Dirac Lie Groups, Dirac Homogeneous Spaces and the Theorem of Drinfeld

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ABSTRACT. The notions of *Poisson Lie group* and *Poisson homogeneous space* are extended to the Dirac category. The theorem of Drinfeld on the one-to-one correspondence between Poisson homogeneous spaces of a Poisson Lie group and a special class of Lagrangian subalgebras of the Lie bialgebra associated to the Poisson Lie group is proved to hold in this more general setting.

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1. INTRODUCTION

A *Poisson Lie group* is a Lie group endowed with a Poisson structure that is compatible with the Lie group structure. Poisson Lie groups were introduced by Drinfeld (1983) and studied by Semenov-Tian-Shansky (1985). Their aim was to understand the Hamiltonian structure of the group of dressing transformations of a completely integrable system. The study of the geometry of Poisson Lie groups was started with the works of Lu and Weinstein (see Lu (1990), Lu and Weinstein (1990), Lu and Weinstein (1989) among others). The notion of Poisson Lie group was generalized to the notion of Poisson Lie groupoids by Weinstein (1988).

A *Poisson homogeneous space* of a Poisson Lie group is a homogeneous space of the Lie group that is endowed with a Poisson structure such that the left action of the Lie group on the homogeneous space is a Poisson map. Poisson homogeneous spaces of Poisson Lie groups are in correspondence with suitable subspaces of the direct sum of the Lie algebra with its dual. We show that this correspondence result fits in a more general and natural context: the one of Dirac manifolds, which are objects generalizing in a sense the Poisson manifolds.

Let G be a Lie group endowed with a bivector field $\pi_G \in \Gamma(\bigwedge^2 TM)$. The bivector field π_G is *multiplicative*, and (G, π_G) is a *Poisson Lie group*, if π_G satisfies

$$\pi_G(gh) = T_g R_h \pi_G(g) + T_h L_g \pi_G(h)$$

for all $g, h \in G$, where R_h is the right multiplication by h on G and L_g is the left multiplication by g. In other words, the group multiplication has to be compatible with the Poisson structure in the sense that the multiplication map

$$m : G \times G \to G$$

is a Poisson map if $G \times G$ is endowed with the product Poisson structure defined by π_G. Equivalently, the graph $\operatorname{Graph}(\pi_G^\sharp) \subseteq TG \oplus T^*G$ of the vector bundle homomorphism $\pi_G^\sharp : T^*G \to TG$, $\pi_G^\sharp(\mathbf{d}f) = \pi_G(\cdot, \mathbf{d}f)$ associated to π_G is a subgroupoid of the Pontryagin groupoid $TG \oplus T^*G \rightrightarrows \mathfrak{g}^*$ defined by G. More generally, a Poisson Lie groupoid is a Lie groupoid $G \rightrightarrows P$ endowed with a Poisson structure π_G such that $\operatorname{Graph}(\pi_G^\sharp)$ is a subgroupoid of the Pontryagin groupoid $TG \oplus T^*G \rightrightarrows TP \oplus A^*G$, where A^*G is the dual of the Lie algebroid $AG \to P$ associated to the Lie groupoid $G \rightrightarrows P$ (see Coste et al. (1987), Pradines (1988), Mackenzie (2005) for the induced Lie groupoid structures $TG \rightrightarrows TP$ and $T^*G \rightrightarrows A^*G$).

A Poisson Lie group (G, π_G) induces a Lie algebra structure on the direct sum of the Lie algebra $\mathfrak{g}$ of G with its dual $\mathfrak{g}^*$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ integrates to a natural action of G on $\mathfrak{g} \oplus \mathfrak{g}^*$, see for example Lu (1990). The generalization of this to Poisson Lie groupoids was done in Mackenzie and Xu (1994): the direct sum $AG \oplus A^*G$ of the Lie algebroid and its dual inherits the structure of a *Lie bialgebroid*, that generalizes the Lie bialgebra of a Poisson Lie group.

A *Poisson homogeneous space* (P, π_P) of a Poisson Lie group (G, π_G) is a homogeneous space P of G endowed with a Poisson structure π_P such that the transitive left action of G on P

$$\sigma : G \times P \to P$$

is a Poisson map, where $G \times P$ is endowed with the product Poisson structure defined by π_G and π_P (see for example Lu (2008)).

Consider the pairing on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by $\langle (x, \xi), (y, \eta) \rangle = \xi(y) + \eta(x)$ for all $(x, \xi), (y, \eta) \in \mathfrak{g} \oplus \mathfrak{g}^*$. A theorem of Drinfeld (see Drinfeld (1993) and Diatta and Medina (1999), Lu (2008) for more details about the proof, see also Liu et al. (1998)) states that there is a one-to-one correspondence between Poisson structures $\pi_{G/H}$ on G/H such that $(G/H, \pi_{G/H})$ is a Poisson homogeneous space of (G, π_G), and Lagrangian subalgebras $\mathfrak{D}$ of $\mathfrak{g} \oplus \mathfrak{g}^*$ satisfying $\mathfrak{D} \cap (\mathfrak{g} \oplus \{0\}) = \mathfrak{h} \oplus \{0\}$ ($\mathfrak{h}$ being the Lie algebra of the Lie subgroup H) which are invariant under the restriction to H of the action of G on $\mathfrak{g} \oplus \mathfrak{g}^*$.

Because of this theorem, it appears natural to try to pass to the category of Dirac manifolds. A Dirac structure on a manifold M is a subbundle D of its *Pontryagin* bundle $\mathsf{P}_M := TM \oplus T^*M$ that is Lagrangian relative to the natural fiberwise pairing $\langle \cdot, \cdot \rangle$ defined on P_M by

$$\langle (v_m, \alpha_m), (w_m, \beta_m) \rangle = \beta_m(v_m) + \alpha_m(w_m)$$

for all $m \in M$ and $(v_m, \alpha_m), (w_m, \beta_m) \in T_mM \times T_m^*M$. The Dirac manifold (M, D) is *integrable* if certain integrability conditions are satisfied. Dirac manifolds generalize Poisson manifolds in the sense that the graph of the homomorphism of vector bundles $\pi^\sharp : T^*M \to TM$ associated to a Poisson bivector field π on M defines an integrable Dirac structure on the manifold M.

In this paper, we study *Dirac homogeneous spaces* of *Dirac Lie groups* and give a generalization of the theorem of Drinfeld (our main theorem 4.17) in this more natural setting.

Dirac Lie groups have been defined independently by Ortiz (2008). The definition is made there in the context of groupoids and is easily shown to be equivalent to the definition made here. An important feature of a Dirac Lie group (G, D_G) is that the characteristic distribution $\mathsf{G}_0 \subseteq TG$ defined by $\mathsf{G}_0 \oplus \{0\} = \mathsf{D}_G \cap (TM \oplus \{0\})$ and the characteristic codistribution $\mathsf{P}_1 = \mathrm{Proj}_{T*M}(\mathsf{D}_G)$ are always left and right invariant and have thus constant dimensional fibers on G. Hence, integrable multiplicative Dirac structures are only a slight generalization of the graphs of multiplicative Poisson bivector fields. The approach in Ortiz (2008) uses this fact for the definition of the Lie bialgebra of a Dirac Lie group. Here, we formulate everything in the Dirac setting and get the known results such as the definition of the Lie bialgebra of a Poisson Lie group as corollaries in the class of examples given by the Poisson Lie groups.

The reason why we chose this approach is because the situation seems to be quite different in the case of a Dirac Lie groupoid. A Dirac Lie groupoid is a groupoid endowed with a Dirac structure that is a subgroupoid of the Pontryagin groupoid $TG \oplus T^*G \rightrightarrows TP \oplus A^*G$ (see Ortiz (2009)). The characteristic distribution G_0 can be more complicated in this case, and the geometry involved is not necessarily induced by an underlying Poisson Lie groupoid (it is even not necessarily the case if G_0 has constant rank, see Jotz (2010)).

The theorem of Drinfeld has been extended in Liu et al. (1998) to a correspondence between a certain class of Dirac subspaces of the Lie bialgebroid of a Poisson Lie groupoid and its Poisson homogeneous spaces. Our next aim is to generalize this result to Dirac Lie groupoids (Jotz (2010)). For this, we will need to construct the object that will play the role of the Lie bialgebroid in this setting. Here, the results known for Poisson Lie groupoids will be the guidelines, but it will not be possible to use them as it is done in Ortiz (2008) in the particular case of Dirac Lie groups.

Outline of the paper. Backgrounds about Dirac manifolds and actions of Lie groups are recalled in Section 2. The definition of a Dirac Lie group is given in Section 3 and compared with the definition in Ortiz (2008). Geometric properties of Dirac Lie groups are proved and the construction of the Lie bialgebra of a Dirac Lie group is made, as well as the definition of the induced action of G on it.

Dirac homogeneous spaces of Dirac Lie groups are defined in Section 4 and their first geometric properties are proved. Our main theorem about the correspondence between (integrable) Dirac homogeneous spaces of a (integrable) Dirac lie group and Lagrangian subspaces (subalgebras) of $\mathfrak{g} \oplus \mathfrak{g}^*$ is proved there, too.

In Section 5, we study the special class of Dirac Lie groups where the characteristic subgroup N is closed in the Lie group G, and the corresponding Dirac homogeneous spaces.

Notation and conventions. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of smooth local sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $\mathsf{E} \to M$, the sheaf of local sections of E will be written $\Gamma(\mathsf{E})$. We will write $\mathrm{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(\mathsf{E})$ is defined.

We will write $\mathcal{D} \oplus \Delta$ for the direct sum of a subbundle $\mathcal{D}$ of TM and a subbundle Δ of T^*M. We choose this notation to distinguish such a direct sum from a direct sum of subbundles of the same vector bundle. Following Yoshimura and Marsden (2006), we will call the direct sum $\mathsf{P}_M := TM \oplus T^*M$ the *Pontryagin bundle* on the manifold M.

2. GENERALITIES ON DIRAC STRUCTURES

2.1. Dirac structures. The Pontryagin bundle $\mathsf{P}_M = TM \oplus T^*M$ of a smooth manifold M is endowed with a non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by

$$\langle (u_m, \alpha_m), (v_m, \beta_m) \rangle := \langle \beta_m, u_m \rangle + \langle \alpha_m, v_m \rangle \tag{2.1}$$

for all $u_m, v_m \in T_mM$ and $\alpha_m, \beta_m \in T^*_mM$. A *Dirac structure* (see Courant (1990)) on M is a Lagrangian vector subbundle $\mathsf{D} \subset \mathsf{P}_M$. That is, D coincides with its orthogonal relative to (2.1) and so its fibers are necessarily $\dim M$-dimensional. The pair (M, D) is then called a *Dirac manifold*.

Let (M, D) be a Dirac manifold. For each $m \in M$, the Dirac structure D defines two subspaces $\mathsf{G}_0(m), \mathsf{G}_1(m) \subset T_mM$ by

$$\mathsf{G}_0(m) := \{v_m \in T_mM \mid (v_m, 0) \in \mathsf{D}(m)\}$$

and

$$\mathsf{G}_1(m) := \{v_m \in T_mM \mid \exists\, \alpha_m \in T^*_mM : (v_m, \alpha_m) \in \mathsf{D}(m)\},$$

and two subspaces $\mathsf{P}_0(m), \mathsf{P}_1(m) \subset T^*M$ defined analogously. The distributions $\mathsf{G}_0 = \bigcup_{m\in M} \mathsf{G}_0(m)$ and $\mathsf{P}_0 = \bigcup_{m\in M} \mathsf{P}_0(m)$ are not necessarily smooth. The distributions $\mathsf{G}_1 = \bigcup_{m\in M} \mathsf{G}_1(m)$ (respectively $\mathsf{P}_1 = \bigcup_{m\in M} \mathsf{P}_1(m)$) are smooth since they are the projections on TM (respectively T^*M) of D.

We have the equalities

$$\mathsf{P}_0(m) = \mathsf{G}_1(m)^\circ, \quad \mathsf{G}_0(m) = \mathsf{P}_1(m)^\circ, \quad \mathsf{P}_1(m) = \mathsf{G}_0(m)^\circ, \quad \mathsf{G}_1(m) \subseteq \mathsf{P}_0(m)^\circ.$$

The space $\Gamma(\mathsf{P}_M)$ of local sections of the Pontryagin bundle is endowed with a skew-symmetric bracket given by (see Courant (1990))

$$\begin{aligned}(2.2)\qquad [(X,\alpha),(Y,\beta)] :&= \left([X,Y], \boldsymbol{\pounds}_X\beta - \boldsymbol{\pounds}_Y\alpha + \frac{1}{2}\mathbf{d}(\alpha(Y) - \beta(X))\right)\\ &= \left([X,Y], \boldsymbol{\pounds}_X\beta - \mathbf{i}_Y\mathbf{d}\alpha - \frac{1}{2}\mathbf{d}\langle (X,\alpha),(Y,\beta)\rangle\right)\end{aligned}$$

This bracket is $\mathbb{R}$-bilinear (in the sense that $[a_1(X_1,\alpha_1) + a_2(X_2,\alpha_2),(Y,\beta)] = a_1[(X_1,\alpha_1),(Y,\beta)] + a_2[(X_2,\alpha_2),(Y,\beta)]$ for all $a_1, a_2 \in \mathbb{R}$ and (X_1,α_1), $(X_2,\alpha_2), (Y,\beta) \in \Gamma(\mathsf{P}_M)$ on the common domain of definition of the three sections) and does not in general satisfy the Jacobi identity.

The Dirac structure D is *integrable* if $[\Gamma(\mathsf{D}), \Gamma(\mathsf{D})] \subset \Gamma(\mathsf{D})$. Due to the fact that $\langle (X,\alpha),(Y,\beta)\rangle = 0$ if $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{D})$, integrability of the Dirac structure is expressed relative to a non-skew-symmetric bracket that differs from (2.2) by eliminating in the second line the third term of the second component. This truncated expression is called the *Courant-Dorfman* or *Dorfman bracket* (Dorfman (1993)):

$$(2.3)\qquad [(X,\alpha),(Y,\beta)] := ([X,Y], \boldsymbol{\pounds}_X\beta - \mathbf{i}_Y\mathbf{d}\alpha).$$

The restriction of the Courant bracket to the sections of an integrable Dirac structure is skew-symmetric and satisfies the Jacobi identity. It satisfies also the Leibnitz-rule:

$$(2.4)\qquad [(X,\alpha), f(Y,\beta)] = f[(X,\alpha),(Y,\beta)] + X(f)\cdot(Y,\beta)$$

for all $(X,\alpha),(Y,\beta)\in\Gamma(\mathsf{D})$ and $f\in C^\infty(M)$.

The Dirac manifold (M,D) is integrable if and only if the *tensor* T_{D} defined on sections $(X,\alpha),(Y,\beta),(Z,\gamma)$ of D by

$$T_{\mathsf{D}}((X,\alpha),(Y,\beta),(Z,\gamma)) = \langle [(X,\alpha),(Y,\beta)],(Z,\gamma)\rangle$$

vanishes identically on M (see Courant (1990)).

The class of Dirac structures presented in the next example will be very important in the following.

Example 2.1. Let M be a smooth manifold endowed with a globally defined bivector field $\pi\in\Gamma(\bigwedge^2 TM)$. Then the subdistribution $\mathsf{D}_\pi\subseteq\mathsf{P}_M$ defined by

$$\mathsf{D}_\pi(m) = \{(\pi^\sharp(\alpha),\alpha)(m) \mid \alpha\in\Omega^1(M), \mathrm{Dom}(\alpha)\ni m\} \quad \text{for all } m\in M,$$

where $\pi^\sharp : T^*M\to TM$ is defined by $\pi^\sharp(\alpha)=\pi(\alpha,\cdot)\in\mathfrak{X}(M)$ for all $\alpha\in\Omega^1(M)$, is a Dirac structure on M. It is integrable if and only if the bivector field satisfies $[\pi,\pi]=0$, that is, if and only if (M,π) is a Poisson manifold.

Note that for this class of Dirac manifolds, T_{D_π} is a 3-tensor on T^*M since each section of $\mathsf{P}_1 = T^*M$ corresponds to exactly one section of D_π. The equality $2\cdot T_{\mathsf{D}_\pi} = [\pi,\pi]$ (compare (1.82) in Dufour and Zung (2005) with Proposition 2.5.3 in Courant (1990)), where $[\cdot,\cdot]$ is the Schouten bracket, shows that π is a Poisson bivector if and only if D_π is integrable.

The product of two Dirac manifolds. Let (M,D_M) and (N,D_N) be Dirac manifolds. Consider the product $M\times N$. We identify in the following always (without mentioning it) the tangent space $T(M\times N)$ with $TM\oplus TN$, and write (v_p,w_q) for the elements of $T_{(p,q)}(M\times N) = T_pM\oplus T_qN$. That is, an element of $\mathfrak{X}(M\times N)$ is written (X,Y) with $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$. We identify in the same manner $T^*(M\times N)$ with $T^*M\oplus T^*N$.

The *product Dirac structure* $\mathsf{D}_M\oplus\mathsf{D}_N$ on $M\times N$ is the direct sum of D_M and D_N: the pair $((X,Y),(\alpha,\beta))$ is a section of $\mathsf{D}_M\oplus\mathsf{D}_N$ if and only if $(X,\alpha)\in\Gamma(\mathsf{D}_M)$ and $(Y,\beta)\in\Gamma(\mathsf{D}_N)$.

The Dirac manifold $(M\times N,\mathsf{D}_M\oplus\mathsf{D}_N)$ is integrable if and only if (M,D_M) and (N,D_N) are integrable.

Maps in the Dirac category. Let (M,D_M) and (N,D_N) be smooth Dirac manifolds and $\varphi: M\to N$ a smooth map. The map φ is said to be *backward Dirac* if for all $(X,\alpha)\in\Gamma(\mathsf{D}_M)$ there exists $(Y,\beta)\in\Gamma(\mathsf{D}_N)$ such that

$$X\sim_\varphi Y \quad\text{and}\quad \alpha=\varphi^*\beta.$$

The map φ is said to be *forward Dirac* if for all $(Y,\beta)\in\Gamma(\mathsf{D}_N)$ there exists $(X,\alpha)\in\Gamma(\mathsf{D}_M)$ such that

$$X\sim_\varphi Y \quad\text{and}\quad \alpha=\varphi^*\beta.$$

Symmetries of a Dirac manifold (M, D)**.** Let G be a Lie group and $\Phi : G \times M \to M$ a smooth left action. Then G is called a *symmetry Lie group of* (M, D) if for every $g \in G$ the condition $(X, \alpha) \in \Gamma(\mathsf{D})$ implies that $(\Phi_g^* X, \Phi_g^* \alpha) \in \Gamma(\mathsf{D})$. We say then that the Lie group G acts *canonically* or *by Dirac actions* on M.

Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g} \mapsto x_M \in \mathfrak{X}(M)$ be a smooth left Lie algebra action, that is, the map $(m, x) \in M \times \mathfrak{g} \mapsto x_M(m) \in TM$ is smooth and $x \in \mathfrak{g} \mapsto x_M \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism. The Lie algebra $\mathfrak{g}$ is said to be a *symmetry Lie algebra of* (M, D) if for every $x \in \mathfrak{g}$ the condition $(X, \alpha) \in \Gamma(\mathsf{D})$ implies that $(\pounds_{x_M} X, \pounds_{x_M} \alpha) \in \Gamma(\mathsf{D})$. Of course, if $\mathfrak{g}$ is the Lie algebra of G and $x \mapsto x_M$ the infinitesimal action map induced by the G-action on M, then if G is a symmetry Lie group of D it follows that $\mathfrak{g}$ is a symmetry Lie algebra of D.

Regular reduction of Dirac structures by Lie group actions. Assume that we have a canonical free and proper G-action on the Dirac manifold (M, D). Let $\mathcal{V}$ be the vertical space of the action, that is, the vector subbundle of TM spanned by the fundamental vector fields x_M, $x \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. Set $\mathcal{K} := \mathcal{V} \oplus \{0\} \subseteq \mathsf{P}_M$ and consider its smooth orthogonal $\mathcal{K}^\perp \subseteq \mathsf{P}_M$ relative to the bracket $< \cdot, \cdot >$ on P_M. Then both vector bundles D and $\mathcal{K}^\perp$ are G-invariant and it is shown in Jotz and Ratiu (2011) following Bursztyn et al. (2007) that, under the assumption that $\mathsf{D} \cap \mathcal{K}^\perp$ is a vector bundle on M, the "quotient"

$$\mathsf{D}_{M/G} = \frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}} / G \tag{2.5}$$

defines a Dirac structure on M/G, called the *reduced* Dirac structure. The formulation of this in terms of smooth sections is the following: the reduced Dirac structure on M/G is given by

$$\mathsf{D}_{M/G} = \operatorname{span}\left\{ (\bar{X}, \bar{\alpha}) \in \Gamma(\mathsf{P}_{M/G}) \,\middle|\, \begin{array}{c} \exists\, X \in \mathfrak{X}(M) \text{ such that} \\ X \sim_q \bar{X} \text{ and } (X, q^*\bar{\alpha}) \in \Gamma(\mathsf{D}) \end{array} \right\}, \tag{2.6}$$

where $q : M \to M/G$ is the quotient map (refer to Blankenstein (2000), Blankenstein and van der Schaft (2001)).

The Dirac structure $\mathsf{D}_{M/G}$ is then the *forward Dirac image* $q(\mathsf{D})$ *of* D *under* q. Note that the *pullback Dirac structure* or *backward Dirac image* $q^*\mathsf{D}_{M/G}$ *of a Dirac bundle* $\mathsf{D}_{M/G} \subseteq \mathsf{P}_{M/G}$ *under* q, defined on M by

$$\Gamma(q^*\mathsf{D}_{M/G}) = \left\{ (X, \alpha) \in \Gamma(\mathsf{P}_M) \,\middle|\, \begin{array}{c} \exists\, (\bar{X}, \bar{\alpha}) \in \Gamma(\mathsf{D}_{M/G}) \text{ such that} \\ X \sim_q \bar{X},\ \alpha = q^*\bar{\alpha} \end{array} \right\},$$

is a Dirac structure on M. It is easy to check that D and $q^*(q(\mathsf{D}))$ are equal if and only if we have the inclusion $\mathcal{V} \subseteq \mathsf{G}_0$. The equality $q(q^*\mathsf{D}_{M/G})) = \mathsf{D}_{M/G}$ holds for any Dirac structure $\mathsf{D}_{M/G}$ on M/G.

2.2. Invariant Dirac structures on a Lie group.

Definition 2.2. A Dirac structure $\mathsf{D} \subseteq TG \oplus TG^*$ on a Lie group G is called *left invariant* if it is invariant under the action of G on $TG \oplus TG^*$ induced from the left action of G on itself. That is, if for all $g, h \in G$ and all $(v_g, \alpha_g) \in \mathsf{D}(g)$ we have $(T_g L_h v_g, (T_{hg} L_{h^{-1}})^* \alpha_g) \in \mathsf{D}(hg)$.

