VB-COURANT ALGEBROIDS AND LIE 2-ALGEBROIDS.

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Abstract. This paper is the second of a series of papers on the geometrisation of $\mathbb{N}$-manifolds of degree 2. An equivalence between the category of linear Courant algebroids and the category of Lie 2-algebroids is explained in detail.

As an application, we prove that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we explain our result geometrically.

Contents

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1. Introduction

Supermanifolds were introduced in the 1970's by physicists, as a formalism to describe supersymmetric field theories, and have been extensively studied since then (see e.g. [29, 34] and references therein). A supermanifold is a smooth manifold the algebra of functions of which is enriched by anti-commuting coordinates. Supermanifolds with an additional $\mathbb{Z}$-grading appeared in the late 1990's in relation to Poisson geometry and Lie and Courant algebroids [30, 28, 35]. An equivalence between Courant algebroids and $\mathbb{N}$-manifolds of degree 2 endowed with a symplectic structure and a compatible homological vector field [28] is at the heart of the current interest in $\mathbb{N}$-graded manifolds, as this algebraic description of Courant algebroids leads to possible paths to their integration [30, 14, 24]. In [10] we showed how the category of $\mathbb{N}$-manifolds of degree 2 is equivalent to a category of double vector bundles endowed with a linear metric. In this paper we extend this correspondence to an equivalence between the category of $\mathbb{N}$-manifolds of degree 2 endowed with a homological vector field and a category of VB-Courant algebroids, i.e. metric double vector bundles endowed with a linear Courant algebroid structure (see also [13]).

Mehta defined and studied $\mathbb{Z}$-graded manifolds in his PhD thesis [23]; here we concentrate on $\mathbb{N}$-graded manifolds. An **$\mathbb{N}$-manifold** $\mathcal{M}$ of degree n and dimension $(p; r_1, \ldots, r_n)$ is a sheaf of $\mathbb{N}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras over a smooth p-dimensional manifold M, that is locally freely generated by $r_1 + \ldots + r_n$ elements $\xi_1^1, \ldots, \xi_1^{r_1}$, $\xi_2^1, \ldots, \xi_2^{r_2}, \ldots$, $\xi_n^1, \ldots, \xi_n^{r_n}$ with ξ_i^j of degree i for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r_i\}$. That is, we can write

$$C^\infty_U(\mathcal{M}) \simeq C^\infty_U(M)[\xi_1^1, \ldots, \xi_1^{r_1}, \xi_2^1, \ldots, \xi_2^{r_2}, \ldots, \xi_n^1, \ldots, \xi_n^{r_n}],$$

with $\xi_i^j \cdot \xi_k^l = (-1)^{ik} \xi_k^l \cdot \xi_i^j$, for $U \subseteq M$ open and small enough. A morphism $\mu\colon \mathcal{N} \dashrightarrow \mathcal{M}$ of $\mathbb{N}$-manifolds over a smooth map $\mu_0\colon N \to M$ of the underlying smooth manifolds is a morphism $\mu^\star\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{N})$ of sheaves of graded algebras over $\mu_0^*\colon C^\infty(M) \to C^\infty(N)$.

For instance, an $\mathbb{N}$-graded manifold of degree 1 is a locally free and finitely generated sheaf of $C^\infty(M)$-modules, hence canonicaly isomorphic to the set of sections of a vector bundle: $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet E^*)$ for a vector bundle $E \to M$. We write then $\mathcal{M} = E^*[1]$.

A vector field ϕ of degree j on an $\mathbb{N}$-graded manifold is a graded derivation that increases the degree by j: $|\phi(\xi)| = j + |\xi|$ for a homogeneous element $\xi \in C^\infty(\mathcal{M})$. The Lie bracket of (homogeneous) graded vector fields is defined by $[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$. An $\mathbb{N}\mathcal{Q}$-manifold of degree 1 is an $\mathbb{N}$-graded manifold $\mathcal{M}$ of degree 1, with a vector field $\mathcal{Q}$ of degree 1 that commutes with itself: $[\mathcal{Q}, \mathcal{Q}] = 0$. The vector field $\mathcal{Q}$ is then called a **homological vector field**. On an $\mathbb{N}$-manifold $\mathcal{M} = E^*[1]$ of degree 1, a graded vector field $\mathcal{Q}$ of degree 1 must be defined by $\mathbf{d}_E\colon \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^{\bullet+1} E^*)$ induced by an anchor $\rho\colon E \to TM$ and a skew-symmetric bracket $[\cdot\,,\cdot]\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ that satisfies a Leibniz identity with ρ_E in both entries. Then $0 = \mathbf{d}_E^2(f) = \mathbf{d}_E(\rho_E^* \mathbf{d} f)$ for all $f \in C^\infty(M)$ is equivalent to ρ_E being compatible with the bracket, and $\mathbf{d}_E^2 \tau = 0$ for all $\tau \in \Gamma(Q^*)$ is equivalent to the Jacobi identity for the bracket. Hence, $\mathbb{N}\mathcal{Q}$-manifolds of degree 1 are equivalent to Lie algebroids. This result, due to Vaintrob [33], is at the origin of the terminology of *Lie n-algebroids* for $\mathbb{N}\mathcal{Q}$-manifolds of degree $n \geq 1$.

In his PhD thesis [13], Li-Bland established abstract correspondences between Lie 2-algebroids and VB-Courant algebroids and between *Poisson* Lie 2-algebroids and LA-Courant algebroids[1] [13]. We make the first of these correspondences more concrete by giving a simple description of the equivalence of the category of VB-Courant algebroids with the category of Lie 2-algebroid. More precisely, this paper explains in detail how to construct a *decomposed* VB-Courant algebroid from a *split* Lie 2-algebroid, and vice-versa. In order to do this, we give a simplified definition of split Lie 2-algebroids, and of their duals, the Dorfman 2-representations, which resemble very much 2-term representations up to homotopy (*2-representations* in our terminology). This analogy shows why Lie 2-algebroids are understood as Lie algebroids up to homotopy.

As an application, we find several new examples of (split) Lie 2-algebroids, and we show in particular that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid (see Theorem 5.1), just as the bicrossproduct of a matched pair of Lie algebroid representations is a Lie algebroid. This result is independently interesting, because it finally unifies in a natural framework the two notions of double of a matched pair of representations: A matched pair of representations of two Lie algebroids A and B over M defines a Lie algebroid structure on $A \oplus B$ [25], called[2] the *bicrossproduct of the matched pair.* The matched pair defines also a double Lie algebroid $A \times_M B$ with sides A and B and trivial core [21]. This double Lie algebroid is called the *double of the matched pair.* Similarly, we know that a matched pair of 2-representations has a split Lie 2-algebroid as bicrossproduct (§5), and a decomposed double Lie algebroid as double [7]. The split Lie 2-algebroid is exactly the $\mathbb{N}$-geometric counterpart of the VB-Courant algebroid that is equivalent to the VB-bialgebroid defined by the double Lie algebroid (see §5.2). In other words, decomposed VB-bialgebroids are equivalent to matched pairs of 2-representations. The case of matched pairs of representations and their bicrossproducts and doubles are in fact a special (degenerate) case of this equivalence; namely the one of a VB-bialgebroid over a trivial base, which is equivalent to a double Lie algebroid with trivial core.

Original motivation. Let us explain in more detail our methodology and our original motivation. A VB-Lie algebroid is a double vector bundle with one side $D \to B$ endowed with a Lie algebroid bracket and an anchor that are *linear*. Gracia-Saz and Mehta prove in [8] that linear decompositions of VB-algebroids are equivalent to super-representations, or in other words, to 2-representations.

