplus A^*)$ for the anchor of $TA \oplus T^*A \to TM \oplus A^*$.

Proposition C.1. *The Lie algebroid $(TA \oplus T^*A, \Theta, [\cdot\,,\cdot])$ is described by the following identities*

$$[a_1^l, a_2^l] = [a_1, a_2]^l, \qquad [a^l, \tau^\dagger] = (\pounds_a\tau)^\dagger, \qquad [\tau_1^\dagger, \tau_2^\dagger] = 0$$
$$[a^l, \widetilde{\phi}] = \widetilde{\pounds_a\phi}, \qquad [\tau^\dagger, \widetilde{\phi}] = \phi(\widetilde{(\rho, \rho^t)}\tau), \qquad [\widetilde{\phi}, \widetilde{\psi}] = \psi \circ \widetilde{(\rho, \rho^t)} \circ \phi - \phi \circ \widetilde{(\rho, \rho^t)} \circ \psi,$$
$$\Theta(a^l) = \widehat{\pounds_a}, \qquad \Theta(\tau^\dagger) = ((\rho, \rho^t)\tau)^\uparrow, \qquad \Theta(\widetilde{\phi}) = \widetilde{(\rho, \rho^t) \circ \phi}$$

for $a, b \in \Gamma(A)$, $\sigma, \tau \in \Gamma(A \oplus T^*M)$ and $\phi, \psi \in \Gamma(\mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$.

Proof. We start by computing the anchor. Note first that for all $f \in C^\infty(M)$ and $\tau' = (a', \theta') \in \Gamma(A \oplus T^*M)$, we have $\pi^* f = \mathrm{pr}_{TM}^* p_M^* f = \mathrm{pr}_{A^*}^* q_*^* f$ and $\ell_{\tau'} = \mathrm{pr}_{TM}^* \ell_{\theta'} + \mathrm{pr}_{A^*}^* \ell_{a'}$. Thus, we get:

$$\begin{aligned}
\Theta(a^l)(\ell_{\tau'}) &= (\rho_{TA} \circ Ta, \rho_{T^*A} \circ a^R)(\mathrm{pr}_{TM}^* \ell_{\theta'} + \mathrm{pr}_{A^*}^* \ell_{a'}) \\
&= \mathrm{pr}_{TM}^*(\rho_{TA} \circ Ta)(\ell_{\theta'}) + \mathrm{pr}_{A^*}^*(\rho_{T^*A} \circ a^R)(\ell_{a'}) \\
&= \mathrm{pr}_{TM}^* \ell_{\pounds_{\rho(a)}\theta'} + \mathrm{pr}_{A^*}^* \ell_{[a,a']} = \ell_{\pounds_a\tau'}, \\
\Theta(a^l)(\pi^* f) &= (\rho_{TA} \circ Ta, \rho_{T^*A} \circ a^R)(\mathrm{pr}_{TM}^* p_M^* f) = \mathrm{pr}_{TM}^* p_M^*(\rho(a)(f)) = \pi^*(\rho(a)(f)), \\
\Theta(\tau^\dagger)(\ell_{\tau'}) &= (\rho_{TA} \circ a^\dagger, \rho_{T^*A} \circ \theta^\dagger)(\mathrm{pr}_{TM}^* \ell_{\theta'} + \mathrm{pr}_{A^*}^* \ell_{a'}) \\
&= \mathrm{pr}_{TM}^*(\rho_{TA} \circ a^\dagger)(\ell_{\theta'}) + \mathrm{pr}_{A^*}^*(\rho_{T^*A} \circ \theta^\dagger)(\ell_{a'}) \\
&= \mathrm{pr}_{TM}^* p_M^* \langle \theta', \rho(a) \rangle + \mathrm{pr}_{A^*}^* q_*^* \langle \theta, \rho(a') \rangle = \pi^* \langle (\rho, \rho^t)\tau, \tau' \rangle, \\
\Theta(\tau^\dagger)(\pi^* f) &= (\rho_{TA} \circ a^\dagger, \rho_{T^*A} \circ \theta^\dagger)(\mathrm{pr}_{TM}^* p_M^* f) = 0.
\end{aligned}$$