In the same manner, a Dirac structure D on a Lie group G is called *right invariant* if it is invariant under the action on $TG \oplus TG^*$ induced from the right action of G on itself.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$, that is, $\mathfrak{D}$ is a vector subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$ that is orthogonal to itself relative to the pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ defined on $\mathfrak{g} \oplus \mathfrak{g}^*$ by $\langle (x, \xi), (y, \eta) \rangle_{\mathfrak{g}} = \eta(x) + \xi(y)$ for all $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. We set

$$\mathfrak{g}_0 := \{x \in \mathfrak{g} \mid (x, 0) \in \mathfrak{D}\}, \qquad \mathfrak{g}_1 := \{x \in \mathfrak{g} \mid \exists\, \xi \in \mathfrak{g}^* : (x, \xi) \in \mathfrak{D}\},$$
$$\mathfrak{p}_0 := \{\xi \in \mathfrak{g}^* \mid (0, \xi) \in \mathfrak{D}\}, \qquad \mathfrak{p}_1 := \{\xi \in \mathfrak{g}^* \mid \exists\, x \in \mathfrak{g} : (x, \xi) \in \mathfrak{D}\}.$$

Then we have $\mathfrak{g}_0^\circ = \mathfrak{p}_1$, $\mathfrak{p}_1^\circ = \mathfrak{g}_0$, $\mathfrak{g}_1^\circ = \mathfrak{p}_0$, and $\mathfrak{p}_0^\circ = \mathfrak{g}_1$.

Let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$, and define $\mathfrak{D}^L$ on G by

$$\mathfrak{D}^L(g) = \{(T_e L_g x, (T_g L_{g^{-1}})^* \xi) \mid (x, \xi) \in \mathfrak{D}\}$$

for all $g \in G$. Then $\mathfrak{D}^L$ is a left invariant Dirac structure on G. Conversely, if D is a left-invariant Dirac structure on a Lie group G, then $\mathsf{D} = \mathfrak{D}^L$, where $\mathfrak{D} := \mathsf{D}(e) \subseteq \mathfrak{g} \oplus \mathfrak{g}^*$.

The next proposition shows that the integrability of $\mathfrak{D}^L$ depends only on $\mathfrak{D}$ (see also Miburn (2007)).

Proposition 2.3. *The Dirac structure $\mathfrak{D}^L$ is integrable if and only if $\zeta([x, y]) + \xi([y, z]) + \eta([z, x]) = 0$ for all pairs (x, ξ), (y, η), and $(z, \zeta) \in \mathfrak{D}$.*

Proof. Recall that $\mathfrak{D}^L$ is integrable if for all sections $(X, \alpha), (Y, \beta) \in \Gamma(\mathfrak{D}^L)$, we have $[(X, \alpha), (Y, \beta)] = ([X, Y], \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha) \in \Gamma(\mathfrak{D}^L)$.

By (2.4), it suffices to show this for a set of spanning sections of $\mathfrak{D}^L$. For $(x, \xi) \in \mathfrak{D}$, the left invariant pair (x^L, ξ^L), which is defined by $(x^L(g), \xi^L(g)) = (T_e L_g x, (T_g L_{g^{-1}})^* \xi)$ for all $g \in G$, is a section of $\mathfrak{D}^L$. Choose (x, ξ), (y, η) and $(z, \zeta) \in \mathfrak{D}$. Then we have $[x^L, y^L] = [x, y]^L$ by definition of the Lie bracket in $\mathfrak{g}$, $\pounds_{x^L} \eta^L = (\mathrm{ad}_x^* \eta)^L$, $\mathbf{i}_{y^L} \mathbf{d} \xi^L = (\mathrm{ad}_y^* \xi)^L$, where for $\xi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, the element $\mathrm{ad}_x^* \xi \in \mathfrak{g}^*$ is defined by $\mathrm{ad}_x^* \xi(y) = \xi([y, x])$ for all $y \in \mathfrak{g}$. We get

$$\langle ([x, y]^L, \pounds_{x^L} \eta^L - \mathbf{i}_{y^L} \mathbf{d} \xi^L), (z^L, \zeta^L) \rangle = \zeta([x, y]) + \eta([z, x]) + \xi([y, z]).$$

Hence, since the sections (z^L, ζ^L), for all $(z, \zeta) \in \mathfrak{D}$, are spanning sections for $\mathfrak{D}^L$, we conclude that $[(x^L, \xi^L), (y^L, \eta^L)] = ([x, y]^L, \mathbf{i}_{x^L} \mathbf{d} \eta^L - \mathbf{i}_{y^L} \mathbf{d} \xi^L)$ is a section of $\mathfrak{D}^L$ if and only if $\zeta([x, y]) + \eta([z, x]) + \xi([y, z]) = 0$ for all $(z, \zeta) \in \mathfrak{D}$. ∎

Note that we have shown simultaneously that if $\mathfrak{D}^L$ is integrable, then $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are Lie subalgebras of $\mathfrak{g}$. Since G_0 and G_1 are here obviously equal to $\mathfrak{g}_0^L$ and $\mathfrak{g}_1^L$, respectively, we recover the fact that both distributions are then integrable.

3. Dirac Lie Groups

3.1. Definitions.

Definition 3.1. A *Dirac Lie group* is a Lie group G endowed with a Dirac structure $\mathsf{D}_G \subseteq TG \oplus T^*G$ such that the group multiplication map

$$m : (G \times G, \mathsf{D}_G \oplus \mathsf{D}_G) \to (G, \mathsf{D}_G)$$

is a *forward Dirac map*. The Dirac structure D_G on G is called *multiplicative* if it satisfies this condition.

More explicitly, there exist for all $g, h \in G$ and pairs (v_{gh}, α_{gh}) in $\mathsf{D}_G(gh)$, two pairs $(w_g, \beta_g) \in \mathsf{D}_G(g)$ and $(u_h, \gamma_h) \in \mathsf{D}_G(h)$ such that

$$T_{(g,h)} m(w_g, u_h) = v_{gh} \quad \text{and} \quad (\beta_g, \gamma_h) = (T_{(g,h)} m)^* \alpha_{gh}.$$

That is, we have

$$T_h L_g u_h + T_g R_h w_g = v_{gh} \in T_{gh} G \quad \text{and} \quad (\beta_g, \gamma_h) = \alpha_{gh} \circ T_{(g,h)} m,$$

which is equivalent to the following: for all $(x_g, y_h) \in T_{(g,h)}(G \times G)$, we have

$$\beta_g(x_g) + \gamma_h(y_h) = (\beta_g, \gamma_h)(x_g, y_h) = \alpha_{gh}(T_h L_g y_h + T_g R_h x_g).$$

Hence, we have in particular $\beta_g = (T_g R_h)^* \alpha_{gh}$ and $\gamma_h = (T_h L_g)^* \alpha_{gh}$.

The next example shows that Dirac Lie groups generalize Poisson Lie groups to the category of Dirac manifolds.

Example 3.2. Let (G, π) be a Lie group endowed with a bivector field $\pi \in \Gamma(\bigwedge^2 TM)$, and let (G, D_π) be the associated Dirac structure as in Example 2.1. We show that π is multiplicative (see Lu and Weinstein (1990)) if and only if (G, D_π) is a Dirac Lie group. The bivector field π is multiplicative if and only if $\pi(gh) = T_h L_g(\pi(h)) + T_g R_h(\pi(g))$ for all $g, h \in G$. For $f \in C^\infty(G)$, we define $X_f := \pi(\cdot, \mathbf{d}f) = -\pi^\sharp(\mathbf{d}f) \in \mathfrak{X}(G)$. Then the pair $(-X_f, \mathbf{d}f)$ is a section of D_π. Choose $(-X_f(gh), \mathbf{d}f_{gh}) \in \mathsf{D}_\pi(gh)$ and consider $\mathbf{d}(R_h^* f)_g \in T_g G^*$ and $\mathbf{d}(L_g^* f)_h \in T_h G^*$. Then we have $(-X_{R_h^* f}(g), \mathbf{d}(R_h^* f)_g) \in \mathsf{D}_\pi(g)$ and $(-X_{L_g^* f}(h), \mathbf{d}(L_g^* f)_h) \in \mathsf{D}_\pi(h)$. This yields for an arbitrary α_{gh} in $T_{gh}^* G$:

$$\begin{aligned}
&\alpha_{gh}(T_g R_h X_{R_h^* f}(g) + T_h L_g X_{L_g^* f}(h)) \\
&\qquad = \pi(g)(\alpha_{gh} \circ T_g R_h, (R_h^* \mathbf{d}f)_g) + \pi(h)(\alpha_{gh} \circ T_h L_g, (L_g^* \mathbf{d}f)_h) \\
&\qquad = (T_g R_h \pi(g))(\alpha_{gh}, \mathbf{d}f_{gh}) + (T_h L_g \pi(h))(\alpha_{gh}, \mathbf{d}f_{gh}).
\end{aligned}$$

The last sum is equal to $\pi(gh)(\alpha_{gh}, \mathbf{d}f_{gh}) = \alpha_{gh}(X_f(gh))$ for all $f \in C^\infty(G)$ if and only if π is multiplicative. Hence, the equality

$$T_gR_h(-X_{R_h^*f}(g)) + T_hL_g(-X_{L_g^*f}(h)) = -X_f(gh)$$

holds for all $f \in C^\infty(G)$ if and only if π is multiplicative. Since $(-X_f(gh), \mathbf{d}f_{gh})$ was an arbitrary element of $\mathsf{D}_\pi(gh)$, we have shown that (G, D_π) is a Dirac Lie group if and only if (G, π) is multiplicative. Thus, (G, D_π) is an integrable Dirac Lie group if and only if (G, π) is a Poisson Lie group.

Note that the Dirac structure associated to the trivial Poisson structure on G is given by $\mathsf{D}_0 = \{0\} \oplus TG^*$. Since a Lie group endowed with the trivial Poisson structure $\pi = 0$ on G is always a Poisson Lie group, this shows that $(G, \{0\} \oplus TG^*)$ is a (integrable) Dirac Lie group. This can also be checked directly from the definition.

The Dirac Lie group as a subgroupoid of the Pontryagin bundle. Another approach to Dirac Lie groups can be found in Ortiz (2008). For the sake of completeness, we show that both definitions are equivalent. For this, we have to introduce the groupoid structure on the Pontryagin bundle of a Lie group.

Let G be a Lie group. Then its Pontryagin bundle P_G has the structure of a Lie groupoid over $\mathfrak{g}^*$ as follows. The *target* and *source* maps t and s are defined by

$$\begin{aligned} \mathsf{t}: \quad & TG \oplus TG^* \to \mathfrak{g}^*, \\ & (v_g, \alpha_g) \in T_gG \times T_gG^* \mapsto (T_eR_g)^*\alpha_g, \end{aligned}$$

and

$$\begin{aligned} \mathsf{s}: \quad & TG \oplus TG^* \to \mathfrak{g}^*, \\ & (v_g, \alpha_g) \in T_gG \times T_gG^* \mapsto (T_eL_g)^*\alpha_g. \end{aligned}$$

If $\mathsf{s}(v_g, \alpha_g) = \mathsf{t}(w_h, \beta_h)$, then the product $(v_g, \alpha_g) \star (w_h, \beta_h)$ makes sense and is equal to

$$\begin{aligned} (v_g, \alpha_g) \star (w_h, \beta_h) &= (T_gR_hv_g + T_hL_gw_h, (T_{gh}R_{h^{-1}})^*\alpha_g) \\ &= (T_gR_hv_g + T_hL_gw_h, (T_{gh}L_{g^{-1}})^*\beta_h). \end{aligned}$$

The identity map $\mathsf{u} : \mathfrak{g}^* \to \mathsf{P}_G$ is given by $\mathsf{u}(\xi) = (0, \xi) \in \mathfrak{g} \oplus \mathfrak{g}^*$ and the inverse map $\mathsf{i} : \mathsf{P}_G \to \mathsf{P}_G$ is defined by

$$\mathsf{i} : (v_g, \alpha_g) \mapsto \left(-T_g(L_{g^{-1}}R_{g^{-1}})v_g, T_g(L_gR_{g^{-1}})^*\alpha_g\right).$$

Given this definition, it is easy to verify that the graph $\mathsf{D}_\pi \subseteq \mathsf{P}_G$ of a Poisson structure on the Lie group G is multiplicative if and only if D_π is a *subgroupoid* of the Pontryagin groupoid.

In Ortiz (2008), a Dirac Lie group is hence defined as follows: a Dirac structure D_G on a Lie group is called *multiplicative* if D_G is a subgroupoid of the Pontryagin groupoid. The pair (G, D_G) is then called a Dirac Lie group.

It is easy to prove that the two definitions of a multiplicative Dirac structure on G are equivalent.

Remark 3.3. The behavior of the *Cartan Dirac structure* under group multiplication is studied in Alekseev et al. (2009). This interesting example of a Dirac structure on a Lie group is defined on a Lie group G with Lie algebra $\mathfrak{g}$ endowed with a bilinear, symmetric Ad-invariant bilinear form (see Alekseev et al. (2009) and references therein).

3.2. Geometric properties of Dirac Lie groups. In this section and the following, (G, D_G) will always be a Dirac Lie group. We denote by $\mathfrak{g}_1 := \mathsf{G}_1(e)$, $\mathfrak{g}_0 := \mathsf{G}_0(e)$, $\mathfrak{p}_1 := \mathsf{P}_1(e)$ and $\mathfrak{p}_0 := \mathsf{P}_0(e)$ the smooth characteristic distributions evaluated at the neutral element e of G.

The following result has been shown independently by Ortiz (2008).

Proposition 3.4. *Let (G, D_G) be a Dirac Lie group. The associated codistribution (respectively distribution) P_1 (respectively G_0) has constant rank on G, and is given by $\mathsf{P}_1 = \mathfrak{p}_1^L = \mathfrak{p}_1^R$ (respectively $\mathsf{G}_0 = \mathfrak{g}_0^L = \mathfrak{g}_0^R$).*

Proof. We use the definition of Ortiz (2009). If v_g is an element of $\mathsf{G}_0(g)$, we have $(v_g, 0_g) \in \mathsf{D}_G(g)$ and $\mathsf{t}(v_g, 0_g) = \mathsf{s}(v_g, 0_g) = 0 \in \mathfrak{g}^*$. Thus, since $(0_{g^{-1}}, 0_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$, we have $(0_{g^{-1}}, 0_{g^{-1}}) \star (v_g, 0_g) \in \mathsf{D}_G(e)$ and $(v_g, 0_g) \star (0_{g^{-1}}, 0_{g^{-1}}) \in \mathsf{D}_G(e)$. But it is easy to see that $0_{g^{-1}} \star v_g = T_{(g^{-1},g)}m(0_{g^{-1}}, v_g) = T_gL_{g^{-1}}v_g$ and $v_g \star 0_{g^{-1}} = T_gR_{g^{-1}}v_g$, and we get $(T_gL_{g^{-1}}v_g, 0_e) \in \mathsf{D}_G(e)$ and $(T_gR_{g^{-1}}v_g, 0_e) \in \mathsf{D}_G(e)$. We have thus shown that $T_gL_{g^{-1}}\mathsf{G}_0(g) \subseteq \mathfrak{g}_0$ and $T_gR_{g^{-1}}\mathsf{G}_0(g) \subseteq \mathfrak{g}_0$. Conversely, if $x \in \mathfrak{g}_0$, then $(x, 0) \in \mathsf{D}_G(e)$ and $(T_eL_gx, 0_g) = (0_g, 0_g) \star (x, 0) \in \mathsf{D}_G(g)$ and $(T_eR_gx, 0_g) = (x, 0) \star (0_g, 0_g) \in \mathsf{D}_G(g)$. Thus, we have shown the equalities $\mathsf{G}_0(g) = T_eL_g\mathfrak{g}_0 = T_eR_g\mathfrak{g}_0$ and G_0 has constant rank on G.

As a consequence, P_1, which is the annihilator of G_0, has also constant rank on G. The following equality follows easily:

$$\mathsf{P}_1 = \mathsf{G}_0^\circ = (\mathfrak{g}_0^L)^\circ = (\mathfrak{g}_0^\circ)^L = \mathfrak{p}_1^L,$$

and we get in the same manner $\mathsf{P}_1 = \mathfrak{p}_1^R$. □

We have the immediate corollaries:

Corollary 3.5. *The subspaces $\mathfrak{g}_0 \subseteq \mathfrak{g}$ and $\mathfrak{p}_1 \subseteq \mathfrak{g}^*$ satisfy $\mathrm{Ad}_g^* \mathfrak{p}_1 = \mathfrak{p}_1$, $\mathrm{Ad}_g \mathfrak{g}_0 = \mathfrak{g}_0$ for all $g \in G$. Consequently, we have $\mathrm{ad}_x^* \mathfrak{p}_1 \subseteq \mathfrak{p}_1$ for all $x \in \mathfrak{g}$ and $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$.*

Proof. We have $\mathsf{P}_1 = \mathfrak{p}_1^R = \mathfrak{p}_1^L$ and $\mathsf{G}_0 = \mathfrak{g}_0^R = \mathfrak{g}_0^L$ by Proposition 3.4. Then, for all $g \in G$ and $\xi \in \mathfrak{p}_1$, the covector $(T_gL_{g^{-1}})^*\xi$ is an element of $\mathsf{P}_1(g)$ and there exists $\eta \in \mathfrak{p}_1$ such that $(T_gL_{g^{-1}})^*\xi = (T_gR_{g^{-1}})^*\eta$. This yields $\mathrm{Ad}_{g^{-1}}^* \xi = \eta \in \mathfrak{p}_1$

and $\mathfrak{p}_1$ is consequently $\mathrm{Ad}^*_{g^{-1}}$-invariant for all $g \in G$. In the same manner, we show that $\mathfrak{g}_0$ is Ad_g-invariant for all $g \in G$.

This yields by derivation $\mathrm{ad}^*_x \xi \in \mathfrak{p}_1$ for all $\xi \in \mathfrak{p}_1$ and $\mathrm{ad}_x z \in \mathfrak{g}_0$ for all $z \in \mathfrak{g}_0$ and $x \in \mathfrak{g}$. The inclusion $[\mathfrak{g}, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$ holds then and shows that $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$. □

If G is a simple Lie group, the ideal $\mathfrak{g}_0$ is either trivial or equal to $\mathfrak{g}$ and we get the following corollary.

Corollary 3.6. *If (G, D_G) is a simple Dirac Lie group, then the Dirac structure D_G is either the graph of the vector bundle homomorphism $T^*G \to TG$ induced by a multiplicative bivector field on G, or the trivial tangent Dirac structure $\mathsf{D}_G = TG \oplus \{0\}$.*

We have also the following proposition:

Proposition 3.7. *For a Dirac Lie group (G, D_G), we have $\mathsf{D}_G(e) = \mathfrak{g}_0 \oplus \mathfrak{p}_1$. Consequently, the equality $\alpha_e(Y(e)) = 0 = \beta_e(X(e))$ holds for all sections (X, α) and (Y, β) of D_G defined on a neighborhood of the neutral element e.*

Proof. Choose $(x, \xi) \in \mathsf{D}_G(e)$. Then we have $\mathsf{s}(x, \xi) = \xi \in \mathfrak{g}^*$ and hence $\mathsf{u}(\xi) = (0, \xi) = (x, \xi)^{-1} \star (x, \xi) \in \mathsf{D}_G(e)$. Hence, $(x, 0) = (x, \xi) - (0, \xi)$ is also an element of $\mathsf{D}_G(e)$ and $x \in \mathfrak{g}_0$. This shows that $\mathsf{D}_G(e) \subseteq \mathfrak{g}_0 \times \mathfrak{p}_1$ and also $\mathfrak{p}_1 = \mathfrak{p}_0$. Because of this last equality, the inclusion $\mathfrak{g}_0 \times \mathfrak{p}_1 \subseteq \mathsf{D}_G(e)$ is obvious. □

Lemma 3.8. *The subbundle $\mathsf{G}_0 \subseteq TG$ is involutive and hence integrable, and P_1 is spanned by exact one-forms.*

Proof. Recall first that $\mathfrak{p}_1^L = \mathsf{P}_1$ and $\mathfrak{g}_0^L = \mathsf{G}_0$ by Proposition 3.4, and that both distributions have hence constant rank on G. Since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$, we have in particular $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$, and G_0 is thus integrable in the sense of Frobenius.

Any $g \in G$ lies in a foliated chart domain U which is described by coordinates $(x^1, \ldots, x^n)$ such that the first k among them define the local integral manifold of G_0 containing g. Thus, for any $g' \in U$ the basis vector fields $\partial_{x^1}, \ldots, \partial_{x^k}$ evaluated at g' span $\mathsf{G}_0(g')$. Since $\mathsf{P}_1 = \mathsf{G}_0^\circ$, the codistribution P_1 is spanned by $\mathbf{d}x^{k+1}, \ldots, \mathbf{d}x^n$ on the neighborhood U of g. □

Remark 3.9. Since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$, the integral leaf N of G_0 through $e \in G$ is a normal subgroup of G. If N is in addition closed in G, its (left- or right-) action on G is proper.

We will see later that in certain cases (for example when the Dirac Lie group is integrable), the induced action of N on (G, D_G) is canonical. Also, since $\mathcal{V}_N := \mathfrak{g}_0^L = \mathfrak{g}_0^R$ is the vertical space of the action of N on G, it is easy to see that $\mathsf{D}_G \cap \mathcal{K}_N^\perp = \mathsf{D}_G$ and this intersection has consequently constant rank on G (recall the paragraph about regular reduction of symmetric Dirac structures in Section ***2.1***). If N is closed in G, we can hence build the quotient $q_N : G \to G/N$ and $(G/N, q_N(\mathsf{D}_G))$ will be shown later to be a Dirac manifold with $q_N(\mathsf{D}_G)$ the graph of a multiplicative bivector field on G/N. In particular, if (G, D_G) is integrable, the quotient $(G/N, q_N(\mathsf{D}_G))$ will be a Poisson Lie group.

Consider a Lie group G and $\tilde{p} : \tilde{G} \to G$ its universal covering. Then there exists a discrete normal subgroup Γ of $\tilde{G}$ such that $G = \tilde{G}/\Gamma$ (see Knapp (2002)). The following proposition is easy to prove.

Proposition 3.10. *Let D_G be a multiplicative (integrable) Dirac structure on G. Then the pullback Dirac structure $\tilde{\mathsf{D}}_G := \tilde{p}^*\mathsf{D}_G$ is a (integrable) multiplicative Dirac structure on $\tilde{G}$.*

Remark 3.11. The integral leaf $\tilde{N}$ through $e \in \tilde{G}$ of the characteristic distribution $\tilde{\mathsf{G}}_0$ defined by $\tilde{\mathsf{D}}_G$ on $\tilde{G}$ is normal in $\tilde{G}$ and hence closed since $\tilde{G}$ is simply connected (see Hilgert and Neeb (1991)). Hence, the quotient $\tilde{G}/\tilde{N}$ is here always well-defined.