The definition of a VB-Courant algebroid is very similar to the one of VB-algebroids. The Courant bracket, the anchor and the non-degenerate pairing all have to be linear. In [9] we prove that the standard Courant algebroid over a vector bundle can be decomposed into a connection, a Dorfman connection, a curvature term and a vector bundle map, in a manner that resembles very much the one in [8]. In other words, as linear splittings of the tangent space TE of a vector bundle E are equivalent to linear connections on the vector bundle, linear splittings of the Pontryagin bundle $TE \oplus T^*E$ over E are equivalent to a certain class of Dorfman connections [9].

[1]We study LA-Courant algebroids in the third part of this work [11].

[2]It is sometimes also called the double of the matched pair, but for clarity we will not use this terminology.

Our original goal in this project was to show that this is in fact a very special case of a general result on linear splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's work [8].

Outline, main results and applications. This paper is organised as follows.
In **Section 2**. we describe the main result in [10] and we recall the background on VB-algebroids and representations up to homotopy that will be necessary for our main application on the bicrossproduct of a matched pair of 2-representations.
In **Section 3**. we start by recalling necessary background on Courant algebroids, Dirac structures and Dorfman connections. Then we formulate in our own manner Sheng and Zhu's definition of split Lie 2-algebroids [31], before dualising it and obtaining the notion of Dorfman 2-representation. Then we write in coordinates the homological vector field corresponding to a split Lie 2-algebroid, showing where the components of the split Lie 2-algebroid appear. In Section 3.4, we give several classes of examples of split Lie 2-algebroids, introducing in particular the standard split Lie 2-algebroids defined by a vector bundle. Finally we describe morphisms of split Lie 2-algebroids.
In **Section 4**. we give the definition of VB-Courant algebroids [13] and we relate Dorfman 2-representations with Lagrangian splittings of VB-Courant algebroids, in the spirit of Gracia-Saz and Mehta's description of split VB-algebroids via 2-term representations up to homotopy [8]. Then we describe the VB-Courant algebroids corresponding to the examples of split Lie 2-algebroids found in the previous section, and we prove that the equivalence of categories established in [10] induces an equivalence of the category of VB-Courant algebroids with the category of Lie 2-algebroids.
In **Section 5**. we construct the bicrossproduct of a matched pair of 2-representations and prove that it is a split Lie 2-algebroid. We then explain geometrically this result by studying VB-bialgebroids and double Lie algebroids.
Appendices. We give in Section A the proof of our main theorem, describing decomposed VB-Courant algebroids via Dorfman 2-representations. In Section B we quickly recall how double vector bundles and their splittings are dualised.

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After this work was completed, the author learned that Fernando del Carpio-Marek has independently found, mostly through different methods, results similar to some of hers in his PhD thesis in preparation [5].

Prerequisites, notation and conventions. We write $p_M\colon TM\to M$, $q_E\colon E\to M$ for vector bundle maps. For a vector bundle $Q\to M$ we often identify without further mentioning the vector bundle $(Q^*)^*$ with Q via the canonical isomorphism. We write $\langle\cdot\,,\cdot\rangle$ or $\langle\cdot\,,\cdot\rangle_M$ the canonical pairing of a vector bundle with its dual; i.e. $\langle a_m,\alpha_m\rangle=\alpha_m(a_m)$ for $a_m\in A$ and $\alpha_m\in A^*$. We use several different pairings; in general, which pairing is used is clear from its arguments. Given a section ε of E^*, we always write $\ell_\varepsilon \colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$.

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of local smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the sheaf of local sections of E will be written $\Gamma(E)$. Let $f \colon M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be ***f*-related** if $Tf \circ X = Y \circ f$ on $\mathrm{Dom}(X) \cap f^{-1}(\mathrm{Dom}(Y))$. We write then $X \sim_f Y$. In the same manner, if $\phi \colon A \to B$ is a vector bundle morphism over $\phi_0 \colon M \to N$, then a section $a \in \Gamma_M(A)$ is ϕ-related to $b \in \Gamma_N(B)$ if $\phi(a(m)) = b(\phi_0(m))$ for all $m \in M$. We write then $a \sim_\phi b$. The dual of the morphism ϕ is in general not a morphism of vector bundles, but a relation $R_{\phi^*} \subseteq A^* \times B^*$ defined as follows:

$$R_{\phi^*} = \{(\phi_m^* \beta_{\phi_0(m)}, \beta_{\phi_0(m)}) \mid m \in M, \beta_{\phi_0(m)} \in B^*_{\phi_0(m)}\},$$

where $\phi_m \colon A_m \to B_{\phi_0(m)}$ is the morphism of vector spaces.

We will say 2-representations for 2-term representations up to homotopy. We write "$[n]$-manifold" for "$\mathbb{N}$-manifolds of degree n". This notation is to avoid confusions with n-manifolds, which are usually understood as smooth manifolds of *dimension* n. Let E_1 and $, E_2$ be smooth vector bundles of finite ranks r_1, r_2 over M and assign the degree i to the fiber coordinates of E_i, for each $i = 1, \ldots, n$. The direct sum $E = E_1 \oplus E_2$ is a graded vector bundle with grading concentrated in degrees 1 and 2. The [2]-manifold $E_1[1] \oplus E_2[2]$ has local basis sections of E_i as local generators of degree i, for $i = 1, 2$, and so dimension $(p; r_1, r_2)$. A [2]-manifold $\mathcal{M} = E_1[1] \oplus E_2[2]$ defined in this manner by a graded vector bundle is called a **split** **[2]-manifold**. In other words, we have $C^\infty(\mathcal{M})^0 = C^\infty(M)$, $C^\infty(\mathcal{M})^1 = \Gamma(E_1)$ and $C^\infty(\mathcal{M})^2 = \Gamma(E_2 \oplus \wedge^2 E_1)$. A morphism $\mu \colon F_1[1] \oplus F_2[2] \dashrightarrow E_1[1] \oplus E_2[2]$ of split [2]-manifolds over the bases M and N, respectively, consists of a smooth map $\mu_0 \colon N \to M$, three vector bundle morphisms $\mu_1 \colon F_1^* \to E_1^*$, $\mu_2 \colon F_2^* \to E_2^*$ and $\mu_{12} \colon \wedge^2 F_1^* \to E_2^*$ over μ_0. The map $\mu^\star$ sends a degree 1 function $\xi \in \Gamma(E_1)$ to ${\mu_1}^\star \xi \in \Gamma(F_1)$ and a degree 2-function $\xi \in \Gamma(E_2)$ to ${\mu_2}^\star \xi + \mu_{12}^\star \xi \in \Gamma(F_2 \oplus \wedge^2 F_1)$.

We refer to [26, 20, 8] for more details on the following notions. We write $(D; A, B; M)$ for a double vector bundle, i.e. a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

with all four sides vector bundles projections, such that π_B is a vector bundle morphism over q_A; such that $+_B : D \times_B D \to D$ is a vector bundle morphism over $+ : A \times_M A \to A$, and such that the scalar multiplication $\mathbb{R} \times D \to D$ in the bundle $D \to B$ is a vector bundle morphism over the scalar multiplication $\mathbb{R} \times A \to A$. The corresponding statements for the operations in the bundle $D \to A$ follow. Recall that the condition that each addition in D is a morphism with respect to the other is exactly

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4) \tag{1}$$

for apropriate $d_1, d_2, d_3, d_4 \in D$. The vector bundles A and B are called the side bundles. The core C of a is the intersection of the kernels of π_A and of π_B. It has a natural vector bundle structure over M, the projection of which we call $q_C\colon C\to M$. We denote by $C_m \ni c \longmapsto \overline{c} \in \pi_A^{-1}(0^A_m)\cap \pi_B^{-1}(0^B_m)$ the inclusion $C\hookrightarrow D$.