For the last equality, note that a section $\phi \in \Gamma(\mathrm{Hom}(TM \oplus A^*, A \oplus T^*M))$ can be written as a sum $\phi = \sum_i \ell_{\chi_i} \cdot \tau_i$ with $\chi_i, \tau_i \in \Gamma(A \oplus T^*M)$. The corresponding section $\widetilde{\phi} \in \Gamma_{TM \oplus A^*}(TA \oplus T^*A)$ is then given by $\widetilde{\phi} = \sum_i \ell_{\chi_i} \cdot \tau_i^\dagger$ and we get

$$\Theta(\widetilde{\phi}) = \sum_i \ell_{\chi_i} \cdot \Theta(\tau_i^\dagger) = \sum_i \ell_{\chi_i} \cdot ((\rho, \rho^t)\tau_i)^\uparrow = \widetilde{(\rho, \rho^t)} \circ \phi.$$

Next we compute the Lie algebroid brackets. For $a_1, a_2 \in \Gamma(A)$ and $\theta_1, \theta_2 \in \Omega^1(M)$, we have $[(Ta_1, {a_1}^R), (Ta_2, {a_2}^R)] = (T[a_1, a_2], [a_1, a_2]^R)$ by the considerations in the previous sections. In the same manner, we show the next two identities: $[a^l, (a', \theta')^\dagger] = ([a, a'], \pounds_{\rho(a)}\theta')^\dagger$ and $[(a_1, \theta_1)^\dagger, (a_2, \theta_2)^\dagger] = (0, 0)$. For the last three brackets we assume without loss of generality that $\phi = \ell_{\chi_1} \cdot \tau_1$ and $\psi = \ell_{\chi_2} \cdot \tau_2$ with $\chi_1, \chi_2, \tau_1, \tau_2 \in \Gamma(A \oplus T^*M)$. Then

$$[a^l, \widetilde{\phi}] = \left[a^l, \ell_{\chi_1} \cdot \tau_1^\dagger\right] = \ell_{\pounds_a \chi_1} \cdot \tau_1^\dagger + \ell_{\chi_1} \cdot (\pounds_a \tau_1)^\dagger = \widetilde{\pounds_a \phi},$$

since

$$\begin{aligned}(\pounds_a \phi)(\nu) &= \pounds_a(\phi(\nu)) - \phi(\pounds_a(\nu)) = \pounds_a(\langle \chi_1, \nu \rangle \cdot \tau_1) - \langle \chi_1, \pounds_a(\nu) \rangle \cdot \tau_1 \\ &= \langle \pounds_a \chi_1, \nu \rangle \cdot \tau_1 + \langle \chi_1, \nu \rangle \cdot \pounds_a \tau_1\end{aligned}$$

for $\nu \in \Gamma(TM \oplus A^*)$, and

$$\begin{aligned}[\tau^\dagger, \widetilde{\phi}] &= \left[\tau^\dagger, \ell_{\chi_1} \cdot {\tau_1}^\dagger\right] = \pi^* \langle (\rho, \rho^t)\tau, \chi_1 \rangle \cdot {\tau_1}^\dagger = \phi((\rho, \rho^t)\tau)^\dagger \\ [\widetilde{\phi}, \widetilde{\psi}] &= \left[\ell_{\chi_1} \cdot {\tau_1}^\dagger, \ell_{\chi_2} \cdot {\tau_2}^\dagger\right] = \ell_{\chi_1} \cdot \psi((\rho, \rho^t)\tau_1)^\dagger - \ell_{\chi_2} \cdot \phi((\rho, \rho^t)\tau_2)^\dagger \\ &= \psi \circ \widetilde{(\rho, \rho^t)} \circ \phi - \phi \circ \widetilde{(\rho, \rho^t)} \circ \psi.\end{aligned}$$

□

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