Example 3.12. Let G be a connected Lie group. The Lie algebra $\mathfrak{g}$ of G can be Levi-decomposed as the semi-direct product $\mathfrak{g} = \mathfrak{s} \oplus_\varphi \operatorname{rad}\mathfrak{g}$ with $\mathfrak{s}$ semi-simple and $\varphi : \mathfrak{s} \to \operatorname{Der}(\operatorname{rad}\mathfrak{g})$ a Lie algebra homomorphism (see for instance Knapp (2002)).

The ideal $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ and its integral leaf R is closed in G (see Hilgert and Neeb (1991)). The quotient G/R is then a semi-simple Lie group. Let $q_R : G \to G/R$ be the projection and π be the standard multiplicative Poisson structure on the semi-simple Lie group G/R (see Etingof and Schiffmann (2002) and Lu (1990)). The pullback $q_R^*\mathsf{D}_\pi$ is an integrable Dirac structure on G. Its characteristic distribution is the left or right invariant image of the ideal $\mathfrak{g}_0 = \operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ and the action of the integral leaf R of G_0 on $(G, q_R^*\mathsf{D}_\pi)$ is canonical, the Poisson Lie group associated to this Dirac Lie group as in Remark 3.9 is obviously $(G/R, \pi)$.

The following lemma will be useful for many proofs in this paper. We will always use the following notation. If ξ is an element of the subspace $\mathfrak{p}_1 \subseteq \mathfrak{g}^*$, then the one-form ξ^L is a section of P_1 by Proposition 3.4. We denote by $X_\xi \in \mathfrak{X}(G)$ a vector field satisfying $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$. The vector field X_ξ is not necessarily unique; all $Y \in X_\xi + \Gamma(\mathsf{G}_0)$ satisfy the condition $(Y, \xi^L) \in \Gamma(\mathsf{D}_G)$.

Lemma 3.13. *Choose $\xi \in \mathfrak{p}_1$ and corresponding vector fields X_ξ and $X_{\operatorname{Ad}^*_{h^{-1}}\xi}$ for $h \in G$. Then the inclusion*

$$X_\xi(gh) \in T_hL_g X_\xi(h) + T_gR_h X_{\operatorname{Ad}^*_{h^{-1}}\xi}(g) + \mathsf{G}_0(gh) \tag{3.1}$$

holds for all $g \in G$.

Remark 3.14. If Y_ξ and $Y_{\operatorname{Ad}^*_h\xi} \in \mathfrak{X}(G)$ are such that

$$(Y_\xi, \xi^R),\ (Y_{\operatorname{Ad}^*_h\xi}, (\operatorname{Ad}^*_h\xi)^R) \in \Gamma(\mathsf{D}_G),$$

then we can show in the same manner, for all $g \in G$:

$$Y_\xi(hg) \in T_gL_h Y_{\operatorname{Ad}^*_h\xi}(g) + T_hR_g Y_\xi(h) + \mathsf{G}_0(hg).$$

Proof. Since (G, D_G) is a Dirac Lie group and $(X_\xi(gh), \xi^L(gh)) \in \mathsf{D}_G(gh)$, there exist $w_g \in T_gG$ and $u_h \in T_hG$ such that

$$(w_g, (T_gR_h)^*\xi^L(gh)) \in \mathsf{D}_G(g), \quad (u_h, (T_hL_g)^*\xi^L(gh)) \in \mathsf{D}_G(h)$$

and

$$T_gR_h w_g + T_hL_g u_h = X_\xi(gh).$$

However, since

$$(T_gR_h)^*\xi^L(gh) = (\mathrm{Ad}^*_{h^{-1}}\xi)^L(g) \quad \text{and} \quad (T_hL_g)^*\xi^L(gh) = \xi^L(h),$$

we have

$$w_g - X_{\mathrm{Ad}^*_{h^{-1}}\xi}(g) \in \mathsf{G}_0(g) \quad \text{and} \quad u_h - X_\xi(h) \in \mathsf{G}_0(h).$$

With the equalities $\mathsf{G}_0 = \mathfrak{g}_0^L = \mathfrak{g}_0^R$, this proves the claim. ◻

Proposition 3.15. *Let ξ and η be elements of $\mathfrak{p}_1$ and $X_\xi, X_\eta \in \mathfrak{X}(G)$ corresponding vector fields. The one-form $\pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\mathbf{d}\xi^L$ is left invariant and equal to $(\mathbf{d}_e(\eta^L(X_\xi)))^L$.*

Proof. Choose $x \in \mathfrak{g}$ and, using the preceding lemma, compute

$$
\begin{aligned}
&(\pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\mathbf{d}\xi^L)(x^L)(g)\\
&= X_\xi(\eta^L(x^L))(g) + \eta^L(\pounds_{x^L}X_\xi)(g) - X_\eta(\xi^L(x^L))(g)\\
&\quad + x^L(\xi^L(X_\eta))(g) - \xi^L(\pounds_{x^L}X_\eta)(g)\\
&= \eta^L(g)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{g\exp(tx)}R_{\exp(-tx)}X_\xi(g\exp(tx))\right) + (\pounds_{x^L}\xi^L)(X_\eta)(g)\\
&\overset{(3.1)}{=} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\eta^L(g)(X_{\mathrm{Ad}^*_{\exp(-tx)}\xi}(g)) + \eta\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{\exp(tx)}R_{\exp(-tx)}X_\xi(\exp(tx))\right)\\
&\quad + (\mathrm{ad}^*_x\xi)^L(X_\eta)(g)\\
&= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\mathrm{Ad}^*_{\exp(-tx)}\xi)^L(g)(X_\eta(g)) + \eta(\pounds_{x^L}X_\xi(e)) + (\mathrm{ad}^*_x\xi)^L(X_\eta)(g)\\
&= -(\mathrm{ad}^*_x\xi)^L(X_\eta)(g) + \pounds_{x^L}(\eta^L(X_\xi))(e) - (\pounds_{x^L}\eta^L)(X_\xi)(e) + (\mathrm{ad}^*_x\xi)^L(X_\eta)(g)\\
&= \mathbf{d}_e(\eta^L(X_\xi))(x),
\end{aligned}
$$

where we have used Proposition 3.7 and $(\pounds_{x^L}\eta^L) = (\mathrm{ad}^*_x\eta)^L \in \Gamma(\mathsf{P}_1)$ by Corollary 3.5. ◻

Definition 3.16. Let (G, D_G) be a Dirac Lie group. Define the bilinear, antisymmetric bracket

$$[\cdot,\cdot] : \mathfrak{p}_1 \times \mathfrak{p}_1 \to \mathfrak{g}^* \quad \text{by} \quad [\xi, \eta] = \mathbf{d}_e(\eta^L(X_\xi)),$$

where $X_\xi \in \mathfrak{X}(G)$ is such that $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$. That is, we set the notation $\pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\mathbf{d}\xi^L =: [\xi,\eta]^L$ and hence $[(X_\xi, \xi^L), (X_\eta, \eta^L)] = ([X_\xi, X_\eta], [\xi,\eta]^L)$ for all $\xi, \eta \in \mathfrak{p}_1$.

Note that $[\xi, \eta]$ does not depend on the choice of the vector field X_ξ. Indeed, if $Y_\xi \in \mathfrak{X}(G)$ is an other vector field such that $(Y_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$, we have $Y_\xi - X_\xi \in \Gamma(\mathsf{G}_0)$ and hence $\eta^L(X_\xi) = \eta^L((X_\xi - Y_\xi) + Y_\xi)) = \eta^L(Y_\xi)$ since $\eta^L \in \Gamma(\mathsf{P}_1)$.

The bilinearity of the bracket is obvious. For the antisymmetry, choose $\xi, \eta \in \mathfrak{p}_1$. Then we have $\xi^L(X_\eta) + \eta^L(X_\xi) = 0$ since (X_ξ, ξ^L) and (X_η, η^L) are sections of D_G, and this leads to

$$[\xi, \eta] = \mathbf{d}_e(\eta^L(X_\xi)) = \mathbf{d}_e(-\xi^L(X_\eta)) = -[\eta, \xi].$$

As a direct corollary of Proposition 3.15 we recover the fact that every multiplicative Dirac structure on a torus is trivial.

Corollary 3.17. *Consider an* Abelian *Dirac Lie group* (G, D_G) *and choose* x *in the Lie algebra* $\mathfrak{g}$. *Then the equality*

$$(\eta^L(X_\xi))(g \cdot \exp(tx)) = [\xi, \eta](tx) + (\eta^L(X_\xi))(g)$$

holds for all $g \in G$ *and* $t \in \mathbb{R}$.

Consequently, if $\mathsf{D}_{\mathbb{T}^n}$ *is a multiplicative Dirac structure on the* n*-torus* $\mathbb{R}^n/\mathbb{Z}^n$, *then* $\mathsf{D}_{\mathbb{T}^n}$ *is the direct sum*

$$\mathsf{D}_{\mathbb{T}^n} = \mathsf{G}_0 \oplus \mathsf{P}_1.$$

Proof. We have shown in Proposition 3.15 that $\pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\xi^L = [\xi, \eta]^L$ is a left-invariant one-form on G. We have for all $\xi, \eta \in \mathfrak{p}_1$ and $x \in \mathfrak{g}$:

$$\begin{aligned}
[\xi,\eta]^L(x^L) &= (\pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\mathbf{d}\xi^L)(x^L) \\
&= \eta^L(\pounds_{x^L}X_\xi) + (\pounds_{x^L}\xi^L)(X_\eta) \quad \text{(see the proof of Proposition 3.15)} \\
&= x^L(\eta^L(X_\xi)) - (\pounds_{x^L}\eta^L)(X_\xi) + (\pounds_{x^L}\xi^L)(X_\eta) \\
&= x^L(\eta^L(X_\xi)) - (\mathrm{ad}_x^*\eta)^L(X_\xi) + (\mathrm{ad}_x^*\xi)^L(X_\eta) \\
&= x^L(\eta^L(X_\xi))
\end{aligned} \tag{3.2}$$

since $\mathrm{ad}_x^*\xi = \mathrm{ad}_x^*\eta = 0$ because $\mathfrak{g}$ is Abelian. We get $\mathbf{d}(\eta^L(X_\xi)) = [\xi,\eta]^L$ and the equality $(\mathrm{d}/\mathrm{d}t)R^*_{\exp(tx)}f = R^*_{\exp(tx)}(\pounds_{x^L}f)$ for all $f \in C^\infty(G)$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\eta^L(X_\xi))(g\exp(tx)) &= R^*_{\exp(tx)}(x^L(\eta^L(X_\xi)))(g)\\ &\overset{(3.2)}{=} R^*_{\exp(tx)}([\xi,\eta](x))\\ &= [\xi,\eta](x)\end{aligned}$$

for all $g \in G$ and $t \in \mathbb{R}$. We get

$$\begin{aligned}(\eta^L(X_\xi))(g\exp(tx)) &= [\xi,\eta](x)\cdot t + (\eta^L(X_\xi))(g)\\ &= [\xi,\eta](tx) + (\eta^L(X_\xi))(g).\end{aligned}$$

On the n-dimensional torus $\mathbb{T}^n$, we have $\exp(tx) = tx + \mathbb{Z}^n$ for all $x \in \mathfrak{g} = \mathbb{R}^n$ and all $t \in \mathbb{R}$. This yields

$$(\eta^L(X_\xi))(\exp(tx)) = [\xi,\eta](tx) + (\eta^L(X_\xi))(0) = [\xi,\eta](tx)$$

for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$. But since the function $(\eta^L(X_\xi))$ is well-defined on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the equality $[\xi,\eta](tx) = [\xi,\eta](tx+z)$ has to hold for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $z \in \mathbb{Z}^n$. This leads to $[\xi,\eta] = 0$ and hence $\eta^L(X_\xi)$ is constant and equal to its value at the neutral element; $\eta^L(X_\xi)(0) = \eta(X_\xi(0)) = 0$ for all $\xi,\eta \in \mathfrak{p}_1$ by Proposition 3.7. Thus, each spanning vector field X_ξ, $\xi \in \mathfrak{p}_1$, of G_1 is annihilated by P_1 and is consequently a section of G_0. □

The next proposition shows that the value of $[\xi,\eta]$, for $\xi,\eta \in \mathfrak{p}_1$, can be computed with any two one-forms in $\Gamma(\mathsf{P}_1)$ taking value ξ,η in e.

Proposition 3.18. *Let $\alpha,\beta \in \Gamma(\mathsf{P}_1)$ be such that $\alpha(e) = \xi$ and $\beta(e) = \eta \in \mathfrak{p}_1$. Then we have*

$$[\xi,\eta] = \mathbf{d}_e(\beta(X_\alpha)),$$

where $X_\alpha \in \mathfrak{X}(G)$ is such that $(X_\alpha,\alpha) \in \Gamma(\mathsf{D}_G)$.

Proof. First, we show that $\pounds_{x^R}\alpha \in \Gamma(\mathsf{P}_1)$ for all $\alpha \in \Gamma(\mathsf{P}_1)$. For all $g,h \in G$ we have $(L_g^*\alpha)(h) = \alpha_{gh} \circ T_hL_g = (T_hL_g)^*\alpha_{gh}$. This is an element of $\mathsf{P}_1(h)$ since $\alpha_{gh} \in \mathsf{P}_1(gh)$ and P_1 is left-invariant by Proposition 3.4. Thus, we get

$$(\pounds_{x^R}\alpha)(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(L^*_{\exp(tx)}\alpha)(h) \in \mathsf{P}_1(h)$$

since $(L^*_{\exp(tx)}\alpha)(h) \in \mathsf{P}_1(h)$ for all t. Choose $x \in \mathfrak{g}$ and compute

$$\begin{aligned}[\xi,\eta](x) &= x^R(\eta^L(X_\xi))(e) = \eta(\pounds_{x^R}X_\xi(e)) = \beta(e)(\pounds_{x^R}X_\xi(e))\\ &= x^R(\beta(X_\xi)) - (\pounds_{x^R}\beta)(X_\xi)(e) = -x^R(\xi^L(X_\beta))(e)\\ &= -\xi(\pounds_{x^R}X_\beta(e)) = -\alpha(e)(\pounds_{x^R}X_\beta(e))\\ &= -\pounds_{x^R}(\alpha(X_\beta))(e) + (\pounds_{x^R}\alpha)(X_\beta)(e) = \mathbf{d}_e(\beta(X_\alpha))(x).\end{aligned}$$

In the fifth and ninth equalities, we have used the fact that $\pounds_{x^R}\alpha, \pounds_{x^R}\beta \in \Gamma(\mathsf{P}_1)$ and Proposition 3.7. ◻

The next lemma holds for integrable Dirac Lie groups, and is in general not true if the Dirac Lie group (G, D_G) is not integrable, as shows the example following it. Recall that N is the normal subgroup of G defined by the integral leaf through e of the integrable subbundle $\mathsf{G}_0 \subseteq TG$.

Lemma 3.19. *If (G, D_G) is integrable, then we have the following: $(\pounds_{x^L}X, \pounds_{x^L}\alpha)$ and $(\pounds_{x^R}X, \pounds_{x^R}\alpha) \in \Gamma(\mathsf{D}_G)$ for all $x \in \mathfrak{g}_0$ and $(X, \alpha) \in \Gamma(\mathsf{D}_G)$, and the pairs $(R_n^*X, R_n^*\alpha)$ and $(L_n^*X, L_n^*\alpha)$ are also elements of $\Gamma(\mathsf{D}_G)$ for all $n \in N$.*

Proof. The right and left invariant vector fields x^R and x^L defined on G by an element of $\mathfrak{g}_0$ are sections of G_0 since we have shown in Proposition 3.4 that $\mathsf{G}_0 = \mathfrak{g}_0^R = \mathfrak{g}_0^L$. If (G, D_G) is integrable, we have $[(x^L, 0), (X, \alpha)] = (\pounds_{x^L}X, \pounds_{x^L}\alpha)$ and $[(x^R, 0), (X, \alpha)] = (\pounds_{x^R}X, \pounds_{x^R}\alpha) \in \Gamma(\mathsf{D}_G)$ for all $(X, \alpha) \in \Gamma(\mathsf{D}_G)$.

In Jotz and Ratiu (2010), it is proved that an integrable Dirac structure D is conserved along the flow of the vector fields $X \in \Gamma(\mathsf{G}_0)$.

For each $x \in \mathfrak{g}_0$, the flow of x^L is $R_{\exp(tx)}$ and the flow of x^R is $L_{\exp(tx)}$. Thus, we have $(R^*_{\exp(tx)}X, R^*_{\exp(tx)}\alpha)$ and $(L^*_{\exp(tx)}X, L^*_{\exp(tx)}\alpha) \in \Gamma(\mathsf{D}_G)$. This yields the claim since N is generated as a group by the elements $\exp(tx)$, $x \in \mathfrak{g}_0$ and small t. ◻

Example 3.20. Consider the Dirac structure $\mathsf{D}_{\mathbb{R}^3}$ defined on the Lie group $\mathbb{R}^3$ as the span of the sections

$$(\partial_z, 0),\ (z\partial_x, \mathbf{d}y),\ (-z\partial_y, \mathbf{d}x)$$

of $\mathsf{P}_{\mathbb{R}^3}$. It is easy to show that $(\mathbb{R}^3, \mathsf{D}_{\mathbb{R}^3})$ is a Dirac Lie group (see also Corollary 3.17 for a description of the multiplicative Dirac structures on $\mathbb{R}^n$). It is not integrable because, for instance, the bracket of $(\partial_z, 0)$ and $(z\partial_x, \mathbf{d}y)$ is equal to $(\partial_x, 0)$, which is not a section of $\mathsf{D}_{\mathbb{R}^3}$. The Dirac structure is obviously not invariant under the action of $N = \{(0,0)\} \times \mathbb{R}$ on $\mathbb{R}^3$.

The following theorem shows how to decide if the action of N on (G, D_G) is canonical.

Theorem 3.21. *The Dirac Lie group (G, D_G) is N-invariant if and only if the bracket $[\cdot,\cdot]$ defined in Definition 3.16 has image in $\mathfrak{p}_1$.*

Example 3.22. Consider again Example 3.20. The bracket on

$$\mathfrak{p}_1 = \operatorname{span}\{\mathbf{d}x(0), \mathbf{d}y(0)\}$$

is given by $[\mathbf{d}y(0), \mathbf{d}x(0)] = \mathbf{d}_0((\mathbf{d}x)(z\partial_x)) = \mathbf{d}z(0) \notin \mathfrak{p}_1$.

For the proof of this theorem, we need to introduce a new notation and show a lemma, that will also be useful in the following.

Definition 3.23. Choose $\xi \in \mathfrak{p}_1$ and $x \in \mathfrak{g}$. Then the elements

$$\mathrm{ad}_x^* \xi \in \mathfrak{p}_1 \quad \text{and} \quad \mathrm{ad}_\xi^* x \in \mathfrak{p}_1^* = \mathfrak{g}/\mathfrak{g}_0$$

are defined by $(\mathrm{ad}_x^* \xi)(y) = \xi([y, x])$ for all $y \in \mathfrak{g}$, and $(\mathrm{ad}_\xi^* x)(\eta) = [\eta, \xi](x)$ for all $\eta \in \mathfrak{p}_1$.

Note that $\mathrm{ad}_x^* \xi$ is an element of $\mathfrak{p}_1$ by Corollary 3.5.

Lemma 3.24. *Choose $\xi \in \mathfrak{p}_1$ and $X_\xi \in \mathfrak{X}(G)$ such that $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$. Then we have for all $x \in \mathfrak{g}$:*

$$(\pounds_{x^L} X_\xi)(e) + \mathfrak{g}_0 = -\,\mathrm{ad}_\xi^* x \in \mathfrak{g}/\mathfrak{g}_0$$

and consequently

$$\pounds_{x^L} X_\xi \in (-\,\mathrm{ad}_\xi^* x)^L + X_{\mathrm{ad}_x^* \xi} + \Gamma(\mathsf{G}_0) \tag{3.3}$$

for all $g \in G$.

Proof. Choose $\eta \in \mathfrak{p}_1$ and compute

$$\begin{aligned} \eta((\pounds_{x^L} X_\xi)(e)) &= \pounds_{x^L}(\eta^L(X_\xi))(e) - (\pounds_{x^L}\eta^L)(X_\xi)(e) \\ &= [\xi, \eta](x) - 0 = \eta\,(-\,\mathrm{ad}_\xi^* x). \end{aligned}$$

This yields the first equality. Using this and the proof of Proposition 3.15, we get

$$\begin{aligned} \eta^L(\pounds_{x^L} X_\xi) &= -(\mathrm{ad}_x^* \xi)^L(X_\eta) + \eta(\pounds_{x^L} X_\xi(e)) \\ &= \eta^L(X_{\mathrm{ad}_x^* \xi}) + \eta^L((-\,\mathrm{ad}_\xi^* x)^L) \\ &= \eta^L(X_{\mathrm{ad}_x^* \xi} + (-\,\mathrm{ad}_\xi^* x)^L). \end{aligned}$$

Since the left-invariant one-forms η^L, for all $\eta \in \mathfrak{p}_1$, span $\Gamma(\mathsf{P}_1)$ as a $C^\infty(G)$-module, we have $\alpha(\pounds_{x^L} X_\xi) = \alpha(X_{\mathrm{ad}_x^* \xi} - (\mathrm{ad}_\xi^* x)^L)$ for all $\alpha \in \Gamma(\mathsf{P}_1)$, and hence we are done because $\mathsf{G}_0 = \mathsf{P}_1^\circ$. ∎

Remark 3.25. Note that if x is an element of $\mathfrak{g}_0$, we have $(\pounds_{x^L} X_\xi, \pounds_{x^L} \xi^L) = [(x^L, 0), (X_\xi, \xi^L)] \in \Gamma(\mathsf{D}_G)$ if D_G is integrable. Since x lies in the ideal $\mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1 = \mathfrak{g}_0^\circ$, we have $\mathrm{ad}_x^* \xi = 0$, thus $\pounds_{x^L} \xi^L = (\mathrm{ad}_x^* \xi)^L = 0$ and we get $\pounds_{x^L} X_\xi \in \Gamma(\mathsf{G}_0)$.

With Lemma 3.24, we can show that this is true without the assumption that D_G is integrable; we need only the hypothesis that the bracket on $\mathfrak{p}_1$ has image in $\mathfrak{p}_1$. We have then

$$\pounds_{x^L} X_\xi \in X_{\mathrm{ad}_x^* \xi} - (\mathrm{ad}_\xi^* x)^L + \Gamma(\mathsf{G}_0) = X_0 - (\mathrm{ad}_\xi^* x)^L + \Gamma(\mathsf{G}_0) = \Gamma(\mathsf{G}_0).$$

The vector field X_0 is indeed an element of $\Gamma(\mathsf{G}_0)$ by definition, and for all $\eta \in \mathfrak{p}_1$, we have $(\mathrm{ad}_\xi^* x)(\eta) = [\eta, \xi](x) = 0$ since $[\eta, \xi] \in \mathfrak{p}_1$ and $x \in \mathfrak{g}_0$, which shows that $\mathrm{ad}_\xi^* x$ is trivial in $\mathfrak{g}/\mathfrak{g}_0$ and thus $(\mathrm{ad}_\xi^* x)^L \in \Gamma(\mathsf{G}_0)$.