For a smooth section $c\colon M\to C$, the corresponding core section $c^\dagger\colon B\to D$ is defined as $c^\dagger(b_m)=\tilde{0}_{b_m}+_A\overline{c(m)}$, $m\in M$, $b_m\in B_m$. We denote the corresponding core section $A\to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B is written $\Gamma^c_B(D)$. A section $\xi\in\Gamma_B(D)$ is linear if $\xi\colon B\to D$ is a bundle morphism from $B\to M$ to $D\to A$ over a section $a\in\Gamma(A)$. The space of linear sections of D over B is written $\Gamma^\ell_B(D)$. A section $\psi\in\Gamma(B^*\otimes C)$ defines a linear section $\widetilde{\psi}\colon B\to D$ over the zero section $0^A\colon M\to A$ by $\widetilde{\psi}(b_m)=\tilde{0}_{b_m}+_A\overline{\psi(b_m)}$ for all $b_m\in B$. We call $\widetilde{\psi}$ a core-linear section. The space of sections of $D\to B$ is generated as a $C^\infty(B)$-module by its linear and core sections.

Let A, B, C be vector bundles over M. The decomposed double vector bundle with sides A and B and core C is $D=A\times_M B\times_M C$ with the vector bundle structures $D=q_A^!(B\oplus C)\to A$ and $D=q_B^!(A\oplus C)\to B$. In particular, the fibered product $A\times_M B$ is a double vector bundle over the sides A and B and its core is the trivial bundle over M.

A linear splitting[3] of $(D;A,B;M)$ is an injective morphism of double vector bundles $\Sigma\colon A\times_M B\hookrightarrow D$ over the identity on the sides A and B. A linear splitting Σ of a double vector bundle D is equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$-modules

$$0\longrightarrow \Gamma(B^*\otimes C)\hookrightarrow \Gamma^\ell_B(D)\longrightarrow\Gamma(A)\longrightarrow 0,$$

where the third map is the map that sends a linear section (ξ,a) to its base section $a\in\Gamma(A)$. The splitting σ_A will be called a horizontal lift or simply a lift. Given Σ, the horizontal lift $\sigma_A\colon\Gamma(A)\to\Gamma^\ell_B(D)$ is given by $\sigma_A(a)(b_m)=\Sigma(a(m),b_m)$ for all $a\in\Gamma(A)$ and $b_m\in B$. By the symmetry of a linear splitting, we find that a lift $\sigma_A\colon\Gamma(A)\to\Gamma^\ell_B(D)$ is equivalent to a lift $\sigma_B\colon\Gamma(B)\to\Gamma^\ell_A(D)$: $\sigma_B(b)(a(m))=\sigma_A(a)(b(m))$ for all $a\in\Gamma(A)$, $b\in\Gamma(B)$. Note finally that two linear splittings $\Sigma^1,\Sigma^2\colon A\times_M B\to D$ differ by a section ϕ of $A^*\otimes B^*\otimes C\simeq \operatorname{Hom}(A,B^*\otimes C)\simeq\operatorname{Hom}(B,A^*\otimes C)$ in the following sense. For each $a\in\Gamma(A)$ the difference $\sigma^2_A(a)-_B\sigma^1_A(a)$ of horizontal lifts is the core-linear section defined by $\phi(a)\in\Gamma(B^*\otimes C)$. By symmetry, $\sigma^2_B(b)-_A\sigma^1_B(b)=\widetilde{\phi(b)}$ for each $b\in\Gamma(B)$.

The space of linear sections of D is a locally free and finitely generated $C^\infty(M)$-module (this follows from the existence of local splittings). Hence, there is a vector bundle $\widehat{A}$ over M such that $\Gamma^l_B(D)$ is isomorphic to $\Gamma(\widehat{A})$ as $C^\infty(M)$-modules. The vector bundle $\widehat{A}$ is called the fat vector bundle defined by $\Gamma^l_B(D)$. The short exact sequence above induces a short exact sequence of vector bundles

$$0\longrightarrow B^*\otimes C\hookrightarrow\widehat{A}\longrightarrow A\longrightarrow 0.$$

2. Preliminaries

In this section we recall the necessary background on VB-algebroids, double Lie algebroids and VB-bialgebroids, and we summarise the main result in [10] on the correspondence between double vector bundles endowed with a linear metric and $\mathbb{N}$-manifolds of degree 2.

[3]Each double vector bundle admits a linear splitting, see [7] for comments on this, and for references.

2.1. **VB-algebroids, double Lie algebroids, VB-bialgebroids.** Let $(D; A, B; M)$ be a double vector bundle

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\big\downarrow & & \big\downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

with core C. Then $(D \to B; A \to M)$ is a **VB-algebroid** ([18]; see also [8]) if $D \to B$ has a Lie algebroid structure the anchor of which is a bundle morphism $\Theta_B \colon D \to TB$ over $\rho_A \colon A \to TM$ and such that the Lie bracket is linear:

$$[\Gamma^\ell_B(D), \Gamma^\ell_B(D)] \subset \Gamma^\ell_B(D), \qquad [\Gamma^\ell_B(D), \Gamma^c_B(D)] \subset \Gamma^c_B(D), \qquad [\Gamma^c_B(D), \Gamma^c_B(D)] = 0.$$

The vector bundle $A \to M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1, \xi_2 \in \Gamma^\ell_B(D)$ are linear over $a_1, a_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]$ is linear over $[a_1, a_2]$. We also say that the Lie algebroid structure on $D \to B$ is linear over the Lie algebroid $A \to M$. Note that since the anchor Θ_B is linear, it sends a core section $c^\dagger$, $c \in \Gamma(C)$ to a vertical vector field on B. This defines the **core-anchor** $\partial_B \colon C \to B$; for $c \in \Gamma(C)$ we have $\Theta_B(c^\dagger) = (\partial_B c)^\uparrow$ (see [17]).

Before we discuss the linear splittings of VB-algebroids, let us discuss the following fundamental class of examples.

Example 2.1 (The tangent double of a vector bundle)**.** Let $q_E \colon E \to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE \to TM$ are the derivatives of the structure maps of $E \to M$. The space TE is a double vector bundle with core bundle $E \to M$. The map$^{-} \colon E \to p_E^{-1}(0^E) \cap (Tq_E)^{-1}(0^{TM})$ sends $e_m \in E_m$ to $\bar{e}_m = \frac{d}{dt}\big|_{t=0} te_m \in T_{0^E_m}E$.

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\big\downarrow & & \big\downarrow{\scriptstyle q_E} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

The core vector field corresponding to $e \in \Gamma(E)$ is the vertical lift $e^\uparrow \colon E \to TE$, i.e. the vector field with flow $\phi \colon E \times \mathbb{R} \to E$, $\phi_t(e'_m) = e'_m + te(m)$. An element of $\Gamma^\ell_E(TE) = \mathfrak{X}^\ell(E)$ is called a **linear vector field**. A linear vector field $\xi \in \mathfrak{X}^l(E)$ covering $X \in \mathfrak{X}(M)$ is equivalent to a derivation $D_\xi \colon \Gamma(E) \to \Gamma(E)$ over $X \in \mathfrak{X}(M)$ (see e.g. [20]). The precise correspondence is given by[4]

$$\xi(\ell_\varepsilon) = \ell_{D^*_\xi(\varepsilon)} \quad \text{and} \quad \xi(q_E^* f) = q_E^*(X(f)) \tag{2}$$

for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$. We will write $\widehat{D}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation D of $\Gamma(E)$. A linear splitting Σ for $(TE; TM, E; M)$ is equivalent to a linear connection on E: $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is defined by $\sigma_{TM}(X) = \widehat{\nabla_X}$ for all $X \in \mathfrak{X}(M)$. Conversely, a connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defines a lift $\sigma^\nabla_{TM} \colon \mathfrak{X}(M) \to \mathfrak{X}^l(E)$ and so a linear

[4]Since its flow is a flow of vector bundle morphisms, a linear vector field sends linear functions to linear functions and pullbacks to pullbacks.

splitting $\Sigma^\nabla\colon TM\times_M E\to TE$. Given ∇, it is easy to see using the equalities in (2) that, writing σ for σ^∇_{TM}:

$$[\sigma(X),\sigma(Y)]=\sigma[X,Y]-\widetilde{R_\nabla(X,Y)},\quad \left[\sigma(X),e^\uparrow\right]=(\nabla_X e)^\uparrow,\quad \left[e^\uparrow,e'^\uparrow\right]=0, \tag{3}$$

for all $X,Y\in\mathfrak{X}(M)$ and $e,e'\in\Gamma(E)$. That is, the Lie bracket of vector fields on M and the connection encode completely the Lie bracket of vector fields on E.

Now let us have a quick look at the other structure on the double vector bundle TE. The lift $\sigma^\nabla_E\colon\Gamma(E)\to\Gamma^\ell_{TM}(TE)$ is given by

$$\sigma^\nabla_E(e)(v_m)=T_m e(v_m)+_{TM}(T_m0^E(v_m)-_E\overline{\nabla_{v_m}e}),\ v_m\in TM,\ e\in\Gamma(E).$$

Further, for $e\in\Gamma(E)$, the core section $e^\dagger\in\Gamma_{TM}(TE)$ is given by

$$e^\dagger(v_m)=T_m0^E(v_m)+_E\left.\frac{d}{dt}\right|_{t=0}te(m). \tag{4}$$

Let $A\to M$ be a Lie algebroid and consider an A-connection ∇ on a vector bundle $E\to M$. Then the space $\Omega^\bullet(A,E)$ of E-valued Lie algebroid forms has an induced operator $\mathbf{d}_\nabla$ given by:

$$\begin{aligned}\mathbf{d}_\nabla\omega(a_1,\dots,a_{k+1})=&\sum_{i<j}(-1)^{i+j}\omega([a_i,a_j],a_1,\dots,\hat a_i,\dots,\hat a_j,\dots,a_{k+1})\\&+\sum_i(-1)^{i+1}\nabla_{a_i}(\omega(a_1,\dots,\hat a_i,\dots,a_{k+1}))\end{aligned}$$

for all $\omega\in\Omega^k(A,E)$ and $a_1,\dots,a_{k+1}\in\Gamma(A)$. The connection is flat if and only if $\mathbf{d}_\nabla^2=0$. Let now consider a vector bundle $\mathcal{E}=E_0\oplus E_1$ and a Lie algebroid A over M. A *2-term representation up to homotopy of A on $\mathcal{E}$* [1] (or a *superrepresentation* [8]) is

(1) a vector bundle map $\partial\colon E_0\to E_1$,
(2) two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial\circ\nabla^0=\nabla^1\circ\partial$,
(3) an element $R\in\Omega^2(A,\operatorname{Hom}(E_1,E_0))$ such that $R_{\nabla^0}=R\circ\partial$, $R_{\nabla^1}=\partial\circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}}R=0$, where ∇^{Hom} is the connection induced on $\operatorname{Hom}(E_1,E_0)$ by ∇^0 and ∇^1.

For brevity we will call such a 2-term representation up to homotopy a **2-representation**.

Let $(D\to B,A\to M)$ be a VB-Lie algebroid and choose a linear splitting $\Sigma\colon A\times_M B\to D$. Since the anchor of a linear section is linear, for each $a\in\Gamma(A)$ the vector field $\Theta_B(\sigma_A(a))$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$ (see §2.1). This defines a linear connection $\nabla^{AB}\colon\Gamma(A)\times\Gamma(B)\to\Gamma(B)$ by $\Theta_B(\sigma_A(a))=\widehat{\nabla^{AB}_a}$ for all $a\in\Gamma(A)$. Since the bracket of a linear section with a core section is again a core section, we find a linear connection $\nabla^{AC}\colon\Gamma(A)\times\Gamma(C)\to\Gamma(C)$ such that $[\sigma_A(a),c^\dagger]=(\nabla^{AC}_ac)^\dagger$ for all $c\in\Gamma(C)$ and $a\in\Gamma(A)$. The difference $\sigma_A[a_1,a_2]-[\sigma_A(a_1),\sigma_A(a_2)]$ is a core-linear section for all $a_1,a_2\in\Gamma(A)$. This defines a form $R\in\Omega^2(A,\operatorname{Hom}(B,C))$ such that $[\sigma_A(a_1),\sigma_A(a_2)]=\sigma_A[a_1,a_2]-\widetilde{R(a_1,a_2)}$, for all $a_1,a_2\in\Gamma(A)$. See [8] for more details on these constructions, and for the proof of the following theorem.

Theorem 2.2. *Let $(D \to B; A \to M)$ be a VB-algebroid and choose a linear splitting $\Sigma\colon A \times_M B \to D$. The triple $(\nabla^{AB}, \nabla^{AC}, R)$ defined as above is a 2-representation of A on the complex $\partial_B\colon C \to B$.*

Conversely, let $(D; A, B; M)$ be a double vector bundle with core C such that A has a Lie algebroid structure, and choose a linear splitting $\Sigma\colon A \times_M B \to D$. Then if $(\nabla^{AB}, \nabla^{AC}, R)$ is a 2-representation of A on a morphism $\partial_B\colon C \to B$, then the three equations above and the core-anchor ∂_B define a VB-algebroid structure on $(D \to B; A \to M)$.

Example 2.3. Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ and consider the corresponding linear splitting Σ^∇ of TE as in Example 2.1. The description of the Lie bracket of vector fields in (3) shows that the 2-representation induced by Σ^∇ is the 2-representation of TM on $\mathrm{Id}_E\colon E \to E$ given by $(\nabla, \nabla, R_\nabla)$.

2.1.1. *Double Lie algebroids and matched pairs of 2-representations.* If D is a VB-algebroid with Lie algebroid structures on $D \to B$ and $A \to M$ the dual vector bundle $D \mathbin{\overset{*}{\vert}} B \to B$ (see Appendix B) has a *Lie-Poisson structure* (a linear Poisson structure), and the structure on $D \mathbin{\overset{*}{\vert}} B$ is also Lie-Poisson with respect to $D \mathbin{\overset{*}{\vert}} B \to C^*$ [21, 3.4]. Dualising this bundle gives a Lie algebroid structure on $D \mathbin{\overset{*}{\vert}} B \mathbin{\overset{*}{\vert}} C^* \to C^*$. This equips the double vector bundle $(D \mathbin{\overset{*}{\vert}} B \mathbin{\overset{*}{\vert}} C^*; C^*, A; M)$ with a VB-algebroid structure. Using the isomorphism defined by $-\langle\cdot,\cdot\rangle$, the double vector bundle $(D \mathbin{\overset{*}{\vert}} A \to C^*; A \to M)$ is also a VB-algebroid. In the same manner, if $(D \to A, B \to M)$ is a VB-algebroid then $(D \mathbin{\overset{*}{\vert}} B \to C^*; B \to M)$ is a VB-algebroid.

Let $\Sigma\colon A \times_M B \to D$ be a linear splitting of D and denote by (∇^B, ∇^C, R) the 2-representation of the Lie algebroid A on $\partial_B\colon C \to B$. The linear splitting Σ induces a linear splitting $\Sigma^\star\colon A \times_M C^* \to D \mathbin{\overset{*}{\vert}} A$ of $D \mathbin{\overset{*}{\vert}} A$ (see Appendix B). The 2-representation of A that is associated to this splitting is then $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ on the complex $\partial_B^*\colon B^* \to C^*$. This is proved in the appendix[5] of [6].

A **double Lie algebroid** [21] is a double vector bundle $(D; A, B; M)$ with core C, and with Lie algebroid structures on each of $A \to M$, $B \to M$, $D \to A$ and $D \to B$ such that each pair of parallel Lie algebroids gives D the structure of a VB-algebroid, and such that the pair $(D \mathbin{\overset{*}{\vert}} A, D \mathbin{\overset{*}{\vert}} B)$ with the induced Lie algebroid structures on base C^* and the pairing $\langle\cdot,\cdot\rangle$, is a Lie bialgebroid.