Proof of Theorem 3.21. If the right action of N on (G, D_G) is canonical, we have $(R_n^* X_\xi, R_n^* \xi^L) \in \Gamma(\mathsf{D}_G)$ for all $n \in N$ and $\xi \in \mathfrak{p}_1$. This yields $(\pounds_{x^L} X_\xi, \pounds_{x^L} \xi^L) \in \Gamma(\mathsf{D}_G)$ for all $x \in \mathfrak{g}_0$. Since $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$, we get $\pounds_{x^L} X_\xi(e) \in \mathfrak{g}_0$. Hence, we have

$$[\xi, \eta](x) = x^L(\eta^L(X_\xi))(e) = (\mathrm{ad}_x^* \eta)(X_\xi(e)) + \eta(\pounds_{x^L} X_\xi(e)) = 0$$

for all $\xi, \eta \in \mathfrak{p}_1$ and $x \in \mathfrak{g}_0$ and consequently $[\xi, \eta] \in \mathfrak{p}_1$.

Conversely, if $[\xi, \eta] \in \mathfrak{p}_1$ for all $\xi, \eta \in \mathfrak{p}_1$, we get $\pounds_{x^L} X_\xi \in \Gamma(\mathsf{G}_0)$ by Remark 3.25. Hence, recalling that $\mathrm{ad}_x^* \xi = 0$ for $x \in \mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1$, we can compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle (R_{\exp(tx)}^* X_\xi, R_{\exp(tx)}^* \xi^L), (X_\eta, \eta^L)\rangle(g) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}(\eta^L(g)(R_{\exp(tx)}^* X_\xi)(g) + (R_{\exp(tx)}^* \xi^L)(g)(X_\eta(g))) \\
&\quad = \eta^L(R_{\exp(tx)}^* (\pounds_{x^L} X_\xi))(g) + (R_{\exp(tx)}^* (\mathrm{ad}_x^* \xi)^L)(X_\eta)(g) \\
&\quad = \eta \circ T_g L_{g^{-1}} \circ T_{g\exp(tx)} R_{\exp(-tx)} (\pounds_{x^L} X_\xi)(g \exp(tx)) \\
&\quad = (\mathrm{Ad}_{\exp(tx)}^* \eta)^L (\pounds_{x^L} X_\xi)(g \exp(tx)) = 0,
\end{aligned}$$

since $\mathrm{Ad}_{\exp(tx)}^* \eta \in \mathfrak{p}_1$ and $\pounds_{x^L} X_\xi \in \Gamma(\mathsf{G}_0)$. However, this yields

$$\begin{aligned}
&\langle (R_{\exp(tx)}^* X_\xi, R_{\exp(tx)}^* \xi^L), (X_\eta, \eta^L)\rangle(g) \\
&\quad = \langle (R_{\exp(0\cdot x)}^* X_\xi, R_{\exp(0\cdot x)}^* \xi^L), (X_\eta, \eta^L)\rangle(g) \\
&\quad = \langle (X_\xi, \xi^L), (X_\eta, \eta^L)\rangle(g) = 0
\end{aligned}$$

for all $t \in \mathbb{R}$, which shows that $(R_{\exp(tx)}^* X_\xi, R_{\exp(tx)}^* \xi^L)$ is a section of D_G for all $t \in \mathbb{R}$. Hence, since N is generated as a group by the elements $\exp(tx)$, for $x \in \mathfrak{g}_0$ and small $t \in \mathbb{R}$, the proof is finished. □

The following theorem will be useful in the next section about integrable Dirac Lie groups.

Theorem 3.26. *The equality*

$$[\xi, \eta]([x, y]) = (\mathrm{ad}_y^* \xi)(\mathrm{ad}_\eta^* x) - (\mathrm{ad}_x^* \xi)(\mathrm{ad}_\eta^* y) + (\mathrm{ad}_x^* \eta)(\mathrm{ad}_\xi^* y) - (\mathrm{ad}_y^* \eta)(\mathrm{ad}_\xi^* x) \tag{3.4}$$

holds for all $\xi, \eta \in \mathfrak{p}_1$ *and* $x, y \in \mathfrak{g}$.

Proof. By Definition 3.16, we have $[\xi,\eta]([x,y]) = [x,y]^L(\eta^L(X_\xi))(e)$ for any $x,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{p}_1$. Hence we can compute

$$\begin{aligned}
[\xi,\eta]([x,y]) &= [x,y]^L(\eta^L(X_\xi))(e)\\
&= \pounds_{x^L}\pounds_{y^L}(\eta^L(X_\xi))(e) - \pounds_{y^L}\pounds_{x^L}(\eta^L(X_\xi))(e)\\
&= \pounds_{x^L}(\pounds_{y^L}\eta^L(X_\xi) + \eta^L(\pounds_{y^L}X_\xi))(e)\\
&\quad - \pounds_{y^L}(\pounds_{x^L}\eta^L(X_\xi) + \eta^L(\pounds_{x^L}X_\xi))(e)\\
&= \pounds_{x^L}((\mathrm{ad}^*_y\,\eta)^L(X_\xi) + \eta^L(-\mathrm{ad}^*_\xi\,y)^L + \eta^L(X_{\mathrm{ad}^*_y\,\xi}))(e)\\
&\quad - \pounds_{y^L}((\mathrm{ad}^*_x\,\eta)^L(X_\xi) + \eta^L(-\mathrm{ad}^*_\xi\,x)^L + \eta^L(X_{\mathrm{ad}^*_x\,\xi}))(e).
\end{aligned}$$

Since $\eta^L(-\mathrm{ad}^*_\xi\,y)^L$ and $\eta^L(-\mathrm{ad}^*_\xi\,x)^L$ are constant functions on G, we get hence

$$\begin{aligned}
[\xi,\eta]([x,y]) &= \pounds_{x^L}((\mathrm{ad}^*_y\,\eta)^L(X_\xi) + \eta^L(X_{\mathrm{ad}^*_y\,\xi}))(e)\\
&\quad - \pounds_{y^L}((\mathrm{ad}_x\,\eta)^L(X_\xi) + \eta^L(X_{\mathrm{ad}^*_x\,\xi}))(e)\\
&= \Big(\pounds_{x^L}(\mathrm{ad}^*_y\,\eta)^L(X_\xi) + (\mathrm{ad}^*_y\,\eta)^L(\pounds_{x^L}X_\xi)\\
&\qquad + (\pounds_{x^L}\eta^L)(X_{\mathrm{ad}^*_y\,\xi}) + \eta^L(\pounds_{x^L}X_{\mathrm{ad}^*_y\,\xi})\Big)(e)\\
&\quad - \Big(\pounds_{y^L}(\mathrm{ad}^*_x\,\eta)^L(X_\xi) + (\mathrm{ad}^*_x\,\eta)^L(\pounds_{y^L}X_\xi)\\
&\qquad + (\pounds_{y^L}\eta^L)(X_{\mathrm{ad}^*_x\,\xi}) + \eta^L(\pounds_{y^L}X_{\mathrm{ad}^*_x\,\xi})\Big)(e)\\
&= (\mathrm{ad}^*_x\,\mathrm{ad}^*_y\,\eta)(X_\xi(e)) - (\mathrm{ad}^*_y\,\eta)(\mathrm{ad}^*_\xi\,x) + (\mathrm{ad}^*_x\,\eta)(X_{\mathrm{ad}^*_y\,\xi}(e))\\
&\quad - \eta(\mathrm{ad}^*_{\mathrm{ad}^*_y\,\xi}\,x) - (\mathrm{ad}^*_y\,\mathrm{ad}^*_x\,\eta)(X_\xi(e)) + (\mathrm{ad}^*_x\,\eta)(\mathrm{ad}^*_\xi\,y)\\
&\quad - (\mathrm{ad}^*_y\,\eta)(X_{\mathrm{ad}^*_x\,\xi}(e)) + \eta(\mathrm{ad}^*_{\mathrm{ad}^*_x\,\xi}\,y).
\end{aligned}$$

Since $\mathsf{D}_G(e) = \mathfrak{g}_0\times\mathfrak{p}_1$ and $\mathfrak{p}_1$ is ad^*_x-invariant for all $x\in\mathfrak{g}$, the first, third, fifth and seventh terms of this sum vanish. Thus, we get

$$\begin{aligned}
&[\xi,\eta]([x,y])\\
&\quad = -(\mathrm{ad}^*_y\,\eta)(\mathrm{ad}^*_\xi\,x) - \eta(\mathrm{ad}^*_{\mathrm{ad}^*_y\,\xi}\,x) + (\mathrm{ad}^*_x\,\eta)(\mathrm{ad}^*_\xi\,y) + \eta(\mathrm{ad}^*_{\mathrm{ad}^*_x\,\xi}\,y)\\
&\quad = -(\mathrm{ad}^*_y\,\eta)(\mathrm{ad}^*_\xi\,x) + [\mathrm{ad}^*_y\,\xi,\eta](x) + (\mathrm{ad}^*_x\,\eta)(\mathrm{ad}^*_\xi\,y) - [\mathrm{ad}^*_x\,\xi,\eta](y)\\
&\quad = -(\mathrm{ad}^*_y\,\eta)(\mathrm{ad}^*_\xi\,x) + (\mathrm{ad}^*_y\,\xi)(\mathrm{ad}^*_\eta\,x) + (\mathrm{ad}^*_x\,\eta)(\mathrm{ad}^*_\xi\,y) - (\mathrm{ad}^*_x\,\xi)(\mathrm{ad}^*_\eta\,y),
\end{aligned}$$

and the proof is complete. □

Remark 3.27. Equation (3.4) is equivalent to either one of the following equations for all $x,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{p}_1$:

$$\mathrm{ad}^*_x([\xi,\eta]) = [\mathrm{ad}^*_x\,\xi,\eta] - [\mathrm{ad}^*_x\,\eta,\xi] + \mathrm{ad}^*_{\mathrm{ad}^*_\eta\,x}\,\xi - \mathrm{ad}^*_{\mathrm{ad}^*_\xi\,x}\,\eta, \tag{3.5}$$

(3.6) $$\mathrm{ad}^*_\xi([x,y]) = [\mathrm{ad}^*_\xi x, y] - [\mathrm{ad}^*_\xi y, x] - \mathrm{ad}^*_{\mathrm{ad}^*_x \xi} y + \mathrm{ad}^*_{\mathrm{ad}^*_y \xi} x \in \mathfrak{p}_1^* = \mathfrak{g}/\mathfrak{g}_0.$$

Let (G, D_G) be a Dirac Lie group. Then the space $\bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is a G-module via

$$g \cdot ((x+\mathfrak{g}_0) \wedge (y+\mathfrak{g}_0)) = (\mathrm{Ad}_g x + \mathfrak{g}_0) \wedge (\mathrm{Ad}_g y + \mathfrak{g}_0)$$

by Proposition 3.4, and by derivation, it is a $\mathfrak{g}$-module via

$$z \cdot ((x+\mathfrak{g}_0) \wedge (y+\mathfrak{g}_0)) = ([z,x]+\mathfrak{g}_0) \wedge (y+\mathfrak{g}_0) + (x+\mathfrak{g}_0) \wedge ([z,y]+\mathfrak{g}_0).$$

Theorem 3.26 states then that the map $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ defined as the dual map of $[\cdot,\cdot] : \bigwedge^2 \mathfrak{p}_1 \to \mathfrak{g}^*$ is a Lie algebra 1-cocycle, that is, we have

$$\delta([x,y]) = x \cdot \delta(y) - y \cdot \delta(x)$$

for all $x, y \in \mathfrak{g}$. Hence, we can associate to each Dirac Lie group (G, D_G) an ideal $\mathfrak{g}_0$ and a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. If (G, D_G) is a Dirac Lie group, then the map $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ defined by $C(g)(\xi,\eta) = \eta^R(Y_\xi)(g) = -\xi^R(Y_\eta)(g)$, where $Y_\xi \in \mathfrak{X}(G)$ is a vector field satisfying $(Y_\xi, \xi^R) \in \Gamma(\mathsf{D}_G)$, is a Lie group 1-cocycle, i.e., it satisfies $C(gh) = C(g) + \mathrm{Ad}_g C(h)$ for all $g, h \in G$. The proof of this uses Remark 3.14. We have $C(e) = 0 \in \mathfrak{g}/\mathfrak{g}_0 \wedge \mathfrak{g}/\mathfrak{g}_0$ by Proposition 3.7 and

$$\begin{aligned}(\mathbf{d}_e C)(\xi,\eta) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} C(\exp(tx))(\xi,\eta)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \eta^R(\exp(tx))\\ &= [\xi,\eta](x) \quad \text{(by Proposition 3.18)}\\ &= \delta(x)(\xi,\eta)\end{aligned}$$

for all $\xi, \eta \in \mathfrak{p}_1$.

Note that if G is connected and $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is a Lie group 1-cocycle integrating δ, that is, with $C(e) = 0$ and $\mathbf{d}_e C = \delta$, then C is unique (see Lu (1990)) and D_G is consequently given on G by

(3.7) $$\mathsf{D}_G(g) = \{(T_e R_g(C(g)^\sharp(\xi) + x), \xi^R(g)) \mid \xi \in \mathfrak{p}_1, x \in \mathfrak{g}_0\}$$

for all $g \in G$, where $C(g)^\sharp : \mathfrak{g}/\mathfrak{g}_0 \to \mathfrak{g}$ is defined as follows.

Choose a vector subspace $W \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus W$; then we have an isomorphism $\varphi_W : W \to \mathfrak{g}/\mathfrak{g}_0$, $w \mapsto w + \mathfrak{g}_0$. Set $C(g)^\sharp(\xi) = \varphi_W^{-1}(C(g)(\xi,\cdot))$

for all $\xi \in \mathfrak{p}_1 = \mathfrak{g}_0^\circ$. Note that by definition, (3.7) does not depend on the choice of W.

Conversely, let G be a connected and simply connected Lie group and $\mathfrak{g}_0$ an ideal in $\mathfrak{g}$. Choose a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. Then there exists a unique Lie group 1-cocycle $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ integrating δ (see Dufour and Zung (2005)). Define $\mathsf{D}_G \subseteq \mathsf{P}_G$ by (3.7). Then it is easy to check that D_G is a multiplicative Dirac structure on G. We have shown the following theorem.

Theorem 3.28. *Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then we have a one-to-one correspondence*

$$\left\{ \left(\mathfrak{g}_0,\, \delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0\right) \,\middle|\, \begin{array}{c} \mathfrak{g}_0 \subseteq \mathfrak{g} \text{ ideal,} \\ \delta \text{ cocycle} \end{array} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{array}{c} \text{multiplicative Dirac} \\ \text{structures on } G \end{array} \right\}.$$

We will see in the next subsection that the *integrable* multiplicative Dirac structures on G correspond via this bijection to the pairs $\left(\mathfrak{g}_0,\, \delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0\right)$ such that the dual $[\cdot,\cdot] := \delta^* : \bigwedge^2 \mathfrak{g}_0^\circ \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}_0^\circ =: \mathfrak{p}_1$.

Example 3.29.

(1) Let $G = \mathbb{R}^n$. Then any vector subspace $V \subseteq \mathbb{R}^n \simeq T_0\mathbb{R}^n$ is an ideal in $\mathfrak{g} = \mathbb{R}^n$ and any $\delta : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ is a Lie algebra 1-cocycle since the cocycle condition is trivial in this particular case. The Lie group 1-cocycle C integrating δ is then the unique linear map $C : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ with $\mathbf{d}_0 C = \delta$, that is, C is equal to δ if we identify $G = \mathbb{R}^n$ with $\mathfrak{g} = T_0\mathbb{R}^n$ via the exponential map. This shows that each multiplicative Dirac structure on $\mathbb{R}^n$ is given by

$$\mathsf{D}_{\mathbb{R}^n}(r) = \{(\delta(r)^\sharp(\xi) + x, \xi) \mid \xi \in V^\circ,\ x \in V\} \subseteq T_r\mathbb{R}^n \times T_r^*\mathbb{R}^n,$$

with V a vector subspace of $\mathbb{R}^n$, $\delta : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ a linear map and $\delta(r)^\sharp$ defined as in (3.7) with a complement W of V in $\mathbb{R}^n$.

(2) Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be the set of upper triangular matrices having non-vanishing determinant. The Lie algebra $\mathfrak{g}$ of G is then the set of upper triangular matrices. Its commutator $\mathfrak{g}_0 := [\mathfrak{g}, \mathfrak{g}]$ is the set of strictly upper triangular matrices, and integrates to the normal subgroup $N \subseteq G$ of upper triangular matrices with all entries on the diagonal equal to 1. Note that G is not connected. The connected component of the neutral element $e \in G$ is the set of upper triangular matrices with strictly positive diagonal entries.

The quotient $\mathfrak{g}/\mathfrak{g}_0$ is isomorphic to the set of diagonal matrices in $\mathfrak{g}$. Since $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$, it is easy to see that the cocycle condition is satisfied for a linear map $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ if and only if $\delta|_{\mathfrak{g}_0}$ vanishes, that is, if and only if δ factors to $\bar{\delta} : \mathfrak{g}/\mathfrak{g}_0 \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. In other words, the dual map $[\cdot\,,\cdot]$ of a Lie algebra 1-cocycle δ has necessarily image in $\mathfrak{p}_1 := \mathfrak{g}_0^\circ$.

Hence, if $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is the Lie algebra 1-cocycle associated to a multiplicative Dirac structure D_G on G, then the dual of its derivative at e has image in $\mathfrak{p}_1$, that is, the bracket on $\mathfrak{p}_1$ defined in Definition 3.16 has automatically image in $\mathfrak{p}_1$. Since N is closed in G, this shows by Remark 3.9 and Theorem 3.21 that any multiplicative Dirac structure on G with $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ is automatically the pullback to G under $q : G \to G/N$ of the graph of a multiplicative bivector field on G/N.

More generally, this result holds for any Dirac Lie group (G, D_G) such that $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$.

3.3. Integrable Dirac Lie groups: induced Lie bialgebra. In continuation of the results in the preceding subsection, we can show that the integrability of (G, D_G) depends only on the properties of the bracket defined in Definition 3.16.

Theorem 3.30. *The Dirac Lie group (G, D_G) is integrable if and only if the bracket $[\cdot\,,\cdot]$ on $\mathfrak{p}_1 \times \mathfrak{p}_1$ defined in Definition 3.16 is a Lie bracket on $\mathfrak{p}_1$.*

In this case, Theorem 3.26 implies that the pair $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ is a Lie bialgebra.

Of course, with Theorem 3.21, we could show this theorem by considering the Lie bialgebra structure defined on $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ by the multiplicative Poisson structure on $\tilde{G}/\tilde{N}$ (recall Proposition 3.10 and Remarks 3.9 and 3.11), but we prefer to do that in the setting of Dirac manifolds.

For the proof of the theorem, we will need the following lemmas concerning the tensor T_{D_G} (see Subsection ***2.1*** about Dirac manifolds).

Lemma 3.31. *Let (G, D_G) be a Dirac Lie group. The tensor T_{D_G} is given by*

$$\begin{aligned}(3.8)\qquad & 2T_{\mathsf{D}_G}((X_\xi, \xi^L), (X_\eta, \eta^L), (X_\zeta, \zeta^L)) \\ &= X_\zeta(\eta^L(X_\xi)) + X_\xi(\zeta^L(X_\eta)) + X_\eta(\xi^L(X_\zeta)) \\ &\quad - [\zeta, \eta]^L(X_\xi) - [\xi, \zeta]^L(X_\eta) - [\eta, \xi]^L(X_\zeta)\end{aligned}$$

for all $\xi, \eta, \zeta \in \mathfrak{p}_1$ and corresponding $X_\xi, X_\eta, X_\zeta \in \mathfrak{X}(G)$, and in particular

$$(3.9)\qquad T_{\mathsf{D}_G}(e)((x, \xi), (y, \eta), (z, \zeta)) = [\xi, \eta](z) + [\eta, \zeta](x) + [\zeta, \xi](y)$$

for any $x, y, z \in \mathfrak{g}_0$.

Proof. Choose $\xi, \eta, \zeta \in \mathfrak{p}_1$ and corresponding vector fields $X_\xi, X_\eta, X_\zeta \in \mathfrak{X}(G)$. Then the following equality is easy to show:

$$\begin{aligned}& T_{\mathsf{D}_G}((X_\xi, \xi^L), (X_\eta, \eta^L), (X_\zeta, \zeta^L)) \\ &= \zeta^L([X_\xi, X_\eta]) + \xi^L([X_\eta, X_\zeta]) + \eta^L([X_\zeta, X_\xi]) \\ &\quad + X_\zeta(\xi^L(X_\eta)) + X_\xi(\eta^L(X_\zeta)) + X_\eta(\zeta^L(X_\xi)).\end{aligned}$$

Using the definition of the bracket, we have also

$$\begin{aligned}
&[\zeta,\eta]^L(X_\xi)+[\xi,\zeta]^L(X_\eta)+[\eta,\xi]^L(X_\zeta)\\
&\quad=(\pounds_{X_\zeta}\eta^L-\mathbf{i}_{X_\eta}\mathbf{d}\zeta^L)(X_\xi)+(\pounds_{X_\xi}\zeta^L-\mathbf{i}_{X_\zeta}\mathbf{d}\xi^L)(X_\eta)\\
&\qquad+(\pounds_{X_\eta}\xi^L-\mathbf{i}_{X_\xi}\mathbf{d}\eta^L)(X_\zeta)\\
&\quad=2\Big(\zeta^L([X_\eta,X_\xi])+\eta^L([X_\xi,X_\zeta])+\xi^L([X_\zeta,X_\eta])\Big)\\
&\qquad+3\left(X_\zeta(\eta^L(X_\xi))+X_\xi(\zeta^L(X_\eta))+X_\eta(\xi^L(X_\zeta))\right)\\
&\quad=-2T_{\mathsf{D}_G}\Big((X_\xi,\xi^L),(X_\eta,\eta^L),(X_\zeta,\zeta^L)\Big)\\
&\qquad+X_\zeta(\eta^L(X_\xi))+X_\xi(\zeta^L(X_\eta))+X_\eta(\xi^L(X_\zeta)).
\end{aligned}$$

Evaluated at e, this leads to

$$\begin{aligned}
&T_{\mathsf{D}_G}(e)\Big((X_\xi(e),\xi),(X_\eta(e),\eta),(X_\zeta(e),\zeta)\Big)\\
&\quad=\frac{1}{2}\Big(\mathbf{d}_e(\eta^L(X_\xi))(X_\zeta(e))+\mathbf{d}_e(\zeta^L(X_\eta))(X_\xi(e))+\mathbf{d}_e(\xi^L(X_\zeta))(X_\eta(e))\\
&\qquad\qquad-[\zeta,\eta](X_\xi(e))-[\xi,\zeta](X_\eta(e))-[\eta,\xi](X_\zeta(e))\Big)\\
&\quad=[\xi,\eta](X_\zeta(e))+[\eta,\zeta](X_\xi(e))+[\zeta,\xi](X_\eta(e)).
\end{aligned}$$

∎

Lemma 3.32. *Assume that the bracket on $\mathfrak{p}_1\times\mathfrak{p}_1$ has image in $\mathfrak{p}_1$. Then, $T_{\mathsf{D}_G}((X_\xi,\xi^L),(X_\eta,\eta^L),(X_\zeta,\zeta^L))$ is independent of the choice of the vector fields $X_\xi,X_\eta,X_\zeta\in\mathfrak{X}(G)$. The tensor T_{D_G} defines in this case a tensor $S_{\mathsf{D}_G}\in\Gamma(\bigwedge^3\mathsf{P}_1^*)$ by*

$$S_{\mathsf{D}_G}(\xi^L,\eta^L,\zeta^L)=T_{\mathsf{D}_G}((X_\xi,\xi^L),(X_\eta,\eta^L),(X_\zeta,\zeta^L))$$

for all $\xi,\eta,\zeta\in\mathfrak{p}_1$ and (G,D_G) is integrable if and only if S_{D_G} vanishes on G.