Definition 2.4. [7] *Let $(A \to M, \rho_A, [\cdot\,,\cdot])$ and $(B \to M, \rho_B, [\cdot\,,\cdot])$ be two Lie algebroids and assume that A acts on $\partial_B\colon C \to B$ up to homotopy via $(\nabla^{AB}, \nabla^{AC}, R_{AB})$ and B acts on $\partial_A\colon C \to A$ up to homotopy via $(\nabla^{BA}, \nabla^{BC}, R_{BA})$[6]. Then we say that the two representations up to homotopy form a matched pair if*

(M1) $\nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1 = -(\nabla_{\partial_A c_2} c_1 - \nabla_{\partial_B c_1} c_2)$,
(M2) $[a, \partial_A c] = \partial_A(\nabla_a c) - \nabla_{\partial_B c} a$,
(M3) $[b, \partial_B c] = \partial_B(\nabla_b c) - \nabla_{\partial_A c} b$,
(M4) $\nabla_b \nabla_a c - \nabla_a \nabla_b c - \nabla_{\nabla_b a} c + \nabla_{\nabla_a b} c = R_{BA}(b, \partial_B c)a - R_{AB}(a, \partial_A c)b$,
(M5) $\partial_A(R_{AB}(a_1, a_2)b) = -\nabla_b[a_1, a_2] + [\nabla_b a_1, a_2] + [a_1, \nabla_b a_2] + \nabla_{\nabla_{a_2} b} a_1 - \nabla_{\nabla_{a_1} b} a_2$,
(M6) $\partial_B(R_{BA}(b_1, b_2)a) = -\nabla_a[b_1, b_2] + [\nabla_a b_1, b_2] + [b_1, \nabla_a b_2] + \nabla_{\nabla_{b_2} a} b_1 - \nabla_{\nabla_{b_1} a} b_2$,

---

[5] The construction of the "dual" linear splitting of $D \mathbin{\overset{*}{\vert}} A$, given a linear splitting of D, is done differently but yields the same result.

[6] For the sake of simplicity, we write in this definition ∇ for all the four connections. It will always be clear from the indexes which connection is meant. We write ∇^A for the A-connection induced by ∇^{AB} and ∇^{AC} on $\wedge^2 B^* \otimes C$ and ∇^B for the B-connection induced on $\wedge^2 A^* \otimes C$.

for all $a, a_1, a_2 \in \Gamma(A)$, $b, b_1, b_2 \in \Gamma(B)$ *and* $c, c_1, c_2 \in \Gamma(C)$, *and*

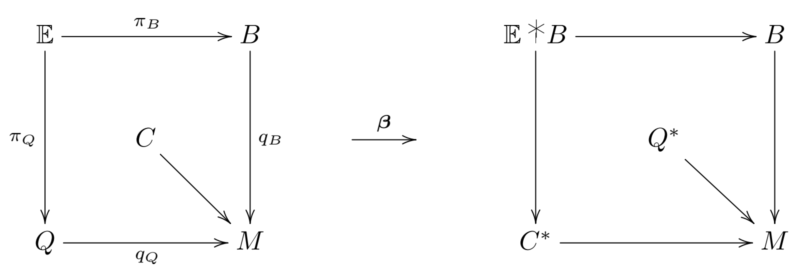
(M7) $\mathbf{d}_{\nabla^A} R_{BA} = \mathbf{d}_{\nabla^B} R_{AB} \in \Omega^2(A, \wedge^2 B^* \otimes C) = \Omega^2(B, \wedge^2 A^* \otimes C)$, *where* R_{AB} *is seen as an element of* $\Omega^1(A, \wedge^2 B^* \otimes C)$ *and* R_{AB} *as an element of* $\Omega^1(B, \wedge^2 A^* \otimes C)$.

Remark 2.5. From these equations follow $\rho_A \circ \partial_A = \rho_B \circ \partial_B$ and $[\rho_A(a), \rho_B(b)] = \rho_B(\nabla_a b) - \rho_A(\nabla_b a)$ for all $a \in \Gamma(A)$ and $b \in \Gamma(B)$. The vector bundle C inherits a Lie algebroid structure with anchor $\rho_A \circ \partial_A = \rho_B \circ \partial_B$ and with bracket given by $[c_1, c_2] = \nabla_{\partial_A c_1} c_2 - \nabla_{\partial_B c_2} c_1$ for all $c_1, c_2 \in \Gamma(C)$.

Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. After the choice of a splitting $\Sigma\colon A \times_M B \to D$, the Lie algebroid structures on the two sides of D are described as above by two 2-representations; the Lie algebroid $D \to B$ is described by $(\partial_B, \nabla^{AB}, \nabla^{AC}, R_{AB} \in \Omega^2(A, \operatorname{Hom}(B, C)))$. The Lie algebroid structure of $D \to A$ is described by $(\partial_A, \nabla^{BA}, \nabla^{BC}, R_{BA} \in \Omega^2(B, \operatorname{Hom}(A, C)))$. We prove in [12] that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any decomposition of D, the two induced 2-representations above form a matched pair.

2.2. **The equivalence of [2]-manifolds with metric double vector bundles.** We quickly recall in this section the main result in [10]. Recall that $\mathbb{N}$-manifolds and their vector fields were defined in the introduction. We refer the reader to [10] for a quick review of split $\mathbb{N}$-manifolds, and for our notation convention.

A **metric double vector bundle** is a double vector bundle $(\mathbb{E}, Q; B, M)$ equipped with a **linear symmetric non-degenerate pairing** $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$, i.e. such that the map



defined by the pairing $\langle \cdot, \cdot \rangle$ is an **isomorphism of double vector bundles**. In particular, the core $C \to M$ of $\mathbb{E}$ is isomorphic to $Q^* \to M$. Note that equivalently, a linear symmetric non-degenerate pairing $\langle \cdot\,, \cdot \rangle$ on $\mathbb{E} \to B$ is linear if the core of $\mathbb{E}$ is isomorphic to Q^* and, via this isomorphism,

(1) $\langle \tau_1^\dagger, \tau_2^\dagger \rangle = 0$ for $\tau_1, \tau_2 \in \Gamma(Q^*)$,
(2) $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for $\chi \in \Gamma_B^l(\mathbb{E})$ linear over $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ and
(3) $\langle \chi_1, \chi_2 \rangle$ is a linear function on B for $\chi_1, \chi_2 \in \Gamma_B^l(\mathbb{E})$.

In the following, we always identify with Q^* the core of a metric double vector bundle $(\mathbb{E}, Q; B, M)$.

Note that the **opposite** $(\overline{\mathbb{E}}; Q; B, M)$ of a metric double vector bundle $(\mathbb{E}; B; Q, M)$ is the metric double vector bundle with

$$\langle \cdot\,, \cdot \rangle_{\overline{\mathbb{E}}} = -\langle \cdot\,, \cdot \rangle_{\mathbb{E}}.$$

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. A linear splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$ is said to be **Lagrangian** if its image is maximal isotropic in $\mathbb{E} \to B$. The corresponding horizontal lifts $\sigma_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$ and $\sigma_B\colon \Gamma(B) \to \Gamma^l_Q(\mathbb{E})$ are then also said to be **Lagrangian**. Hence, by definition, a horizontal lift $\sigma_Q\colon \Gamma(Q) \to \Gamma^l_B(\mathbb{E})$ is Lagrangian if and only if $\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle = 0$ for all $q_1, q_2 \in \Gamma(Q)$.

Showing the existence of a Lagrangian splitting of $\mathbb{E}$ is relatively easy [10]. Further, if Σ^1 and $\Sigma^2\colon Q \times_M B \to \mathbb{E}$ are Lagrangian, then the change of splitting $\phi_{12} \in \Gamma(Q^* \otimes Q^* \otimes B^*)$,

$$\Sigma^2(q, b) = \Sigma^1(q, b) + \widetilde{\phi(q, b)} \quad \text{for all}(q, b) \in Q \times_M B,$$

is skew-symmetric, i.e. a section of $Q^* \wedge Q^* \otimes B^*$.

Example 2.6. Let $E \to M$ be a metric vector bundle, i.e. a vector bundle endowed with a symmetric non-degenerate pairing $\langle \cdot\,, \cdot\rangle\colon E \times_M E \to \mathbb{R}$. Then $E \simeq E^*$ and the tangent double is a metric double vector bundle $(TE, E; TM, M)$ with pairing $TE \times_{TM} TE \to \mathbb{R}$ the tangent of the pairing $E \times_M E \to \mathbb{R}$. In particular, we have

$$\langle Te_1, Te_2\rangle_{TE} = \ell_{\mathbf{d}\langle e_1, e_2\rangle}, \quad \langle Te_1, e_2^\dagger\rangle_{TE} = p_M^*\langle e_1, e_2\rangle \quad \text{ and } \langle e_1^\dagger, e_2^\dagger\rangle_{TE} = 0$$

for $e_1, e_2 \in \Gamma(E)$.

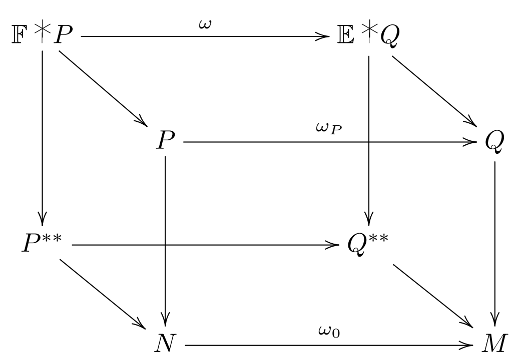
Recall that linear splittings of TE are equivalent to linear connections $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. The Lagrangian splittings of TE are exactly the linear splittings that correspond to **metric** connections, i.e. linear connections $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ that preserve the metric: $\langle \nabla_\cdot e_1, e_2\rangle + \langle e_1, \nabla_\cdot e_2\rangle = \mathbf{d}\langle e_1, e_2\rangle$ for $e_1, e_2 \in \Gamma(E)$.

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Choose a Lagrangian splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$ and set

$$\mathcal{A}^2(\mathbb{E}) := \sigma_B(\Gamma(B)) + \{\tilde{\omega} \mid \omega \in \Gamma(Q^* \wedge Q^*)\}.$$

In other words, $\mathcal{A}^2(\mathbb{E})$ is the $C^\infty(M)$-module generated by all Lagrangian horizontal lifts of sections of B. Note that by the considerations above, $\mathcal{A}^2(\mathbb{E})$ does not depend on the choice of Lagrangian splitting. Note also that $\mathcal{A}^2(\mathbb{E})$ together with $\Gamma^c_Q(\mathbb{E}) \simeq \Gamma(Q^*)$ span $\mathbb{E}$ as a vector bundle over Q.

A **morphism** $\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$ of **metric double vector bundles** is an isotropic relation $\Omega \subseteq \overline{\mathbb{F}} \times \mathbb{E}$ that is the dual of a double vector bundle morphism $\omega\colon \mathbb{F} \ast P \to \mathbb{E} \ast Q$.



The morphism $\Omega\colon \mathbb{F} \dashrightarrow \mathbb{E}$ is equivalent to a triple of maps

$$\omega^\star\colon \mathcal{A}^2(\mathbb{E}) \to \mathcal{A}^2(\mathbb{F}), \quad \omega_P^\star\colon \Gamma(Q^*) \to \Gamma(P^*), \quad \text{ and a smooth map } \omega_0\colon N \to M$$

such that

1. $\omega^\star\left(\widetilde{\tau_1\wedge\tau_2}\right)=\omega_P^\star\widetilde{\tau_1\wedge\omega_P^\star}\tau_2$,
2. $\omega^\star(q_Q^*f\cdot\chi)=q_P^*(\omega_0^*f)\cdot\omega^\star(\chi)$ and
3. $\omega_P^\star(f\cdot\tau)=\omega_0^*f\cdot\omega_P^\star\tau$

for all $\tau,\tau_1,\tau_2\in\Gamma(Q^*)$, $f\in C^\infty(M)$ and $\chi\in\mathcal{A}^2(\mathbb{E})$.

Theorem 2.7 ([10])**.** *The category defined above of metric double vector bundles and the category of* [2]*-manifolds are equivalent.*

Let $(\mathbb{E},B;Q,M)$ be a metric double vector bundle. We define a [2]-manifold $\mathcal{M}(\mathbb{E})$ as follows. The sheaf $\mathcal{A}(\mathbb{E})=C^\infty(\mathcal{M}(\mathbb{E}))$ of $\mathbb{N}$-graded commutative associative unital $\mathbb{R}$-algebras is generated by $\mathcal{A}^0(\mathbb{E})=C^\infty(M)$ in degree 0, by $\mathcal{A}^1(\mathbb{E})=\Gamma(Q^*)$ in degree 1 and the sheaf of degree 2-functions is $\mathcal{A}^2(\mathbb{E})$. The product of τ_1 with $\tau_2\in\Gamma(Q^*)$ is $\widetilde{\tau_1\wedge\tau_2}\in\mathcal{A}^2(\mathbb{E})$. The product of $f\in C^\infty(M)$ with $\xi\in\mathcal{A}^2(\mathbb{E})$ is $q_Q^*f\cdot\xi\in\mathcal{A}^2(\mathbb{E})$ and the product of elements of $\mathcal{A}^0$ with elements of $\mathcal{A}^1$ is obvious since $\Gamma(Q^*)$ is a sheaf of $C^\infty(M)$-modules.

Conversely, we construct explicitly a metric double vector bundle associated to a given [2]-manifold $\mathcal{M}$. The idea is to adapt the construction of the equivalence of locally free and finitely generated sheaves of $C^\infty(M)$-modules with vector bundles over M.

Let M be the smooth manifold underlying $\mathcal{M}$ and assume that $\mathcal{M}$ has dimension $(l;m,n)$. Choose a maximal open covering $\{U_\alpha\}$ of M such that $C^\infty_{U_\alpha}\mathcal{M})$ is freely generated by $\xi_1^\alpha,\ldots,\xi_m^\alpha$ (in degree 1) and $\eta_1^\alpha,\ldots,\eta_n^\alpha$ (degree 2 generators). Choose now α,β such that $U_\alpha\cap U_\beta\neq\emptyset$. Then each generator ξ_i^β can be written in a unique manner as $\sum_{j=1}^m\omega_{\alpha\beta}^{ji}\xi_j^\alpha$ with $\omega^{ji}\in C^\infty(U_\alpha\cap U_\beta)$. Each generator η_i^β can be written

$$\eta_i^\beta=\sum_{j=1}^n\psi_{\alpha\beta}^{ji}\cdot\left(\eta_j^\alpha+\sum_{1\leq k<l\leq m}\rho_{\alpha\beta}^{jkl}\cdot\xi_k^\alpha\wedge\xi_l^\alpha\right)$$

with $\psi_{\alpha\beta}^{ij},\rho_{\alpha\beta}^{ikl}\in C^\infty(U_\alpha\cap U_\beta)$. Set $A_1^{\alpha\beta}=(\omega_{\alpha\beta}^{ij})_{i,j}\in C^\infty(M,\mathrm{Gl}(\mathbb{R}^{m*}))$, $A_2^{\alpha\beta}=(\psi_{\alpha\beta}^{ij})_{i,j}\in C^\infty(M,\mathrm{Gl}(\mathbb{R}^n))$. Define $\nu^{\alpha\beta}\in C^\infty(M,\mathrm{Hom}(\mathbb{R}^m\otimes\mathbb{R}^n,\mathbb{R}^{m*}))$ by $\nu^{\alpha\beta}(e_i,e_j)(e_l)=\rho_{\alpha\beta}^{jil}$ for $1\leq i<l\leq m$ and $j=1,\ldots,n$. Then by construction

$$A_1^{\gamma\alpha}\cdot A_1^{\alpha\beta}=A_1^{\gamma\beta},\qquad A_2^{\gamma\alpha}\cdot A_2^{\alpha\beta}=A_2^{\gamma\beta}\quad\text{ and}$$

$$\begin{aligned}\nu^{\gamma\beta}(A_1^{\beta\gamma^*}(e_i),A_2^{\gamma\beta}(e_j))(e_l)&=\nu^{\gamma\alpha}(A_1^{\beta\gamma^*}(e_i),A_2^{\gamma\beta}(e_j),e_l)\\&\quad+\nu^{\alpha\beta}(A_1^{\beta\alpha^*}(e_i),A_2^{\alpha\beta}(e_j))(A_1^{\gamma\alpha^*}e_l).\end{aligned}\tag{5}$$