Proof. Consider (3.8). We have $[\xi,\eta]^L(X_\zeta+Z)=[\xi,\eta]^L(X_\zeta)$ for all $Z\in\Gamma(\mathsf{G}_0)$, since $[\xi,\eta]\in\mathfrak{p}_1$ Thus, we have only to show that $X_\xi(\eta^L(X_\zeta)$ is independent of the choices of X_ξ,X_ζ. Choose Z and $W\in\Gamma(\mathsf{G}_0)$ and compute

$$\begin{aligned}
(X_\xi+Z)(\eta^L(X_\zeta+W))&=X_\xi(\eta^L(X_\zeta))+Z(\eta^L(X_\zeta))+(X_\xi+Z)(\eta^L(W))\\
&=X_\xi(\eta^L(X_\zeta)+Z(\eta^L(X_\zeta))
\end{aligned}$$

since $\eta^L(W)=0$. For any $x\in\mathfrak{g}_0$, we have

$$x^L(\eta^L(X_\zeta))=(\mathrm{ad}_x^*\eta)^L(X_\zeta)+\eta^L(\pounds_{x^L}X_\zeta)=0$$

since $\mathrm{ad}_x^*\eta=0$ and $\pounds_{x^L}X_\zeta\in\Gamma(\mathsf{G}_0)$ by Remark 3.25. Since $\Gamma(\mathsf{G}_0)$ is spanned as a $C^\infty(G)$-module by $\{x^L\mid x\in\mathfrak{g}_0\}$, we are done.

Recall that the pairs $(x^L, 0)$ and (X_ξ, ξ^L), for all $x \in \mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1$ span the Dirac bundle D_G. Hence, to prove integrability of D_G, we have only to show that the Courant bracket of two sections of D_G of this type is a section of D_G. We have already $[(x_1^L, 0), (x_2^L, 0)] \in \Gamma(\mathsf{D}_G)$ for all $x_1, x_2 \in \mathfrak{g}_0$ since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$ and $[(x^L, 0), (X_\xi, \xi^L)] = (\pounds_{x^L} X_\xi, (\mathrm{ad}_x^* \xi)^L) = (\pounds_{x^L} X_\xi, 0) \in \Gamma(\mathsf{D}_G)$ by Remark 3.25 . Thus, we have only to show that $([X_\xi, X_\eta], [\xi, \eta]^L)$ is a section of D_G for all $\xi, \eta \in \mathfrak{p}_1$. Since $[\xi, \eta] \in \mathfrak{p}_1$, we have $\langle (x^L, 0), ([X_\xi, X_\eta], [\xi, \eta]^L) \rangle = [\xi, \eta](x) = 0$ for all $x \in \mathfrak{g}_0$. The Dirac structure D_G is thus integrable if and only if $\langle ([X_\xi, X_\eta], [\xi, \eta]^L), (X_\zeta, \zeta^L) \rangle = 0$ for all $\xi, \eta, \zeta \in \mathfrak{p}_1$, that is, if and only if $S_{\mathsf{D}_G} = 0$. ◻

Proof of Theorem 3.30. We have to show that the bracket has image in $\mathfrak{p}_1$ and satisfies the Jacobi identity if and only if the Dirac Lie group (G, D_G) is integrable.

Assume first that (G, D_G) is integrable. The tensor T_{D_G} vanishes identically on G and

$$[(X_\xi, \xi^L), (X_\eta, \eta^L)] = ([X_\xi, X_\eta], \pounds_{X_\xi} \eta^L - \mathbf{i}_{X_\eta} \mathbf{d} \xi^L) = ([X_\xi, X_\eta], [\xi, \eta]^L)$$

is a section of D_G for any $\xi, \eta \in \mathfrak{p}_1$. Hence the covector $[\xi, \eta] = \mathbf{d}_e(\eta^L(X_\xi))$ is an element of $\mathfrak{p}_1$ for all $\xi, \eta \in \mathfrak{p}_1$. We get then using (3.8)

$$\begin{aligned}
&[\xi, [\zeta, \eta]] + [\eta, [\xi, \zeta]] + [\zeta, [\eta, \xi]] \\
&\quad = \mathbf{d}_e([\zeta, \eta]^L(X_\xi)) + \mathbf{d}_e([\xi, \zeta]^L(X_\eta)) + \mathbf{d}_e([\eta, \xi]^L(X_\zeta)) \\
&\quad = -2\mathbf{d}_e\Big(T_{\mathsf{D}_G}((X_\xi, \xi^L), (X_\eta, \eta^L), (X_\zeta, \zeta^L))\Big) \\
&\qquad + \mathbf{d}_e\Big(X_\zeta(\eta^L(X_\xi)) + X_\xi(\zeta^L(X_\eta)) + X_\eta(\xi^L(X_\zeta))\Big)
\end{aligned}$$

for all $\xi, \zeta, \eta \in \mathfrak{p}_1$. We have for any $x \in \mathfrak{g}$:

$$\begin{aligned}
\mathbf{d}_e(X_\zeta(\eta^L(X_\xi)))(x) &= x^L(X_\zeta(\eta^L(X_\xi)))(e) \\
&\overset{(3.3)}{=} \mathbf{d}_e(\eta^L(X_\xi))(-\mathrm{ad}_\zeta^* x) + \mathbf{d}_e((\mathrm{ad}_x^* \eta)^L(X_\xi))(X_\zeta(e)) \\
&\quad + X_\zeta\Big(\eta^L((-\mathrm{ad}_\xi^* x)^L + X_{\mathrm{ad}_x^* \xi})\Big)(e) \\
&= [\xi, \eta](-\mathrm{ad}_\zeta^* x) + [\xi, \mathrm{ad}_x^* \eta](X_\zeta(e)) + [\mathrm{ad}_x^* \xi, \eta](X_\zeta(e)) \\
&= [\zeta, [\xi, \eta]](x).
\end{aligned}$$

We have used the equality $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ and $[\xi, \mathrm{ad}_x^* \eta], [\mathrm{ad}_x^* \xi, \eta] \in \mathfrak{p}_1$ as we have seen above. This leads to

$$\begin{aligned}
&[\xi, [\zeta, \eta]] + [\eta, [\xi, \zeta]] + [\zeta, [\eta, \xi]] \\
&\quad = -\mathbf{d}_e\Big(T_{\mathsf{D}_G}((X_\xi, \xi^L), (X_\eta, \eta^L), (X_\zeta, \zeta^L))\Big) = 0.
\end{aligned}$$

For the converse implication, we know by Lemma 3.32 and the hypothesis that the Lie bracket has image in $\mathfrak{p}_1$ that we have only to show the equality $S_{\mathsf{D}_G} = 0$. We compute $\pounds_{x^L}(S_{\mathsf{D}_G})$ for any $x \in \mathfrak{g}$. It is given for any $g \in G$ and $\xi, \eta, \zeta \in \mathfrak{p}_1$ by

$$\begin{aligned}(\pounds_{x^L} S_{\mathsf{D}_G})(g)(\xi^L(g), \eta^L(g), \zeta^L(g)) \\ &= \pounds_{x^L}(S_{\mathsf{D}_G}(\xi^L, \eta^L, \zeta^L))(g) - S_{\mathsf{D}_G}((\mathrm{ad}_x^* \xi)^L, \eta^L, \zeta^L)(g) \\ &\quad - S_{\mathsf{D}_G}(\xi^L, (\mathrm{ad}_x^* \eta)^L, \zeta^L)(g) - S_{\mathsf{D}_G}(\xi^L, \eta^L, (\mathrm{ad}_x^* \zeta)^L)(g).\end{aligned}$$

A long but straightforward calculation using the definition of S_{D_G}, (3.8), (3.3), and (3.5) yields that

$$\begin{aligned}\pounds_{x^L}(S_{\mathsf{D}_G}(\xi^L, \eta^L, \zeta^L)) &= S_{\mathsf{D}_G}((\mathrm{ad}_x^* \xi)^L, \eta^L, \zeta^L) + S_{\mathsf{D}_G}(\xi^L, (\mathrm{ad}_x^* \eta)^L, \zeta^L) \\ &+ S_{\mathsf{D}_G}(\xi^L, \eta^L, (\mathrm{ad}_x^* \zeta)^L) + ([[\zeta, \eta], \xi] + [[\xi, \zeta], \eta] + [[\eta, \xi], \zeta])(x).\end{aligned}$$

Since $[\cdot, \cdot] : \mathfrak{p}_1 \times \mathfrak{p}_1 \to \mathfrak{p}_1$ satisfies the Jacobi identity by hypothesis, this shows that $\pounds_{x^L} S_{\mathsf{D}_G} = 0$ for all $x \in \mathfrak{g}$ and consequently that S_{D_G} is right invariant. Thus, we get

$$\begin{aligned}S_{\mathsf{D}_G}(g)(\xi^R(g), \zeta^R(g), \eta^R(g)) &= S_{\mathsf{D}_G}(e)(\xi, \zeta, \eta) \\ &\overset{(3.9)}{=} [\zeta, \eta](X_\xi(e)) + [\xi, \zeta](X_\eta(e)) + [\eta, \xi](X_\zeta(e)) \\ &= 0\end{aligned}$$

since $[\cdot, \cdot]$ has image in $\mathfrak{p}_1$ and $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$. Hence, we have shown that S_{D_G} vanishes identically on G and the Dirac Lie group (G, D_G) is consequently integrable by Lemma 3.32. ◻

Remark 3.33.

(1) We can see from the last proof that

$$(\pounds_{x^L} S_{\mathsf{D}_G})(\xi^L, \zeta^L, \eta^L) = [[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta]$$

if the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ has image in $\mathfrak{p}_1$. This shows that $\pounds_{x^L} S_{\mathsf{D}_G}$ is left-invariant and we can see using (3.9) and $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ that $S_{\mathsf{D}_G}(e) = 0$. Thus, $S_{\mathsf{D}_G} \in \Gamma(\bigwedge^3 \mathsf{P}_1^*)$ is multiplicative (see for instance Lu (1990)).

(2) Note that if (G, D_G) is integrable, then $[\xi, \eta] \in \mathfrak{p}_1$ and we get $X_{[\xi,\eta]} \in [X_\xi, X_\eta] + \Gamma(\mathsf{G}_0)$ for all $\xi, \eta \in \mathfrak{p}_1$ and any vector field $X_{[\xi,\eta]}$ such that $(X_{[\xi,\eta]}, [\xi, \eta]^L) \in \Gamma(\mathsf{D}_G)$.

(3) If (G, D_G) is a Poisson Lie group, that is, D_G is the graph of the map $\pi^\sharp : T^*G \to TG$ induced by a multiplicative Poisson bivector field π_G on

G, we recover the Lie bracket defined by a Poisson Lie group on the dual of its Lie algebra (see Lu (1990)). For all $\xi, \eta \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$:

$$\begin{aligned}[\xi,\eta](x) &= \mathbf{d}_e(\eta^L(X_\xi))(x) = x^R(\eta^L(X_\xi))(e) = x^R(\pi(\xi^L,\eta^L))(e)\\ &= (\pounds_{x^R}\pi)(e)(\xi,\eta) + \pi(e)(\pounds_{x^R}\xi^L(e),\eta) + \pi(e)(\xi,\pounds_{x^R}\eta^L(e))\\ &= (\pounds_{x^R}\pi)(e)(\xi,\eta) = (\mathbf{d}_e\pi)^*(\xi,\eta)(x).\end{aligned}$$

Recall that since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{g}_0$ has a canonical Lie algebra structure given by $[x+\mathfrak{g}_0, y+\mathfrak{g}_0] = [x,y]+\mathfrak{g}_0$ for all $x, y \in \mathfrak{g}$. Recall from Theorems 3.26 and 3.30 that if (G, D_G) is integrable, then the pair $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ is a Lie bialgebra. The following theorem is a general fact about Lie bialgebras (see for instance Lu and Weinstein (1990)).

Theorem 3.34. *Assume that the Dirac Lie group (G, D_G) is integrable. The Lie algebra structures on $\mathfrak{g}/\mathfrak{g}_0$ and $\mathfrak{p}_1$ induce a Lie algebra structure on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$, with the bracket given by*

$$[(x+\mathfrak{g}_0,\xi),(y+\mathfrak{g}_0,\eta)] = ([x,y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y + \mathfrak{g}_0, [\xi,\eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi),$$

for all $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{p}_1$.

3.4. The action of G on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Theorem 3.35. *Let (G, D_G) be a Dirac Lie group. Define*

$$A : G \times (\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1) \to \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$$

by

$$A(g,(x+\mathfrak{g}_0,\xi)) = (\mathrm{Ad}_g x + T_gR_{g^{-1}}X_\xi(g) + \mathfrak{g}_0, \mathrm{Ad}^*_{g^{-1}}\xi)$$

for all $g \in G$, where $X_\xi \in \mathfrak{X}(G)$ is a vector field such that $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$. The map A is a well-defined action of G on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Proof. We prove first the fact that the action is well-defined, that is, that it doesn't depend on the choices of x and X_ξ. Choose $x' \in \mathfrak{g}$ such that $x' + \mathfrak{g}_0 = x + \mathfrak{g}_0$. Then $x' - x =: x_0 \in \mathfrak{g}_0$ and

$$\mathrm{Ad}_g x' = \mathrm{Ad}_g(x + x_0) = \mathrm{Ad}_g x + \mathrm{Ad}_g x_0 \in \mathrm{Ad}_g x + \mathfrak{g}_0$$

for all $g \in G$, since $\mathfrak{g}_0$ is Ad_g-invariant for all $g \in G$.

Next, if X_ξ and $X'_\xi \in \mathfrak{X}(G)$ are such that (X_ξ, ξ^L) and $(X'_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$, the difference $X'_\xi - X_\xi$ is a section of G_0 and hence we can write $(X'_\xi - X_\xi)(g) = T_eR_g x_0$ with $x_0 \in \mathfrak{g}_0$. This leads to

$$T_gR_{g^{-1}}X'_\xi(g) = T_gR_{g^{-1}}X_\xi(g) + x_0 \in T_gR_{g^{-1}}X_\xi(g) + \mathfrak{g}_0.$$

The map A is hence shown to be well-defined. We show next that A is an action of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$. We have to show that

$$A(g',A(g,(x+\mathfrak{g}_0,\xi)))=A(g'g,(x+\mathfrak{g}_0,\xi))$$

for all $g,g'\in G$, $x\in\mathfrak{g}$ and $\xi\in\mathfrak{p}_1$.

We have with the same arguments as above

$$\begin{aligned}
&A_{g'g}(x+\mathfrak{g}_0,\xi)\\
&=\left(\operatorname{Ad}_{g'}(\operatorname{Ad}_g x)+T_{g'g}R_{g^{-1}g'^{-1}}X_\xi(g'g)+\mathfrak{g}_0,\ \operatorname{Ad}^*_{g'^{-1}}(\operatorname{Ad}^*_{g^{-1}}\xi)\right)\\
&\overset{(3.1)}{=}\Big(\operatorname{Ad}_{g'}(\operatorname{Ad}_g x)+T_{g'g}R_{g^{-1}g'^{-1}}(T_gL_{g'}X_\xi(g)+T_{g'}R_gX_{\operatorname{Ad}^*_{g^{-1}}\xi}(g'))+\mathfrak{g}_0,\\
&\qquad\qquad \operatorname{Ad}^*_{g'^{-1}}(\operatorname{Ad}^*_{g^{-1}}\xi)\Big)\\
&=\Big(\operatorname{Ad}_{g'}\left(\operatorname{Ad}_g x+T_gR_{g^{-1}}X_\xi(g)\right)+T_{g'}R_{g'^{-1}}X_{\operatorname{Ad}^*_{g^{-1}}\xi}(g')+\mathfrak{g}_0,\\
&\qquad\qquad \operatorname{Ad}^*_{g'^{-1}}(\operatorname{Ad}^*_{g^{-1}}\xi)\Big)\\
&=A_{g'}\left(\operatorname{Ad}_g x+T_gR_{g^{-1}}X_\xi(g)+\mathfrak{g}_0,\ \operatorname{Ad}^*_{g^{-1}}\xi\right)=A_{g'}(A_g(x+\mathfrak{g}_0,\xi)).
\end{aligned}$$
□

Remark 3.36. Assume that the bracket on $\mathfrak{p}_1\times\mathfrak{p}_1$ has image in $\mathfrak{p}_1$.

(1) We have $N\subseteq G_{(x+\mathfrak{g}_0,\xi)}$, where $G_{(x+\mathfrak{g}_0,\xi)}$ is the isotropy group of $(x+\mathfrak{g}_0,\xi)\in\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$.

Indeed, for $n\in N$, we have $\operatorname{Ad}_n x\in x+\mathfrak{g}_0$, for all $x\in\mathfrak{g}$ and hence $\operatorname{Ad}^*_{n^{-1}}\xi=\xi$ for all $\xi\in\mathfrak{p}_1$. The proof of this is easy, see also Ortega and Ratiu (2004, Lemma 2.1.13). Since $(X_\xi,\xi^L)\in\Gamma(\mathsf{D}_G)$ and $n\in N$, we know by Theorem 3.21 that $(R_n^*X_\xi,R_n^*\xi^L)\in\Gamma(\mathsf{D}_G)$. Hence, we have $R_n^*X_\xi(e)\in\mathfrak{g}_0$ because $\mathsf{D}_G(e)=\mathfrak{g}_0\times\mathfrak{p}_1$, that is, $T_nR_{n^{-1}}X_\xi(n)\in\mathfrak{g}_0$. Using this and $\operatorname{Ad}_n x\in x+\mathfrak{g}_0$, we get $\operatorname{Ad}_n x+T_nR_{n^{-1}}X_\xi(n)\in x+\mathfrak{g}_0$.

(2) Thus, we get a well-defined action $\bar{A}$ of G/N on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$, that is given by $\bar{A}(gN,(x+\mathfrak{g}_0,\xi))=A(g,(x+\mathfrak{g}_0,\xi))$ for all $g\in G$.

In fact (G,D_G) is integrable and N is closed in G, the next theorem shows that $\bar{A}$ is the adjoint action of G/N on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ integrating the adjoint action of $\mathfrak{g}/\mathfrak{g}_0$ defined by the bracket on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$.

Theorem 3.37. *Assume that (G,D_G) is an integrable Dirac Lie group. The adjoint action of $\mathfrak{g}/\mathfrak{g}_0\simeq\mathfrak{g}/\mathfrak{g}_0\times\{0\}\subseteq\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ "integrates" to the action A of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ in the sense that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}A(\exp(ty),(x+\mathfrak{g}_0,\xi))=[(y+\mathfrak{g}_0,0),(x+\mathfrak{g}_0,\xi)]$$

for all $y\in\mathfrak{g}$ and $(x+\mathfrak{g}_0,\xi)\in\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$.

Proof. Choose $x, y \in \mathfrak{g}$ and $\xi \in \mathfrak{p}_1$ and compute

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} A(\exp(ty), (x + \mathfrak{g}_0, \xi)) \\
&\quad = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{Ad}_{\exp(ty)} x + T_{\exp(ty)} R_{\exp(-ty)} X_\xi(\exp(ty)) + \mathfrak{g}_0, \mathrm{Ad}^*_{\exp(-ty)} \xi\right) \\
&\quad = ([y, x] + \pounds_{y^L} X_\xi(e) + \mathfrak{g}_0, \mathrm{ad}^*_y \xi) \\
&\quad \overset{(3.3)}{=} ([y, x] - \mathrm{ad}^*_\xi y + \mathfrak{g}_0, \mathrm{ad}^*_y \xi) \\
&\quad = [(y + \mathfrak{g}_0, 0), (x + \mathfrak{g}_0, \xi)]. \qquad \square
\end{aligned}$$

4. Dirac Homogeneous Spaces

4.1. Definition and properties. Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. Let $G/H = \{gH \mid g \in G\}$ be the homogeneous space defined as the quotient space by the right action of H on G. Let $q : G \to G/H$ be the quotient map. For $g \in G$, let $\sigma_g : G/H \to G/H$ be the map defined by $\sigma_g(g'H) = gg'H$.

Definition 4.1. Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. Let G/H be endowed with a Dirac structure $\mathsf{D}_{G/H}$. The pair $(G/H, \mathsf{D}_{G/H})$ is a *Dirac homogeneous space of* (G, D_G) if the left action

$$\sigma : G \times G/H \to G/H, \quad \sigma_g(g'H) = gg'H$$

is a forward Dirac map, where $G \times G/H$ is endowed with the product Dirac structure $\mathsf{D}_G \oplus \mathsf{D}_{G/H}$.

Remark 4.2. If G/H is a homogeneous space of a Lie group G, there is an induced *Lie groupoid action* of $TG \oplus T^*G \rightrightarrows \mathfrak{g}^*$ on $J : T(G/H) \oplus T^*(G/H) \to \mathfrak{g}^*$, $(v_{gH}, \alpha_{gH}) \mapsto (T_e(q \circ R_g))^* \alpha_{gH}$. The Dirac manifold $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G, D_G) if and only if this groupoid action restricts to a Lie groupoid action of $\mathsf{D}_G \rightrightarrows \mathfrak{p}_1$ on $J|_{\mathsf{D}_{G/H}} : \mathsf{D}_{G/H} \to \mathfrak{p}_1$. This will be shown in Jotz (2010), where Dirac homogeneous spaces of Dirac Lie groupoids will be defined in this manner.

Remark 4.3. The definition is also easily shown to be equivalent to the following: for all $gH \in G/H$ and $(v_{gH}, \alpha_{gH}) \in \mathsf{D}_{G/H}(gH)$, there exist $(w_g, \beta_g) \in \mathsf{D}_G(g)$ and $(u_{eH}, \gamma_{eH}) \in \mathsf{D}_{G/H}(eH)$ such that

$$\beta_g = (T_g q)^*(\alpha_{gH}), \quad \gamma_{eH} = (T_{eH}\sigma_g)^*(\alpha_{gH}), \quad v_{gH} = T_g q w_g + T_{eH}\sigma_g u_{eH}.$$

This yields immediately: for all $h \in H$ and $(v_{eH}, \alpha_{eH}) \in \mathsf{D}_{G/H}(eH) = \mathsf{D}_{G/H}(hH)$, there exist $(w_h, \beta_h) \in \mathsf{D}_G(h)$ and $(u_{eH}, \gamma_{eH}) \in \mathsf{D}_{G/H}(eH)$ such that $\beta_h = (T_h q)^*(\alpha_{eH})$, $\gamma_{eH} = (T_{eH}\sigma_h)^*(\alpha_{eH})$, and $v_{eH} = T_h q w_h + T_{eH}\sigma_h u_{eH}$.

Definition 4.4. Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. We say that a subspace $S \subseteq \mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ has property $(*)$ if for all $h \in H$ and $(\bar{x}, \bar{\xi}) \in S$, there exist $(w_h, \beta_h) \in \mathsf{D}_G(h)$ and $(\bar{y}, \bar{\eta}) \in S$ such that $\beta_h = (T_h q)^*(\bar{\xi})$, $\bar{\eta} = (T_{eH}\sigma_h)^*(\bar{\xi})$, and $\bar{x} = T_h q w_h + T_{eH}\sigma_h \bar{y}$.

By Remark 4.3, if $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of the Dirac Lie group (G, D_G), then $\mathsf{D}_{G/H}(e)$ has property $(*)$. This leads to the following lemma.

Lemma 4.5. *Let $\mathfrak{D}_{G/H}$ be a Dirac subspace of $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ with the property $(*)$. Then the inclusions $(T_e q)^*\bar{\mathfrak{p}_1} \subseteq \mathfrak{p}_1$ and $T_e q \mathfrak{g}_0 \subseteq \bar{\mathfrak{g}_0}$ hold, where $\bar{\mathfrak{p}_1} \subseteq (\mathfrak{g}/\mathfrak{h})^*$ and $\bar{\mathfrak{g}_0} \subseteq \mathfrak{g}/\mathfrak{h}$ are the subspaces defined by $\mathfrak{D}_{G/H}$.*

Proof. Choose $\alpha \in \bar{\mathfrak{p}_1}$, then there exists $v \in \mathfrak{g}/\mathfrak{h}$ such that $(v, \alpha) \in \mathfrak{D}_{G/H}$. By $(*)$, there exist $(w_e, \beta_e) \in \mathsf{D}_G(e)$ and $(u_{eH}, \gamma_{eH}) \in \mathfrak{D}_{G/H}$ such that $\beta_e = (T_e q)^*\alpha$, $\gamma_{eH} = (T_{eH}\sigma_h)^*\alpha$, and $v = T_e q w_e + u_{eH}$. The covector $\beta_e = (T_e q)^*\alpha$ is an element of $\mathfrak{p}_1$. The second inclusion is a consequence of the first. ◻

We call in the following $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ the pullback Dirac subspace $\mathfrak{D} = (T_e q)^*\mathfrak{D}_{G/H}$ of $\mathfrak{D}_{G/H} \subseteq \mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ with Property $(*)$, that is,

$$\mathfrak{D} = \{(x, \xi) \mid \exists \bar{\xi} \in (\mathfrak{g}/\mathfrak{h})^* \text{ such that } (T_e q)^*\bar{\xi} = \xi \text{ and } (T_e q x, \bar{\xi}) \in \mathfrak{D}_{G/H}\}.$$

Lemma 4.6. *Let $\mathfrak{p}_1' \subseteq \mathfrak{g}^*$ and $\mathfrak{g}_0' \subseteq \mathfrak{g}$ be the vector subspaces associated to the Dirac subspace $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$. Then we have the inclusions*

$$\mathfrak{g}_0 + \mathfrak{h} \subseteq \mathfrak{g}_0' \quad \text{and} \quad \mathfrak{p}_1' \subseteq \mathfrak{p}_1 \cap \mathfrak{h}^\circ.$$

Hence, we have $\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{p}_1$ and the vector space $\bar{\mathfrak{D}} := \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ can be seen as a subset of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Proof. We know from Lemma 4.5 that $T_e q \mathfrak{g}_0 \subseteq \bar{\mathfrak{g}_0}$ and $(T_e q)^*\bar{\mathfrak{p}_1} \subseteq \mathfrak{p}_1$. The inclusions here follow directly from this and the definition of $\mathfrak{D}$. ◻

As in the case of a Dirac Lie group, the distributions G_0 and P_1 are regular. This fact is proved in the next proposition. In order to simplify the notation, we write G_0 and P_1, respectively, for both the distributions, respectively codistributions, defined by D_G on G and by $\mathsf{D}_{G/H}$ on G/H. It will always be clear from the context which object is to be considered.

Proposition 4.7. *Let $(G/H, \mathsf{D}_{G/H})$ be a Dirac homogeneous space of (G, D_G). Then the distribution G_0 defined by $\mathsf{D}_{G/H}$ on G/H is a subbundle of $T(G/H)$. Consequently, the codistribution P_1 also has constant rank on G/H. More explicitly, the distribution G_0 and the codistribution P_1 are given by*

$$\mathsf{G}_0(gH) = T_{eH}\sigma_g \mathsf{G}_0(eH) \quad \text{and} \quad \mathsf{P}_1(gH) = (T_{gH}\sigma_{g^{-1}})^*\mathsf{P}_1(eH)$$

for all $gH \in G/H$.

Proof. We show that $\mathsf{P}_1(gH) = (T_{gH}\sigma_{g^{-1}})^*\mathsf{P}_1(eH)$ for all $g \in G$: choose first $\bar{\xi} \in \mathsf{P}_1(eH)$, then there exists $\bar{x} \in T_{eH}(G/H)$ such that $(\bar{x}, \bar{\xi}) \in \mathsf{D}_{G/H}(eH)$ and hence $w_{g^{-1}} \in T_{g^{-1}}G$ and $u_{gH} \in T_{gH}(G/H)$ such that $(w_{g^{-1}}, (T_{g^{-1}}q_H)^*\bar{\xi}) \in \mathsf{D}_G(g^{-1})$, $(u_{gH}, (T_{gH}\sigma_{g^{-1}})^*\bar{\xi}) \in \mathsf{D}_{G/H}(gH)$, and $T_{g^{-1}}q_H w_{g^{-1}} + T_{eH}\sigma_{g^{-1}}u_{gH} = \bar{x}$. This yields immediately $(T_{gH}\sigma_{g^{-1}})^*\mathsf{P}_1(eH) \subseteq \mathsf{P}_1(gH)$. The other implication is a direct consequence of Remark 4.3.

Thus, the codistribution P_1 is a subbundle of $T^*(G/H)$ and its annihilator is equal to G_0, which is consequently given by $\mathsf{G}_0(gH) = T_{eH}\sigma_g\mathsf{G}_0(eH)$ for all $g \in G$. ◻

Finally, we see that the notion of Dirac homogeneous spaces generalizes the Poisson homogeneous spaces. The proof can be done as in Example 3.2.

Example 4.8. Let (G, π_G) be a Poisson Lie group, H a closed subgroup of G and π a Poisson bivector on G/H. Let (G, D_G) and $(G/H, \mathsf{D}_{G/H})$ be the Dirac Lie group and Dirac manifold induced by (G, π_G) and $(G/H, \pi)$, respectively. The Dirac manifold $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G, D_G) if and only if $(G/H, \pi)$ is a Poisson homogeneous space of (G, π_G).

4.2. The pullback to** G **of a homogeneous Dirac structure. Consider a Dirac Lie group (G, D_G) and let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g} \times \mathfrak{g}^*$ satisfying

$$(**) \qquad \mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{p}_1.$$

We denote by $\mathfrak{g}_0' \subseteq \mathfrak{g}$ and $\mathfrak{p}_1' \subseteq \mathfrak{g}^*$ the subspaces defined by $\mathfrak{D}$. We have $\mathfrak{g}_0 \subseteq \mathfrak{g}_0'$ and $\mathfrak{p}_1' \subseteq \mathfrak{p}_1$ and we can define the generalized distribution $\mathsf{D}' \subseteq \mathsf{P}_G$ by

$$(4.1) \quad \mathsf{D}'(g) := \{(x^L(g) + v_g, \xi^L(g)) \in \mathsf{P}_G(g) \mid (x, \xi) \in \mathfrak{D} \text{ and } (v_g, \xi^L(g)) \in \mathsf{D}_G(g)\}$$

for all $g \in G$. Note that D' is smooth since it is spanned by the smooth sections $(X_\xi + x^L, \xi^L)$, with $(x, \xi) \in \mathfrak{D}$ and $X_\xi \in \mathfrak{X}(G)$ such that $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$.

Proposition 4.9. *Let* (G, D_G) *be a Dirac Lie group and* $\mathfrak{D}$ *a Dirac subspace of* $\mathfrak{g} \times \mathfrak{g}^*$ *satisfying* $(**)$. *The induced subset* $\mathsf{D}' \subseteq TG \oplus T^*G$ *as in* (4.1) *is a Dirac structure on* G.

The construction of the Dirac structure D' is inspired by Diatta and Medina (1999).

Note that the codistribution P_1' induced by D' on G is equal to $\mathsf{P}_1' = \mathfrak{p}_1'^L$ by definition and consequently the distribution G_0' induced by D' on G is equal to G_0'. We have $\mathsf{G}_0 \subseteq \mathsf{G}_0'$ and $\mathsf{P}_1' \subseteq \mathsf{P}_1$.

Proof. Choose $g \in G$ and $(x^L(g) + v_g, \xi^L(g)), (y^L(g) + w_g, \eta^L(g)) \in \mathsf{D}'(g)$, i.e., with $(x, \xi), (y, \eta) \in \mathfrak{D}$ and $(v_g, \xi^L(g)), (w_g, \eta^L(g)) \in \mathsf{D}_G(g)$. We have

$$\begin{aligned} &\langle (x^L(g) + v_g, \xi^L(g)), (y^L(g) + w_g, \eta^L(g)) \rangle \\ &\quad = \langle (v_g, \xi^L(g)), (w_g, \eta^L(g)) \rangle + \langle (x, \xi), (y, \eta) \rangle = 0, \end{aligned}$$

where we have used the equalities $\mathsf{D}_G = \mathsf{D}_G^\perp$ and $\mathfrak{D} = \mathfrak{D}^\perp$. This shows $\mathsf{D}'(g) \subseteq \mathsf{D}'(g)^\perp$.

Conversely, let $(u_g, \gamma_g) \in \mathsf{P}_G(g)$ be an element of $\mathsf{D}'(g)^\perp$. Then we have $\gamma_g(x^L(g)) = \langle (x^L(g), 0), (u_g, \gamma_g)\rangle = 0$ for all $x \in \mathfrak{g}_0'$, and γ_g is thus an element of $\mathsf{P}_1'(g) \subseteq \mathsf{P}_1(g)$. Choose $v_g \in T_gG$ such that $(v_g, \gamma_g) \in \mathsf{D}_G(g)$ and set $w_g = u_g - v_g$. Then we get for any $(x, \xi) \in \mathfrak{D}$:

$$\begin{aligned}\langle (w_g, \gamma_g), (x^L, \xi^L)(g)\rangle &= \gamma_g(x^L(g)) + \xi^L(g)(w_g)\\ &= \gamma_g(x^L(g)) + \xi^L(g)(w_g) + \xi^L(g)(v_g) + \gamma_g(X_\xi(g)),\end{aligned}$$

where $X_\xi \in \mathfrak{X}(G)$ is such that (X_ξ, ξ^L) is a section of D_G that is defined at g. We have used the identity $\xi^L(v_g) + \gamma_g(X_\xi(g)) = 0$ which holds because $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$ and $(v_g, \gamma_g) \in \mathsf{D}_G(g)$. However, this equals

$$\langle (v_g + w_g, \gamma_g), (X_\xi + x^L, \xi^L)(g)\rangle = \langle (u_g, \gamma_g), (X_\xi + x^L, \xi^L)(g)\rangle = 0,$$

since $(X_\xi + x^L, \xi^L)$ is by definition a section of D' and $(u_g, \gamma_g) \in \mathsf{D}'(g)^\perp$. This shows that $(w_g, \gamma_g) \in \mathfrak{D}^L(g)^\perp = \mathfrak{D}^L(g)$. Hence, we have shown $(u_g, \gamma_g) = (w_g + v_g, \gamma_g) \in \mathsf{D}'(g)$. ◻

Remark 4.10. If (Z, α) is a section of D', then we have $(Z, \alpha) = (X_\alpha + Y_\alpha, \alpha)$ with X_α and $Y_\alpha \in \mathfrak{X}(G)$ such that $(X_\alpha, \alpha) \in \Gamma(\mathsf{D}_G)$ and $(Y_\alpha, \alpha) \in \Gamma(\mathfrak{D}^L)$. Hence, we have

$$(Z(e), \alpha(e)) = (Y_\alpha(e), \alpha(e)) + (X_\alpha(e), 0) \in \mathfrak{D} + (\mathfrak{g}_0 \times \{0\}) = \mathfrak{D}$$

because $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ and $\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D}$. Since $\mathfrak{D}$ and $\mathsf{D}'(e)$ are Lagrangian, this shows that $\mathfrak{D} = \mathsf{D}'(e)$.

Let now H be a closed subgroup of G with Lie algebra $\mathfrak{h}$, and denote by $q_H : G \to G/H$ the smooth surjective submersion. Let $\mathfrak{D}_{G/H} \subseteq \mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ be a Dirac subspace, such that $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ defined by $\mathfrak{D} = (T_e q_H)^* \mathfrak{D}_{G/H}$ satisfies $(**)$. Recall that property $(*)$ has been defined in Definition 4.4.

Theorem 4.11. *The following are equivalent for $\mathfrak{D}_{G/H}$ and $\mathfrak{D}$ as above and D' as in (4.1):*

1. *$\mathfrak{D}_{G/H}$ satisfies property $(*)$*
2. *$A_h(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})) \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ for all $h \in H$*
3. *D' is invariant under the right action of H on G*
4. *(G, D') projects under q_H to a Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ such that $\mathsf{D}_{G/H}(eH) = \mathfrak{D}_{G/H}$.*

Proof. Assume that $\mathfrak{D}_{G/H}$ satisfies $(*)$ and choose $(x + \mathfrak{g}_0, \xi) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$. We have then $(x, \xi) \in \mathfrak{D}$ and hence there exists $\bar\xi \in (\mathfrak{g}/\mathfrak{h})^*$ such that $(T_e q_H)^*\bar\xi = \xi$ and $(T_e q_H x, \bar\xi) \in \mathfrak{D}_{G/H}$. By $(*)$ for $(T_e q_H x, \bar\xi) \in \mathfrak{D}_{G/H}$ and $h^{-1} \in H$, there exists $(w_{h^{-1}}, \beta_{h^{-1}}) \in \mathsf{D}_G(h^{-1})$ and $(\bar{y}, \bar{\eta}) \in \mathfrak{D}_{G/H}$ such that $\beta_{h^{-1}} = (T_{h^{-1}}q_H)^*\bar{\xi}$, $\bar{\eta} = (T_{eH}\sigma_{h^{-1}})^*\bar{\xi}$, and $T_e q_H x = T_{eH}\sigma_{h^{-1}}\bar{y} + T_{h^{-1}}q_H w_{h^{-1}}$. We compute

$$\begin{aligned}(T_eL_{h^{-1}})^*\beta_{h^{-1}} &= (T_{h^{-1}}q_H \circ T_eL_{h^{-1}})^*\bar{\xi}\\ &= (T_eq_H \circ T_{h^{-1}}R_h \circ T_eL_{h^{-1}})^*\bar{\xi} = \mathrm{Ad}^*_{h^{-1}}\xi\end{aligned}$$

and also

$$\begin{aligned}\mathrm{Ad}^*_{h^{-1}}\xi = (T_eL_{h^{-1}})^*\beta_{h^{-1}} &= (T_{h^{-1}}q_H \circ T_eL_{h^{-1}})^*\bar{\xi}\\ &= (T_{eH}\sigma_{h^{-1}} \circ T_eq_H)^*\bar{\xi} = (T_eq_H)^*\bar{\eta} =: \eta.\end{aligned}$$

This yields also $\eta \in \mathfrak{p}_1$, and there exists a vector field $X_\eta \in \mathfrak{X}(G)$ such that $(X_\eta, \eta^L) \in \Gamma(\mathsf{D}_G)$ and $X_\eta(h^{-1}) = w_{h^{-1}}$. We have

$$\begin{aligned}X_\eta(e) &= X_{\mathrm{Ad}^*_{h^{-1}}\xi}(hh^{-1})\\ &\overset{(3.1)}{=} T_{h^{-1}}L_hX_{\mathrm{Ad}^*_{h^{-1}}\xi}(h^{-1}) + T_hR_{h^{-1}}X_\xi(h) + z\end{aligned}$$

with $z \in \mathfrak{g}_0$. We get

$$\begin{aligned}\bar{y} &= T_{eH}\sigma_hT_eq_Hx - T_{eH}\sigma_hT_{h^{-1}}q_Hw_{h^{-1}}\\ &= T_hq_HT_eL_hx - T_eq_HT_{h^{-1}}L_hX_\eta(h^{-1})\\ &= T_eq_H\big(T_hR_{h^{-1}}T_eL_hx + T_hR_{h^{-1}}X_\xi(h) - X_{\mathrm{Ad}^*_{h^{-1}}\xi}(e) + z\big).\end{aligned}$$

Since $(\bar{y}, \bar{\eta})$ is an element of $\mathfrak{D}_{G/H}$, we have $(y, \eta) \in \mathfrak{D}$ for any $y \in \mathfrak{g}$ such that $T_eq_Hy = \bar{y}$. Hence, the pair $(\mathrm{Ad}_h x + T_hR_{h^{-1}}X_\xi(h) - X_{\mathrm{Ad}^*_{h^{-1}}\xi}(e) + z,\ \eta) = (\mathrm{Ad}_h x + T_hR_{h^{-1}}X_\xi(h) - X_{\mathrm{Ad}^*_{h^{-1}}\xi}(e) + z,\ \mathrm{Ad}^*_{h^{-1}}\xi)$ is an element of $\mathfrak{D}$. With $X_{\mathrm{Ad}^*_{h^{-1}}\xi}(e) + z \in \mathfrak{g}_0$, this shows that

$$A(h, (x + \mathfrak{g}_0, \xi)) = \big(\mathrm{Ad}_h x + T_hR_{h^{-1}}X_\xi(h) + \mathfrak{g}_0,\ \mathrm{Ad}^*_{h^{-1}}\xi\big) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}).$$

Assume next that $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is H-invariant and choose a spanning section $(X_\xi + x^L, \xi^L)$ of D', hence with $(x, \xi) \in \mathfrak{D}$ and $X_\xi \in \mathfrak{X}(G)$ a vector field satisfying $(X_\xi, \xi^L) \in \Gamma(\mathsf{D}_G)$. Since $(x + \mathfrak{g}_0, \xi)$ is an element of $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$, we get for an arbitrary $h \in H$,

$$A_h(x + \mathfrak{g}_0, \xi) = \big(\mathrm{Ad}_h x + T_hR_{h^{-1}}X_\xi(h) + \mathfrak{g}_0,\ \mathrm{Ad}^*_{h^{-1}}\xi\big) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}),$$

and hence

$$\big(\mathrm{Ad}_h x + T_hR_{h^{-1}}X_\xi(h),\ \mathrm{Ad}^*_{h^{-1}}\xi\big) \in \mathfrak{D}. \tag{4.2}$$

Then we can compute for $g \in G$:

$$
\begin{aligned}
&\left(R_h^*(X_\xi + x^L)(g), R_h^*(\xi^L)(g)\right) \\
&= \left(T_{gh}R_{h^{-1}}T_eL_{gh}x + T_{gh}R_{h^{-1}}X_\xi(gh), \xi \circ T_{gh}L_{h^{-1}g^{-1}} \circ T_gR_h\right) \\
&\overset{(3.1)}{=} \left(T_eL_g \operatorname{Ad}_h x + T_{gh}R_{h^{-1}}(T_gR_hX_{\operatorname{Ad}^*_{h^{-1}}\xi}(g) + T_hL_gX_\xi(h) + T_gR_hT_eL_gz),\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.(\operatorname{Ad}^*_{h^{-1}}\xi)^L(g)\right)
\end{aligned}
$$

for some $z \in \mathfrak{g}_0$ by Lemma 3.13. Thus, we get

$$
\begin{aligned}
&\left(R_h^*(X_\xi + x^L)(g), R_h^*(\xi^L)(g)\right) \\
&= \left((\operatorname{Ad}_h x)^L(g) + X_{\operatorname{Ad}^*_{h^{-1}}\xi}(g) + (T_hR_{h^{-1}}X_\xi(h))^L(g) + z^L(g), (\operatorname{Ad}^*_{h^{-1}}\xi)^L(g)\right) \\
&= \left(\left(\operatorname{Ad}_h x + T_hR_{h^{-1}}X_\xi(h)\right)^L(g) + X_{\operatorname{Ad}^*_{h^{-1}}\xi}(g), (\operatorname{Ad}^*_{h^{-1}}\xi)^L(g)\right) + (z^L, 0)(g).
\end{aligned}
$$

By the definition of D' and (4.2), we get consequently that

$$
(R_h^*(X_\xi + x^L)(g), R_h^*(\xi^L)(g)) \in \mathsf{D}'(g)
$$

(note that z^L is a section of $\mathsf{G}_0 \subseteq \mathsf{G}_0'$), and hence that the right-action of H on (G, D') is canonical.