Set $\tilde{\mathbb{E}}=\bigsqcup_\alpha U_\alpha\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{m*}$ (the disjoint union) and identify

$$(x,v_1,v_2,l_0)\in U_\beta\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{m*}$$

with

$$\left(x,(A_1^{\beta\alpha}(x))^*(v_1),A_2^{\alpha\beta}(x)(v_2),A_1^{\alpha\beta}(x)(l_0)+\nu^{\alpha\beta}(x)((A_1^{\beta\alpha}(x))^*(v_1),A_2^{\alpha\beta}(x)(v_2))\right)$$

in $U_\alpha\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{m*}$ for $x\in U_\alpha\cap U_\beta$. The cocycle equations (5) imply that this defines an equivalence relation on $\tilde{\mathbb{E}}$. The quotient space is $\mathbb{E}$, a double vector bundle. The linear metric on $\mathbb{E}$ is defined over a chart domain U_α by

$$\langle(x,v_1,v_2,l_0),(x,v_1',v_2,l_0')\rangle=l_0(v_1')+l_0'(v_1).$$

By construction, this does not depend on the choice of α with $x \in U_\alpha$. The equivalence of morphisms of metric double vector bundles with morphisms of [2]-manifolds is then easy to see [10].

3. Split Lie 2-algebroids and Dorfman 2-representations

In this section we recall the notions of Courant algebroids, Dirac structures, dull algebroids and Dorfman connections. Then we discuss (split) Lie 2-algebroids and the dual Dorfman 2-representations. We give several classes of examples of split Lie 2-algebroids.

3.1. **Preliminaries.** We introduce in this section a slight generalisation of the notion of Courant algebroid, namely the one of *Courant algebroids with pairing in a vector bundle* (called E-Courant algebroids in [4]). We will see that the fat bundle associated to a VB-Courant algebroid will carry a natural Courant algebroid structure with pairing in a vector bundle.

In the following, an anchored vector bundle is a vector bundle $Q \to M$ endowed with a vector bundle morphism $\rho_Q\colon Q \to TM$ over the identity. An anchored vector bundle $(Q \to M, \rho_Q)$ and a vector bundle $B \to M$ are said to be paired if there exists a fibrewise pairing $\langle \cdot\,,\cdot\rangle\colon Q \times_M B \to \mathbb{R}$ and a map $\mathbf{d}_B\colon C^\infty(M) \to \Gamma(B)$ such that

$$\langle q, \mathbf{d}_B f\rangle = \rho_Q(q)(f) \tag{6}$$

for all $q \in \Gamma(Q)$ and $f \in C^\infty(M)$. The triple $(B, \mathbf{d}_B, \langle \cdot\,,\cdot\rangle)$ will be called a **pre-dual** of Q and Q and B are said to be **paired by** $\langle \cdot\,,\cdot\rangle$.

Example 3.1. Take $B = Q^*$ and $\langle \cdot\,,\cdot\rangle$ to be the canonical pairing of Q with Q^*. Then (B, ρ_Q) is paired with $(Q^*, \langle \cdot\,,\cdot\rangle, \rho_Q^*\mathbf{d})$.

Consider an anchored vector bundle $(\mathsf{E} \to M, \rho)$ and a vector bundle V over the same base M together with a map $\tilde{\rho}\colon \Gamma(\mathsf{E}) \to \operatorname{Der}(V)$, such that the symbol of $\tilde{\rho}(e)$ is $\rho(e) \in \mathfrak{X}(M)$ for all $e \in \Gamma(\mathsf{E})$. Assume that E is paired with itself via a nondegenerate pairing $\langle \cdot\,,\cdot\rangle\colon \mathsf{E} \times_M \mathsf{E} \to V$ with values in V and that there exists a map $\mathcal{D}\colon \Gamma(V) \to \Gamma(\mathsf{E})$ such that $\langle \mathcal{D}v, e\rangle = \tilde{\rho}(e)(v)$ for all $v \in \Gamma(V)$.

Then $\mathsf{E} \to M$ is a **Courant algebroid with pairing in** V over the manifold M if E is in addition equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot\,,\cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$ such that the following conditions are satisfied:

(CA1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
(CA2) $\tilde{\rho}(e_1)\langle e_2, e_3\rangle = \langle [\![e_1, e_2]\!], e_3\rangle + \langle e_2, [\![e_1, e_3]\!]\rangle$,
(CA3) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2\rangle$,
(CA4) $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$,
(CA5) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$. If the pairing $\langle \cdot\,,\cdot\rangle$ is nondegenerate, then $(\mathsf{E} \to M, \tilde{\rho}, \langle \cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is a **Courant algebroid with pairing in V**. If $V = \mathbb{R} \times M \to M$ is the trivial bundle, then $\mathcal{D} = \beta^{-1} \circ \rho^* \circ \mathbf{d}\colon C^\infty(M) \to \Gamma(\mathsf{E})$, where β is the isomorphism $\mathsf{E} \to \mathsf{E}^*$ given by $\boldsymbol{\beta}(e) = \langle e, \cdot\rangle$ for all $e \in \mathsf{E}$. The quadruple $(\mathsf{E} \to M, \rho, \langle \cdot\,,\cdot\rangle, [\![\cdot\,,\cdot]\!])$ is then a **Courant algebroid** [15, 27] and Conditions (CA4) and (CA5) follow then from (CA1), (CA2) and (CA3) (see [32] and also [9] for a quicker proof).

In our study of VB-Courant algebroids, we will need the following two lemmas.

Lemma 3.2 ([28])**.** *Let* $(\mathsf{E}\to M,\rho,\langle\cdot\,,\cdot\rangle,[\![\cdot\,,\cdot]\!])$ *be a Courant algebroid. For all* $\theta\in\Omega^1(M)$ *and* $e\in\Gamma(\mathsf{E})$*, we have:*

$$[\![e,\beta^{-1}\rho^*\theta]\!]=\beta^{-1}\rho^*(\pounds_{\rho(e)}\theta),\qquad [\![\beta^{-1}\rho^*\theta,e]\!]=-\beta^{-1}\rho^*(\mathbf{i}_{\rho(e)}\mathbf{d}\theta)$$

and

$$\rho(\beta^{-1}\rho^*\theta)=0. \tag{7}$$

In particular, it follows from (7) that

$$\rho\circ\mathcal{D}=0. \tag{8}$$

Lemma 3.3 ([13])**.** *Let* $\mathsf{E}\to M$ *be a vector bundle,* $\rho\colon\mathsf{E}\to TM$ *be a bundle map,* $\langle\cdot,\cdot\rangle$ *a nondegenerate pairing on* E*, and let* $\mathcal{S}\subseteq\Gamma(\mathsf{E})$ *be a subspace of sections which generates* $\Gamma(\mathsf{E})$ *as a* $C^\infty(M)$*-module. Suppose that* $[\![\cdot\,,\cdot]\!]:\mathcal{S}\times\mathcal{S}\to\mathcal{S}$ *is a bracket which satisfies*