Assume that the right action of H on (G, D') is canonical. The vertical space $\mathcal{V}_H$ of the right action of H on G is $\mathcal{V}_H = \mathfrak{h}^L \subseteq \mathsf{G}_0'$ since by definition of $\mathfrak{D}$ and $\mathfrak{g}_0'$, we have $\mathfrak{h} \subseteq \mathfrak{g}_0'$. Thus, we have $\mathsf{P}_1' \subseteq \mathcal{V}_H{}^\circ$ and hence, $\mathsf{D}' \cap \mathcal{K}_H^\perp = \mathsf{D}'$ (recall the notation for (2.5)). The reduced Dirac structure $\mathsf{D}_{G/H}$ is then given by

$$
\mathsf{D}_{G/H} := q_H(\mathsf{D}') = \left(\frac{\mathsf{D}' + \mathcal{K}_H}{\mathcal{K}_H}\right)/H = \frac{\mathsf{D}'}{\mathcal{K}_H}/H.
$$

We have to show that this defines a Dirac homogeneous space of (G, D_G). Note first that if $(\bar{x}, \bar{\xi}) \in \mathsf{D}_{G/H}(eH)$, then there exists $(x, \xi) \in \mathsf{D}'(e) = \mathfrak{D}$ (see Remark 4.10) such that $T_eq_Hx = \bar{x}$ and $(T_eq_H)^*\bar{\xi} = \xi$. But then $(\bar{x}, \bar{\xi})$ is an element of $\mathfrak{D}_{G/H}$. The other inclusion can be shown in the same manner and we get $\mathsf{D}_{G/H}(eH) = \mathfrak{D}_{G/H}$. Choose then $gH \in G/H$ and $(\bar{v}, \bar{\alpha}) \in \mathsf{D}_{G/H}(gH)$, that is, $(\bar{v}, \bar{\alpha}) \in T_{gH}(G/H) \times T_{gH}(G/H)^*$ such that there exists $v \in T_gG$ with $T_gq_Hv = \bar{v}$ and $(v, (T_gq_H)^*\bar{\alpha}) \in \mathsf{D}'(g)$. Then we can write v as a sum $v = w + u$ with $w, u \in T_gG$ such that $(w, (T_gq_H)^*\bar{\alpha}) \in \mathsf{D}_G(g)$ and $(u, (T_gq_H)^*\bar{\alpha}) \in \mathfrak{D}^L(g)$, i.e., $(T_gL_{g^{-1}}u, (T_eL_g)^* \circ (T_gq_H)^*\bar{\alpha}) \in \mathfrak{D}$. Since $(T_eL_g)^* \circ (T_gq_H)^*\bar{\alpha} = (T_eq_H)^*(T_{eH}\sigma_g)^*\bar{\alpha}$, we get $(T_eq_HT_gL_{g^{-1}}u, (T_{eH}\sigma_g)^*\bar{\alpha}) \in \mathfrak{D}_{G/H} = \mathsf{D}_{G/H}(eH)$. Set $\bar{u} := T_eq_HT_gL_{g^{-1}}u$, then we have $T_{eH}\sigma_g\bar{u} = T_gq_Hu$ and hence

$$
\bar{v} = T_gq_Hw + T_gq_Hu = T_gq_Hw + T_{eH}\sigma_g\bar{u}.
$$

The proof of the last implication (4) ⇒ (1) is given by Remark 4.3. ◻

We have immediately the following corollary, which, together with the preceding theorem, classifies the Dirac structures on G/H that make $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of (G, D_G).

Corollary 4.12. *Let (G, D_G) be a Dirac Lie group, H a closed Lie subgroup of G and $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of (G, D_G). The Dirac structure $\mathsf{D}_{G/H}$ on G/H is then uniquely determined by $\mathsf{D}_{G/H}(eH)$ and (G, D_G).*

Proof. Since $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G, D_G), the subspace $\mathsf{D}_{G/H}(eH)$ satisfies $(*)$ by Remark 4.3, and $\mathfrak{D} = T_e q_H^* \mathsf{D}_{G/H}(eH)$ satisfies $(**)$ by Lemma 4.6. Define D' as above. Then, by the preceding theorem, we get that D' is right H-invariant and projects under q_H to a Dirac structure $q_H(\mathsf{D}')$. It is easy to check that $q_H(\mathsf{D}') = \mathsf{D}_{G/H}$. □

Remark 4.13.

(1) Since the vertical space of the right action of H on (G, D') denoted here by $\mathcal{V}_H$ is equal to $\mathfrak{h}^L$ and hence contained in G_0', the Dirac structure D' is the backward Dirac image of $\mathsf{D}_{G/H}$ defined on G by q_H (see Subsection ***2.1***).

(2) The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is easily shown to be a Lagrangian subspace of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ if and only if $\mathfrak{D}_{G/H}$ is a Lagrangian subspace of $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ satisfying $(**)$.

Corollary 4.14. *The Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ is integrable if and only if the smooth Dirac manifold (G, D') defined by $\mathsf{D}_{G/H}(eH)$ as in (4.1) is integrable.*

Proof. It is known by the theory of Dirac reduction that if an integrable Dirac manifold (M, D) is acted upon in a free and proper canonical way by a Lie group H, then the quotient Dirac manifold $(M/H, q_H(\mathsf{D}))$ is also integrable. Hence, if (G, D') is integrable, then $(G/H, \mathsf{D}_{G/H})$ is also integrable (see for instance Jotz and Ratiu (2011)).

For the converse implication, we deduce from the proof of Theorem 4.11 that $(\pounds_{\xi^L} X, \pounds_{\xi^L} \alpha)$ is an element of $\Gamma(\mathsf{D}')$ for all sections (X, α) of D' and Lie algebra elements $\xi \in \mathfrak{h}$. This yields that $\mathsf{D}' = \mathsf{D}' \cap \mathcal{K}_H^{\perp}$ satisfies

$$[\Gamma(\mathcal{K}_H), \Gamma(\mathsf{D}')] \subseteq \Gamma(\mathsf{D}' + \mathcal{K}_H).$$

We get from a result in Jotz et al. (2011) that D' is spanned by *right H-descending sections* $(X, \alpha) \in \Gamma(\mathsf{D}')$, that is, with $[X, \Gamma(\mathcal{V}_H)] \subseteq \Gamma(\mathcal{V}_H)$ and $\alpha \in \Gamma(\mathcal{V}_H{}^\circ)^H$. Hence, it suffices to show that if (X, α) and (Y, β) are such elements of $\Gamma(\mathsf{D}')$, then their bracket $[(X, \alpha), (Y, \beta)]$ is a section of D'.

Since (X, α) and (Y, β) are H-descending and $(G/H, \mathsf{D}_{G/H})$ is the Dirac quotient space of (G, D'), we find $(\bar{X}, \bar{\alpha})$ and $(\bar{Y}, \bar{\beta}) \in \Gamma(\mathsf{D}_{G/H})$ such that $X \sim_{q_H} \bar{X}$, $Y \sim_{q_H} \bar{Y}$, $\alpha = q_H^* \bar{\alpha}$, and $\beta = q_H^* \bar{\beta}$.

We have then $[X, Y] \sim_{q_H} [\bar{X}, \bar{Y}]$ and $\pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha = q_H^*(\pounds_{\bar{X}} \bar{\beta} - \mathbf{i}_{\bar{Y}} \mathbf{d}\bar{\alpha})$. If $(G/H, \mathsf{D}_{G/H})$ is integrable, the pair $[(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})] = ([\bar{X}, \bar{Y}], \pounds_{\bar{X}} \bar{\beta} - \mathbf{i}_{\bar{Y}} \mathbf{d}\bar{\alpha})$ is a section of $\mathsf{D}_{G/H}$. By construction of the Dirac quotient of a Dirac manifold by a smooth Dirac action, there exists a smooth vector field $Z \in \mathfrak{X}(G)$ such that $(Z, q_H^*(\pounds_{\bar{X}}\bar{\beta} - \mathbf{i}_{\bar{Y}}\mathbf{d}\bar{\alpha}))$ is an element of $\Gamma(\mathsf{D}')$ and $Z \sim_{q_H} [\bar{X}, \bar{Y}]$. But then there exists a smooth section $V \in \Gamma(\mathcal{V}_H) = \Gamma(\mathfrak{h}^L)$ such that $Z + V = [X, Y]$. Since $\mathfrak{h}^L \subseteq \mathsf{G}_0'$, this yields

$$[(X,\alpha),(Y,\beta)] = ([X,Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha) = (Z, \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha) + (V, 0) \in \Gamma(\mathsf{D}'),$$

and thus the Dirac manifold (G, D') is integrable. □

Remark 4.15.

(1) If $(G/H, \mathsf{D}_{G/H})$ is integrable, the Dirac structure D' is also integrable as we have seen above and the subbundle $\mathsf{G}_0' = {\mathfrak{g}_0'}^L$ is integrable in the sense of Frobenius. The vector subspace $\mathfrak{g}_0' \subseteq \mathfrak{g}$ is then a subalgebra and the integral leaf of G_0' through e is a Lie subgroup of G, which will be called J in the following. As in the proof of Lemma 3.19, we can show that the right action of J on G is canonical on (G, D'). Theorem 4.11 yields then

$$A_j(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})) \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$$

for all $j \in J$.

(2) We can also show that if (G, D_G) is N-invariant, then (G, D') is N-invariant. By Theorem 3.21, the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ defined in Definition 3.16 has image in $\mathfrak{p}_1$, and by Remark 3.36, we know then that the action of N on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ is trivial. Hence, we have $A_n(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})) = \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ for all $n \in N$, and we can apply Theorem 4.11.

4.3. Integrable Dirac homogeneous spaces. We consider here an *integrable* Dirac Lie group (G, D_G), a closed Lie subgroup H of G (with Lie algebra $\mathfrak{h}$) and a Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ of (G, D_G). As above, we consider the backward image $\mathfrak{D} = (T_eq_H)^*\mathsf{D}_{G/H}(eH)$ of $\mathsf{D}_{G/H}(eH)$ under T_eq_H, i.e.,

$$\mathfrak{D} = \{(x, (T_eq)^*\bar{\xi}) \mid x \in \mathfrak{g}, \bar{\xi} \in (\mathfrak{g}/\mathfrak{h})^* \text{ such that } (T_eqx, \bar{\xi}) \in \mathsf{D}_{G/H}(eH)\},$$

and the Dirac structure D' defined by $\mathfrak{D}$ on G as in (4.1) and Proposition 4.9.

Theorem 4.16. *The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ if and only if (G, D') (or equivalently $(G/H, \mathsf{D}_{G/H})$) is integrable.*

Proof. Choose $(x, \xi), (y, \eta)$ in $\mathfrak{D}$; then it follows that the pairs $(X_\xi + x^L, \xi^L)$ and $(X_\eta + y^L, \eta^L)$ are sections of D'. We have by Proposition 3.15, Definition 3.16, Proposition 3.24, and Remark 3.33:

(4.3)

$$
\begin{aligned}
&[(X_\xi + x^L, \xi^L), (X_\eta + y^L, \eta^L)]\\
&\quad = \left([X_\xi, X_\eta] + \pounds_{x^L} X_\eta - \pounds_{y^L} X_\xi + [x, y]^L,\ \pounds_{X_\xi}\eta^L - \mathbf{i}_{X_\eta}\mathbf{d}\xi^L + \pounds_{x^L}\eta^L - \mathbf{i}_{y^L}\mathbf{d}\xi^L\right)\\
&\quad \overset{(3.3)}{=} \Big(X_{[\xi,\eta]} + X_{\mathrm{ad}^*_x \eta} - (\mathrm{ad}^*_\eta x)^L - X_{\mathrm{ad}^*_y \xi} + (\mathrm{ad}^*_\xi y)^L + [x, y]^L,\\
&\qquad\qquad ([\xi, \eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi)^L\Big) + (Z, 0)\\
&\qquad\qquad \text{for some } Z \in \Gamma(\mathsf{G}_0)\\
&\quad = \left(X_{[\xi,\eta]+\mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi} + ([x, y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y)^L, ([\xi, \eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi)^L\right),
\end{aligned}
$$

where we have chosen the vector field $X_{[\xi,\eta]+\mathrm{ad}^*_x \eta-\mathrm{ad}^*_y \xi} := X_{[\xi,\eta]} - X_{\mathrm{ad}^*_y \xi} - X_{\mathrm{ad}^*_y \xi} + Z$.

If (G, D') is integrable, we have $[(X_\xi + x^L, \xi^L), (X_\eta + y^L, \eta^L)] \in \Gamma(\mathsf{D}')$, and hence its value at the neutral element e is an element of $\mathfrak{D}$ by Remark 4.10. But since $X_{[\xi,\eta]+\mathrm{ad}^*_x \eta-\mathrm{ad}^*_y \xi}(e)$ is an element of $\mathfrak{g}_0$, (4.3) yields

$$
\begin{aligned}
&[(X_\xi + x^L, \xi^L), (X_\eta + y^L, \eta^L)](e)\\
&\qquad \in ([x, y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y, [\xi, \eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi) + (\mathfrak{g}_0 \times \{0\}).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
&[(x + \mathfrak{g}_0, \xi), (y + \mathfrak{g}_0, \eta)]\\
&\qquad = ([x, y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y + \mathfrak{g}_0, [\xi, \eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}).
\end{aligned}
$$

For the converse implication, it is sufficient to show that for all $(x, \xi), (y, \eta) \in \mathfrak{D}$, we have

$$[(X_\xi + x^L, \xi^L), (X_\eta + y^L, \eta^L)] \in \Gamma(\mathsf{D}')$$

since D' is spanned by these sections. By hypothesis, we have

$$
\begin{aligned}
(4.4)\quad &[(x + \mathfrak{g}_0, \xi), (y + \mathfrak{g}_0, \eta)]\\
&\qquad = ([x, y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y + \mathfrak{g}_0, [\xi, \eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})
\end{aligned}
$$

for all $(x, \xi), (y, \eta) \in \mathfrak{D}$, and the claim follows using (4.3). □

We have proved the following theorem which is a generalization of the theorem in Drinfeld (1993).

Theorem 4.17. *Let (G, D_G) be a Dirac Lie group and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. The assignment*

$$\mathsf{D}_{G/H} \mapsto \mathfrak{D} = (T_e q_H)^* \mathsf{D}_{G/H}(eH)$$

gives a one-to-one correspondence between (G, D_G)-Dirac homogeneous structures on

G/H and Dirac subspaces $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ such that

(1) $(\mathfrak{g}_0 + \mathfrak{h}) \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times (\mathfrak{p}_1 \cap \mathfrak{h}^\circ)$
(2) $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ *is Lagrangian in* $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$
(3) $A_h(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})) \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ *for all* $h \in H$.

If the Dirac Lie group is integrable, then $(G/H, \mathsf{D}_{G/H})$ *is integrable if and only if* $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ *is a* subalgebra *of* $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Example 4.18.

(1) Let (G, D_G) be a Dirac Lie group and $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of (G, D_G) corresponding by Theorem 4.17 to the Lagrangian subspace $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$. Then, again by Theorem 4.17 applied to the Lie subgroup $\{e\}$ of G and the Dirac subspace $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$, we get that (G, D') is a Dirac homogeneous space of (G, D_G). If (G, D_G) is integrable, we recover the fact that (G, D') is integrable if and only if $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a *subalgebra* of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$, that is, if and only if $(G/H, \mathsf{D}_{G/H})$ is integrable.
(2) Choose a Dirac Lie group (G, D_G) and assume that the corresponding bracket on $\mathfrak{p}_1$ has image in $\mathfrak{p}_1$ and that the Lie subgroup N is closed in G. The Lagrangian subspace $\mathfrak{g}_0 \times \mathfrak{p}_1$ of $\mathfrak{g} \times \mathfrak{g}^*$ satisfies ($**$) and the corresponding Dirac structure D' is equal to D_G by definition. Since N corresponds to the Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ and fixes $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ pointwise by Remark 4.15, we get from Theorem 4.11 that the quotient $(G/N, q_N(\mathsf{D}_G))$ is a Dirac homogeneous space of the Dirac Lie group (G, D_G). We will study this particular Dirac homogeneous space in Section 5.

Remark 4.19. The previous theorem does not reduce, in the case of Poisson Lie groups, to the same theorem but with $\mathfrak{g}_0$ set to be $\{0\}$, as in many other statements of this work. Indeed, the theorem of Drinfeld (Drinfeld (1993)) gives a correspondence between *Poisson* homogeneous structures on G/H of a *Poisson* Lie group $(G, \{\cdot\,,\cdot\})$ and Lagrangian subalgebras $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ satisfying $A_h\mathfrak{D} \subseteq \mathfrak{D}$ for all $h \in H$ and *the equality* $\mathfrak{D} \cap (\mathfrak{g} \times \{0\}) = \mathfrak{h} \times \{0\}$ (see Drinfeld (1993)), that is, $\mathfrak{g}_0' = \mathfrak{h}$.

Here, we have $\mathfrak{D} \cap (\mathfrak{g} \times \{0\}) = \mathfrak{g}_0' \times \{0\}$ and we get the following cases:

(1) $\mathfrak{g}_0 \subseteq \mathfrak{h} = \mathfrak{g}_0'$: the Dirac homogeneous space is a Poisson homogeneous space of the Dirac Lie group (G, D_G). Furthermore, if $\mathfrak{g}_0 = \{0\}$, the Dirac Lie group is a Poisson Lie group and we are in the situation of Drinfeld's theorem. The case $\mathfrak{g}_0 = \mathfrak{h}$ (see the second part of Example 4.18) will be studied in Section 5.
(2) $\mathfrak{h} \subsetneq \mathfrak{g}_0'$: the Dirac homogeneous space has non-trivial G_0-distribution and is hence not a Poisson homogeneous space of (G, D_G). Therefore, in the case where $\mathfrak{g}_0 = \{0\}$, we obtain a Dirac homogeneous space of a Poisson Lie group.

Example 4.20. Consider an n-dimensional torus $G := \mathbb{T}^n$. In Corollary 3.17, we have recovered the fact that the only multiplicative Poisson structure on $\mathbb{T}^n$ is the trivial Poisson structure $\pi = 0$, that is $\mathsf{D}_\pi = \{0\} \oplus T^*\mathbb{T}^n$. The Lie algebra structure on $\mathfrak{g} \times \mathfrak{g}^*$ is given by $[(x, \xi), (y, \eta)] = ([x, y], \mathrm{ad}_x^* \eta - \mathrm{ad}_y^* \xi) = (0, 0)$ since the Lie group $\mathbb{T}^n$ is Abelian. Hence, every Dirac subspace of $\mathfrak{g} \times \mathfrak{g}^*$ is a Lagrangian subalgebra. Indeed, it is easy to verify that each left invariant Dirac structure on $\mathbb{T}^n$ is an integrable Dirac homogeneous space of the trivial Poisson Lie group $(\mathbb{T}^n, \pi = 0)$.

In general, if $(G, \pi = 0)$ is a trivial Poisson Lie group, the Lie algebra structure on $\mathfrak{g} \times \mathfrak{g}^*$ is given by $[(x, \xi), (y, \eta)] = ([x, y], \mathrm{ad}_x^* \eta - \mathrm{ad}_y^* \xi)$. The $(G, \pi = 0)$-homogeneous structures on G are here the left invariant Dirac structures $\mathfrak{D}^L$ on G. Hence, the integrable homogeneous Dirac structures on G are the left invariant Dirac structures $\mathfrak{D}^L$ such that $\mathfrak{D}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{g}^*$. But $\mathfrak{D}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{g}^*$ if and only if

$$\langle ([x, y], \mathrm{ad}_x^* \eta - \mathrm{ad}_y^* \xi), (z, \zeta) \rangle = \xi([y, z]) + \eta([z, x]) + \zeta([x, y]) = 0$$

for all (x, ξ), (y, η), and $(z, \zeta) \in \mathfrak{D}$. We recover Proposition 2.3 about integrability of a left-invariant Dirac structure on G, see also Miburn (2007).

5. The Poisson Lie Group Induced as a Dirac Homogeneous Space of a Dirac Lie Group if N is Closed in G

We will see in this section that if (G, D_G) is an integrable Dirac Lie group, such that the leaf N of the involutive subbundle G_0 through the neutral element e is a *closed* normal subgroup of G, then the Lie bialgebra $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1) \simeq (\mathfrak{g}/\mathfrak{g}_0, (\mathfrak{g}/\mathfrak{g}_0)^*)$ arises from a natural multiplicative Poisson structure π on the quotient G/N, that makes $(G/N, \pi)$ a Poisson homogeneous space of (G, D_G).

Theorem 5.1. *Let (G, D_G) be a Dirac Lie group such that the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ has image in $\mathfrak{p}_1$ and N is closed in G. The reduced Dirac structure $\mathsf{D}_{G/N} = q_N(\mathsf{D}_G)$ on G/N (that is a homogeneous Dirac structure of (G, D_G), see Example 4.18) is the graph of a skew-symmetric multiplicative bivector field π on G/N (as in Example 2.1). If (G, D_G) is integrable, the quotient $(G/N, \mathsf{D}_{G/N}) =: (G/N, \pi)$ is a Poisson Lie group, and the induced Lie bialgebra $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1) \simeq (\mathfrak{g}/\mathfrak{g}_0, (\mathfrak{g}/\mathfrak{g}_0)^*)$ as in Remark 3.33 is the Lie bialgebra defined by (G, D_G) as in Theorems 3.26 and 3.30.*

Since each normal subgroup of a simply connected Lie group G is closed (see Hilgert and Neeb (1991)), we have the following immediate corollary.

Corollary 5.2. *Let (G, D_G) be an integrable, simply connected Dirac Lie group. Then D_G is the pullback Dirac structure defined on G by $q_N : G \to G/N$ and a multiplicative Poisson bracket on G/N.*

Proof of Theorem 5.1. Since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$, the Lie subgroup N is normal in G. If it is closed in G, the left or right action of N on G is free and proper and the reduced space G/N is a Lie group. Let $q_N : G \to G/N$ be the projection.

The vertical distribution $\mathcal{V}_N$ of the left (right) action of N on G is the span of the right-invariant vector fields x^R, for all $x \in \mathfrak{g}_0$, that is, $\mathcal{V}_N = \mathsf{G}_0$. This yields $\mathcal{K}_N = \mathcal{V}_N \oplus \{0\} = \mathsf{G}_0 \oplus \{0\}$, and hence $\mathcal{K}_N^\perp = TG \oplus \mathsf{P}_1$. The intersection $\mathsf{D}_G \cap \mathcal{K}_N^\perp$ is consequently equal to D_G and has constant rank on G. Recall that the set of smooth local sections of $\mathsf{D}_{G/N}$ is given by

$$\left\{ (\bar{X}, \bar{\alpha}) \in \mathfrak{X}(G/N) \times \Omega^1(G/N) \middle| \begin{array}{c} \exists\, X \in \mathfrak{X}(G) \text{ such that } X \sim_{q_N} \bar{X} \\ \text{and } (X, q_N^* \bar{\alpha}) \in \Gamma(\mathsf{D}_G) \end{array} \right\}.$$

Since N lets (G, D_G) invariant by Theorem 3.21 and is connected by definition, we have $[\Gamma(\mathcal{K}_N), \Gamma(\mathsf{D}_G)] \subseteq \Gamma(\mathsf{D}_G)$ and we get using a result in Jotz et al. (2011) that D_G is spanned by its N-descending sections, that is, the pairs $(X, \alpha) \in \Gamma(\mathsf{D}_G)$ with $[X, \Gamma(\mathcal{V}_N)] \subseteq \Gamma(\mathcal{V}_N)$ and $\alpha \in \Gamma(\mathcal{V}_N^\circ)^N$ (see Jotz et al. (2011) or Jotz et al. (2011)). The vector subbundle $\mathsf{P}_1 = \mathcal{V}_N^\circ$ of T^*G is spanned by its descending sections since $\mathcal{V}_N$ is a smooth integrable subbundle of TG (see Jotz et al. (2011)), and the push-forwards of the descending sections of $\mathcal{V}_N^\circ$ are exactly the sections of the cotangent space $T^*(G/N)$ of G/N. Since D_G is spanned by its descending sections $(X, \alpha) \in \Gamma(\mathsf{D}_G)$, P_1 is in particular spanned by descending sections belonging to descending pairs in $\Gamma(\mathsf{D}_G)$. This shows that the cotangent distribution $\bar{\mathsf{P}}_1$ defined by $\mathsf{D}_{G/N}$ on G/N is equal to $T^*(G/N)$, and $(T_g q_N)^*(T^*_{gN} G/N) = \mathsf{P}_1(g)$ for all $g \in G$. This yields that $\mathsf{D}_{G/N}$ is the graph of a skew-symmetric bivector field π on G/N. Thus, if we show that $(G/N, \mathsf{D}_{G/N})$ is a Dirac Lie group, we will have simultaneously proved that $(G/N, \pi)$ is multiplicative by Example 3.2.

We thus show that $\mathsf{D}_{G/N}$ is multiplicative. Choose a product $gNg'N = gg'N \in G/N$ and $(\bar{v}_{gg'N}, \bar{\alpha}_{gg'N}) \in \mathsf{D}_{G/N}(gg'N)$. Then there exists a pair $(v_{gg'}, \alpha_{gg'}) \in \mathsf{D}_G(gg')$ such that

$$T_{gg'} q_N v_{gg'} = \bar{v}_{gg'N} \quad \text{and} \quad (T_{gg'} q_N)^* \bar{\alpha}_{gg'N} = \alpha_{gg'}.$$

Since D_G is multiplicative, we can find $w_g \in T_g G$ and $u_{g'} \in T_{g'} G$ such that

$$T_g R_{g'} w_g + T_{g'} L_g u_{g'} = v_{gg'}, \quad (w_g, (T_g R_{g'})^* \alpha(gg')) \in \mathsf{D}_G(g),$$

and

$$(u_{g'}, (T_{g'} L_g)^* \alpha_{gg'}) \in \mathsf{D}_G(g').$$

We have $\gamma_{g'} := (T_{g'} L_g)^* \alpha_{gg'} \in \mathsf{P}_1(g')$ and $\beta_g := (T_g R_{g'})^* \alpha_{gg'} \in \mathsf{P}_1(g)$ and hence, by the considerations above, there exist $\bar{\beta}_{gN} \in \bar{\mathsf{P}}_1(gN)$ and $\bar{\gamma}_{g'N} \in \bar{\mathsf{P}}_1(g'N)$ satisfying $(T_g q_N)^* \bar{\beta}_{gN} = \beta_g$ and $(T_{g'} q_N)^* \bar{\gamma}_{g'N} = \gamma_{g'}$. By construction of $\mathsf{D}_{G/N}$, we have then $(T_g q_N w_g, \bar{\beta}_{gN}) \in \mathsf{D}_{G/N}(gN)$ and $(T_{g'} q_N u_{g'}, \bar{\gamma}_{g'N}) \in \mathsf{D}_{G/N}(g'N)$.

We compute

$$\begin{aligned}&T_{g'N}L_{gN}T_{g'}q_N u_{g'} + T_{gN}R_{g'N}T_g q_N w_g\\ &\qquad = T_{g'g}q_N(T_{g'}L_g u_{g'} + T_g R_{g'} w_g) = T_{gg'}q_N v_{gg'} = \bar{v}_{gg'N},\end{aligned}$$

$$\begin{aligned}&(T_{g'}q_N)^*\big((T_{g'N}L_{gN})^*\bar{\alpha}_{gg'N}\big)\\ &\qquad = (T_{g'}L_g)^*\big((T_{gg'}q_N)^*\bar{\alpha}_{gg'N}\big) = (T_{g'}L_g)^*\,\alpha_{gg'} = \gamma_{g'} = (T_{g'}q_N)^*\bar{\gamma}_{g'N},\end{aligned}$$

and in the same manner, $(T_g q_N)^*((T_{gN}R_{g'N})^*\bar{\alpha}_{gg'N}) = \beta_g = (T_g q_N)^*\bar{\beta}_{gN}$. This leads to

$$(T_{g'N}L_{gN})^*(\bar{\alpha}_{gg'N}) = \bar{\gamma}_{g'N} \quad \text{and} \quad (T_{gN}R_{g'N})^*(\bar{\alpha}_{gg'N}) = \bar{\beta}_{gN}$$

since q_N is a smooth surjective submersion.

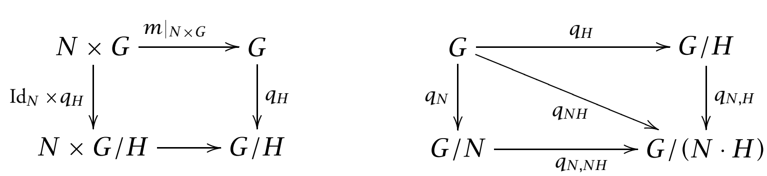
Hence, we have shown that $(G/N, \mathsf{D}_{G/N})$ is a Dirac Lie group, which we will sometimes write $(G/N, \pi)$ in the following, since $\mathsf{D}_{G/N}$ is the graph of a multiplicative skew-symmetric bivector field π on G/N.

The last statement is obvious with the considerations above and Proposition 3.18. □

Furthermore, we can show that each Dirac homogeneous structure on G/H, H a closed subgroup of G, can be assigned to a unique Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$ if the product $N \cdot H$ remains closed in G.

Let $(G/H, \mathsf{D}_{G/H})$ be a Dirac homogeneous space. We assume that the Lie subgroup $N \cdot H$ (with Lie algebra $\mathfrak{g}_0 + \mathfrak{h}$) is closed in G. The Lie group N acts by smooth left actions given by $n \cdot gH = ngH$ for all $n \in N$ and $g \in G$ on the homogeneous space G/H. This is well-defined since if $g^{-1}g' \in H$, we have $g^{-1}n^{-1}ng' \in H$ and hence $ngH = ng'H$. It is easy to check that the quotient of G/H by the left action of N is equal to the quotient of G by the right action of $N \cdot H$. Indeed, the class of gH in $(G/H)/N$ is the set $\{ngH \mid n \in N\} = NgH$. But since N is normal in G, this class is equal to gNH, which is the class of the element $g \in G$ in the quotient by the right action of $N \cdot H$ on G. Since $G/(N \cdot H)$ has the structure of a smooth regular quotient manifold and the maps q_H and $q_{N \cdot H}$ are smooth surjective submersions, the projection $q_{N,H} : G/H \to (G/H)/N$ is also a smooth surjective submersion.

In the second diagram, we have $(G/N)/(NH/N) \simeq G/(N \cdot H) \simeq (G/H)/N$.



We have the following theorem. We assume here for simplicity that the Dirac Lie group (G, D_G) is integrable, but analogous results can be shown for a Dirac Lie group that is invariant under the action of the induced Lie subgroup N.

Theorem 5.3. *Let $(G/H, \mathsf{D}_{G/H})$ be an integrable Dirac homogeneous space of the integrable Dirac Lie group (G, D_G) such that N and NH are closed in G.*

The Lie group N acts smoothly on the left on $(G/H, \mathsf{D}_{G/H})$ by Dirac actions, and the Lie group $N \cdot H$ acts smoothly on the right on the Dirac manifold (G, D') by Dirac actions.

The quotient Dirac structures on $G/(N \cdot H) \simeq (G/H)/N$ are equal and will be called $\mathsf{D}_{G/(NH)}$. The pair $(G/(N \cdot H), \mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$ and of the Dirac Lie group (G, D_G).

Conversely, if $(G/(NH), \mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$, then the pullbacks

$$(G/H, q_{N,H}^*(\mathsf{D}_{G/(NH)})) \quad \text{and} \quad (G, q_{NH}^*(\mathsf{D}_{G/(NH)}))$$

are Dirac homogeneous spaces of the Dirac Lie group (G, D_G).

Proof. Consider again the Dirac subspace $\mathfrak{D} = (T_e q_H)^* \mathsf{D}_{G/H}(eH) \subseteq \mathfrak{g} \times \mathfrak{p}_1$. We write $\bar{\mathfrak{D}}$ for the quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$. Since (G, D_G) and $(G/H, \mathsf{D}_{G/H})$ are integrable, we get from Theorem 4.17 that $\bar{\mathfrak{D}}$ is a subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ and from Remark 4.15 that $\bar{\mathfrak{D}}$ is N-invariant. We have then $A_{nh}\bar{\mathfrak{D}} \subseteq \bar{\mathfrak{D}}$ for all $nh \in N \cdot H$ and, by Theorem 4.17, the group $N \cdot H$ acts on (G, D') by Dirac actions and the quotient $(G/(N \cdot H), q_{NH}(\mathsf{D}')) =: (G/(N \cdot H), \mathsf{D}_{G/NH})$ is an integrable Dirac homogeneous space of the Dirac Lie group (G, D_G).

Next we show that the left action Φ of N on $(G/H, \mathsf{D}_{G/H})$ is canonical. Let $(\bar{X}, \bar{\alpha})$ be a section of $\mathsf{D}_{G/H}$. Then there exists $(X, \alpha) \in \Gamma(\mathsf{D}')$ such that $X \sim_{q_H} \bar{X}$ and $\alpha = q_H^* \bar{\alpha}$. We have $q_H \circ L_n = \Phi_n \circ q_H$ for all $n \in N$ and hence $L_n^* X \sim_{q_H} \Phi_n^* \bar{X}$ and $L_n^* \alpha = q_H^* \Phi_n^* \bar{\alpha}$. Since the action of N on (G, D') is canonical, we have $(L_n^* X, L_n^* \alpha) \in \Gamma(\mathsf{D}')$ and the pair $(\Phi_n^* \bar{X}, \Phi_n^* \bar{\alpha})$ is consequently a section of $\mathsf{D}_{G/H}$.

Let $\mathcal{V}$ be the vertical space of the action Φ of N on G/H and $\mathcal{K} = \mathcal{V} \oplus \{0\}$. The subbundle $\mathcal{V}$ of $T(G/H)$ is spanned by the projections to G/H of the right-invariant vector fields x^R on G, for all $x \in \mathfrak{g}_0$, and $\mathcal{V}^\circ$ is spanned by the push-forwards of the one-forms ξ^R, for all $\xi \in \mathfrak{p}_1 \cap \mathfrak{h}^\circ$. But since $\mathsf{D}' \cap \mathcal{K}_H^\perp = \mathsf{D}' \subseteq TG \oplus (\mathfrak{p}_1 \cap \mathfrak{h}^\circ)^R$, and $\mathsf{D}_{G/H} = q_H(\mathsf{D}')$, we get easily $\mathsf{D}_{G/H} \cap \mathcal{K}^\perp = \mathsf{D}_{G/H}$, which has consequently constant dimensional fibers on G/H. Thus, by the regular reduction theorem for Dirac manifolds, the quotient $((G/H)/N, q_{N,H}(\mathsf{D}_{G/H}))$ is a smooth Dirac manifold.

We have then to show that the quotient Dirac structure $q_{N,H}(\mathsf{D}_{G/H})$ is equal to $\mathsf{D}_{G/(NH)}$. If $(\tilde{X}, \tilde{\alpha})$ is a section of $q_{N,H}(\mathsf{D}_{G/H})$, then there exists $(\bar{X}, \bar{\alpha})$ in $\Gamma(\mathsf{D}_{G/H})$ such that $\bar{X} \sim_{q_{N,H}} \tilde{X}$ and $q_{N,H}^* \tilde{\alpha} = \bar{\alpha}$, but then there exists $(X, \alpha) \in \Gamma(\mathsf{D}')$ such that $X \sim_{q_H} \bar{X}$ and $\alpha = q_H^* \bar{\alpha}$. Then we have $\alpha = q_H^* q_{N,H}^* \tilde{\alpha} = q_{NH}^* \tilde{\alpha}$, $X \sim_{q_{NH}} \tilde{X}$, and $(\tilde{X}, \tilde{\alpha})$ is a section of $\mathsf{D}_{G/NH}$. This shows $q_{N,H}(\mathsf{D}_{G/H}) \subseteq \mathsf{D}_{G/NH}$ and hence equality since both Dirac structures have the same rank.

Finally, we show that $(G/(N\cdot H), \mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$. The Lie bialgebra of the Poisson Lie group $(G/N, \pi)$ is $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ with the bracket as in Theorem 3.34. We have

$$(T_{eN}q_{N,NH})^*\mathsf{D}_{G/(NH)}(eNH) = (T_e q_N)\big((T_e q_{NH})^*\mathsf{D}_{G/(NH)}(eNH)\big)$$
$$= \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}) \subseteq \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1.$$

By Remark 3.36, the action of G on $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ induces an action of G/N on $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$; this is exactly the action of G/N defined by the Poisson Lie group $(G/N, \pi)$ on its Lie bialgebra. Since $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is NH-invariant, it is NH/N-invariant under $\bar{A}$. Since $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a Lagrangian subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ and $(\mathfrak{g}_0 + \mathfrak{h})/\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}) \subseteq \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{h}^\circ \cap \mathfrak{p}_1$, we are done by Theorem 4.17.

For the converse statement, we use Remark 3.36 about the action $\bar{A}$ of G/N on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ and apply the first part of Example 4.18 to the Dirac Lie group $(G/N, \pi)$ and the closed subgroup NH/N of G/N and to the Dirac Lie group (G, D_G) and the closed subgroup NH of G. □

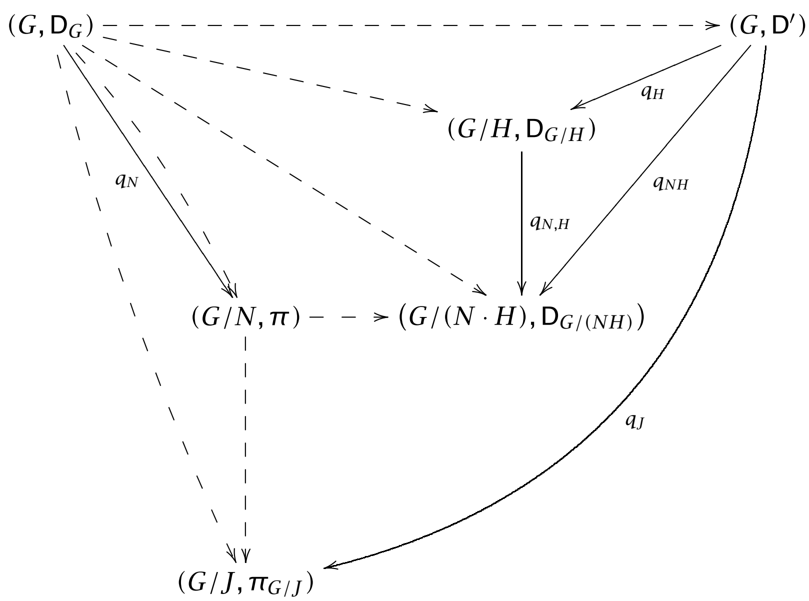
Now choose an integrable Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ of (G, D_G) and let (G, D') be the Dirac structure on G defined as in the preceding section. Since D' is integrable and G_0' is left invariant, it is an involutive subbundle of TG which is consequently integrable in the sense of Frobenius. Then the integral leaf J of G_0' through the neutral element e, which was defined in Remark 4.15, is a Lie subgroup of G.

Lemma 5.4. *If the Lie subgroup J is closed in G, it acts properly on the right on (G, D') by Dirac actions. The intersection $\mathsf{D}' \cap \mathcal{K}_J^\perp$, with $\mathcal{K}_J = \mathcal{V}_J \oplus \{0\} = \mathfrak{g}_0'^L \oplus \{0\}$, is equal to D' by definition of J and we can build the quotient $(G/J, q_J(\mathsf{D}'))$, where $q_J : G \to G/J$ is the projection.*

Furthermore, since $N \subseteq J$ is a normal subgroup, we can build the quotient Lie group J/N if N is closed in G. It acts properly on the right on G/N and we can see that $G/J \simeq (G/N)/(J/N)$ as a homogeneous space of G/N.

Proof. We have seen in Remark 4.15 that if (G, D') is integrable, we have $A_j\bar{\mathfrak{D}} = \bar{\mathfrak{D}}$ for all $j \in J$. By Theorem 4.11 (note that all the hypotheses are satisfied since $\mathfrak{g}_0' \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{p}_1'$, the Dirac subspace $\mathfrak{D}$ is equal to the pullback $\mathfrak{D} = T_e q_J^*(T_e q_J \mathfrak{D})$), the Dirac manifold (G, D') is right J-invariant. □

In the following diagram, the dashed arrows join Dirac Lie groups to their Dirac homogeneous spaces.



Theorem 5.5. *Under the hypotheses of Lemma 5.4, the pair* $(G/J, q_J(\mathsf{D}')) =:$ $(G/J, \pi_{G/J})$ *is a Poisson homogeneous space of the Dirac Lie group* (G, D_G) *and of the Poisson Lie group* $(G/N, \pi)$.

Proof. By Theorem 4.17 and Lemma 5.4, $(G/J, \pi_{G/J})$ is a Dirac homogeneous space of the integrable Dirac Lie group (G, D_G). The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a Lagrangian subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ with $(\mathfrak{g}_0 + \mathfrak{g}_0') \times \{0\} = \mathfrak{g}_0' \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{p}_1' = \mathfrak{g} \times ((\mathfrak{g}_0')^\circ \cap \mathfrak{p}_1)$. Furthermore, $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is A_j-invariant and hence also $\bar{A}_{jN}$-invariant by Remark 3.36 for all $j \in J$ (see also the proof of the preceding theorem). Hence, $(G/J, \pi_{G/J})$ is also a Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$.

Note that since $\mathsf{G}_0' = \mathcal{V}_J$, the quotient Dirac structure $\mathsf{D}_{G/J} := q_J(\mathsf{D}')$ has vanishing characteristic distribution and is hence a Poisson manifold (see the proof of Theorem 5.1) □

Finally, we give examples where it is not possible to build the diverse quotients as above. Let $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ be the universal covering of the Lie group $\mathrm{SL}_2(\mathbb{R})$.

Example 5.6.

(1) Consider the Lie group

$$G = \left(\mathbb{T}^2 \times \widetilde{\mathrm{SL}_2(\mathbb{R})}\right) / \Gamma,$$

where Γ is the group homomorphism $Z(\widetilde{\mathrm{SL}_2(\mathbb{R})}) \simeq \mathbb{Z} \to \mathbb{T}^2$ given by $\Gamma(z) = (e^{i\sqrt{2}z}, e^{iz})$ for all $z \in Z(\widetilde{\mathrm{SL}_2(\mathbb{R})})$ (or more generally a group homomorphism with dense image in $\mathbb{T}^2$).

The graph of Γ is a discrete normal subgroup of $\mathbb{T}^2 \times \widetilde{\mathrm{SL}_2(\mathbb{R})}$ and hence the quotient G is a Lie group. The Lie algebra of G is equal to the direct sum of Lie algebras $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{sl}_2(\mathbb{R})$ and has hence $\mathfrak{g}_0 := \mathfrak{sl}_2(\mathbb{R})$ as an ideal. The corresponding Lie subgroup N of G corresponds to the images of the elements of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ in G and is hence by construction not closed in G. Let G be endowed with the trivial Dirac structure such that $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$; the quotient Poisson Lie group $(G/N, \pi)$ does not exist here.

(2) Consider $G = \mathbb{T}^4 \times \mathbb{R}$ with coordinates s_1, s_2, s_3, s_4, t and the integrable Dirac structure given by $\mathfrak{g}_0 = \operatorname{span}\{x_1\}$, $\mathfrak{p}_1 = \operatorname{span}\{\xi_2, \xi_3, \xi_4, \xi_5\}$ and D_G the (necessarily) trivial multiplicative Dirac structure

$$\mathsf{D}_G = \mathfrak{g}_0^L \oplus \mathfrak{p}_1^L = \operatorname{span}\{(x_1^L, 0), (0, \xi_2^L), (0, \xi_3^L), (0, \xi_4^L), (0, \xi_5^L)\},$$

where

$$\begin{aligned}
\xi_2 &= \sqrt{2}\mathbf{d}s_1(0) - \mathbf{d}s_2(0), & x_1 &= \partial_{s_1}(0) + \sqrt{2}\partial_{s_2}(0) + \partial_t(0),\\
\xi_3 &= \mathbf{d}s_1(0) - \mathbf{d}t(0), & x_2 &= -\partial_{s_2}(0),\\
\xi_4 &= \mathbf{d}s_3(0), & x_3 &= -\partial_t(0),\\
\xi_5 &= \mathbf{d}s_4(0), & x_4 &= \partial_{s_3}(0),\\
\xi_1 &= \mathbf{d}s_1(0), & x_5 &= \partial_{s_4}(0).
\end{aligned}$$

The group N is then equal to $N = \{(e^{ti}, e^{\sqrt{2}ti}, 1, 1, t) \mid t \in \mathbb{R}\}$ and is closed in G as the graph of a smooth map $\mathbb{R} \to \mathbb{T}^4$. The quotient $(G/N, \pi)$ is a torus $\mathbb{T}^4$ with trivial Poisson Lie group structure. Consider the subgroup $H = \{(e^{2ti}, e^{2\sqrt{2}ti}, 1, 1, t) \mid t \in \mathbb{R}\}$ of G. Then H is closed in G and each Dirac subspace $\mathfrak{D} \subseteq \mathbb{R}^5 \times \mathbb{R}^{5*}$ with $(\mathfrak{h} + \mathfrak{g}_0) \times \{0\} \subseteq \mathfrak{g} \times (\mathfrak{h}^\circ \cap \mathfrak{p}_1)$ induces a homogeneous Dirac structure on $G/H \simeq \mathbb{T}^4$.

The subgroup $N \cdot H$ of G is dense in $\mathbb{T}^2 \times \{1\}^2 \times \mathbb{R} \subseteq G$ and is hence not closed in G.

Consider now the Dirac subspace

$$\begin{aligned}\mathfrak{D} := \operatorname{span}\Big\{&(\partial_{s_1}(0), 0),\ (\partial_{s_2}(0), 0),\ (\sqrt{3}\partial_{s_3}(0) + \partial_{s_4}(0), 0),\\ &(\partial_t(0), 0),\ (X, \mathbf{d}s_3(0) - \sqrt{3}\mathbf{d}s_4(0))\Big\} \text{ of } \mathfrak{g} \times \mathfrak{g}^*\end{aligned}$$

with $X \in \mathfrak{g}$ an arbitrary vector satisfying $(\mathbf{d}s_3(0) - \sqrt{3}\mathbf{d}s_4(0))(X) = 0$. The Dirac structure $\mathsf{D}' = \mathfrak{D}^L$ defines a (G, D_G)-homogeneous Dirac structure of since $\mathfrak{g}_0 + \mathfrak{h} \subseteq \mathfrak{g}_0'$, but the leaf J of G_0' through the neutral element 0 is equal to $J = \{(e^{\theta i}, e^{\varphi i}, e^{\sqrt{3}ti}, e^{ti}, s) \mid \theta, \varphi, t, s \in \mathbb{R}\}$ and thus dense in G.

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