(1) $[\![s_1,[\![s_2,s_3]\!]]\!]=[\![[\![s_1,s_2]\!],s_3]\!]+[\![s_2,[\![s_1,s_3]\!]]\!]$,
(2) $\rho(s_1)\langle s_2,s_3\rangle=\langle[\![s_1,s_2]\!],s_3\rangle+\langle s_2,[\![s_1,s_3]\!]\rangle$,
(3) $[\![s_1,s_2]\!]+[\![s_2,s_1]\!]=\rho^*\mathbf{d}\langle s_1,s_2\rangle$,
(4) $\rho[\![s_1,s_2]\!]=[\rho(s_1),\rho(s_2)]$,

for any $s_i\in\mathcal{S}$*, and that* $\rho\circ\rho^*=0$*. Then there is a unique extension of* $[\![\cdot\,,\cdot]\!]$ *to a bracket on all of* $\Gamma(\mathsf{E})$ *such that* $(\mathsf{E},\rho,\langle\cdot\,,\cdot\rangle,[\![\cdot\,,\cdot]\!])$ *is a Courant algebroid.*

A **Dirac structure** with support in a Courant algebroid $\mathsf{E}\to M$ is a subbundle $D\to S$ over a sub-manifold S of M, such that $D(s)$ is maximal isotropic in $\mathsf{E}(s)$ for all $s\in S$ and

$$e_1|_S\in\Gamma_S(D),e_2|_S\in\Gamma_S(D)\quad\Rightarrow\quad[\![e_1,e_2]\!]|_S\in\Gamma_S(D)$$

for all $e_1,e_2\in\Gamma(\mathsf{E})$.

We will use the following lemma involving Dirac structures with support. We leave the proof to the reader.

Lemma 3.4. *Let* $\mathsf{E}\to M$ *be a Courant algebroid and* $D\to S$ *a subbundle; with* S *a sub-manifold of* M*. Assume that* $D\to S$ *is spanned by the restrictions to* S *of a family* $\mathcal{S}\subseteq\Gamma(\mathsf{E})$ *of sections of* E*. Then* D *is a Dirac structure with support* S *if and only if*

(1) $\rho_{\mathsf{E}}(e)(s)\in T_sS$ *for all* $e\in\mathcal{S}$ *and* $s\in S$*,*
(2) D_s *is Lagrangian in* $\mathbb{E}_s$ *for all* $s\in S$ *and*
(3) $[\![e_1,e_2]\!]|_S\in\Gamma_S(D)$ *for all* $e_1,e_2\in\mathcal{S}$*.*

Next we recall the notion of Dorfman connection [9].

Definition 3.5. *Let* $(Q\to M,\rho_Q)$ *be an anchored vector bundle and let* $(B\to M,\mathbf{d}_B,\langle\cdot\,,\cdot\rangle)$ *be paired with* (Q,ρ_Q)*. A* ***Dorfman (Q-)connection on*** B *is an* $\mathbb{R}$*-linear map*

$$\Delta\colon\Gamma(Q)\to\operatorname{Der}(B)$$

such that

(1) Δ_q *is a derivation over* $\rho_Q(q)\in\mathfrak{X}(M)$*,*
(2) $\Delta_{fq}b=f\Delta_qb+\langle q,b\rangle\cdot\mathbf{d}_Bf$ *and*
(3) $\Delta_q\mathbf{d}_Bf=\mathbf{d}_B(\rho_Q(q)f)$

for all $f\in C^\infty(M)$*,* $q,q'\in\Gamma(Q)$*,* $b\in\Gamma(B)$*.*

In the last definition, the map $\langle q,\cdot\rangle \mathbf{d}_B f\colon B\to B$ is seen as a section of $\operatorname{Hom}(B,B)$, i.e. as a derivation over $0\in\mathfrak{X}(M)$.

Remark 3.6. Note that if the pairing $\langle\cdot\,,\cdot\rangle : Q\times_M B\to\mathbb{R}$ is nondegenerate, then $B\simeq Q^*$ and the map $\mathbf{d}_B=\mathbf{d}_{Q^*}\colon C^\infty(M)\to\Gamma(Q^*)$ is *defined* by (6): we have then $\mathbf{d}_{Q^*}f=\rho_Q^*\mathbf{d}f$ for all $f\in C^\infty(M)$.

The map $\Delta^*:\Gamma(Q)\times\Gamma(Q)\to\Gamma(Q)$ that is dual to Δ in the sense of dual derivations, i.e. $\langle\Delta^*_{q_1}q_2,\tau\rangle=\rho_Q(q_1)\langle q_2,\tau\rangle-\langle q_2,\Delta_{q_1}\tau\rangle$ for all $q_1,q_2\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$ defines then a *dull bracket* on $\Gamma(Q)$:

$$[\![q_1,q_2]\!]_\Delta=\Delta^*_{q_1}q_2$$

in the sense of the following definition.

Definition 3.7. *A* ***dull algebroid*** *is an anchored vector bundle $(Q\to M,\rho_Q)$ with a bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(Q)$ such that*

$$\rho_Q[\![q_1,q_2]\!]=[\rho_Q(q_1),\rho_Q(q_2)] \tag{9}$$

and (the Leibniz identity)

$$[\![f_1q_1,f_2q_2]\!]=f_1f_2[\![q_1,q_2]\!]+f_1\rho_Q(q_1)(f_2)q_2-f_2\rho_Q(q_2)(f_1)q_1$$

for all $f_1,f_2\in C^\infty(M)$, $q_1,q_2\in\Gamma(Q)$.

In other words, a dull algebroid is a **Lie algebroid** if its bracket is in addition skew-symmetric and satisfies the Jacobi identity. Note that a skew symmetric dull bracket can be constructed as follows from an arbitrary dull bracket on Q; the skew-symmetrisation $[\![\cdot\,,\cdot]\!]'$ of $[\![\cdot\,,\cdot]\!]$ is defined by $[\![q_1,q_2]\!]'=\frac{1}{2}\left([\![q_1,q_2]\!]-[\![q_2,q_1]\!]\right)$ for all $q_1,q_2\in\Gamma(Q)$.

Assume that $\Delta\colon\Gamma(Q)\to\Gamma(Q^*)\to\Gamma(Q^*)$ is a Dorfman connection and let $[\![\cdot\,,\cdot]\!]_\Delta$ be the dual dull bracket. Note that (9) is equivalent to (3) in Definition 3.5 and the Leibniz identity corresponds to (1) and (2).

The *curvature* of a general Dorfman connection $\Delta\colon\Gamma(Q)\times\Gamma(B)\to\Gamma(B)$ is the map

$$R_\Delta\colon\Gamma(Q)\times\Gamma(Q)\to\Gamma(B^*\otimes B),$$

defined on $q,q'\in\Gamma(Q)$ by $R_\Delta(q,q'):=\Delta_q\Delta_{q'}-\Delta_{q'}\Delta_q-\Delta_{[q,q']_Q}$. If $B=Q^*$ and the pairing is the natural one, the curvature is equivalent to the Jacobiator of the dull bracket:

$$\langle\tau,\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta}(q_1,q_2,q_3)\rangle=\langle R_\Delta(q_1,q_2)\tau,q_3\rangle \tag{10}$$

for $q_1,q_2,q_3\in\Gamma(Q)$ and $b\in\Gamma(B)$, where

$$\operatorname{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta}(q_1,q_2,q_3)=[\![[\![q_1,q_2]\!]_\Delta,q_3]\!]_\Delta+[\![q_2,[\![q_1,q_3]\!]_\Delta]\!]_\Delta-[\![q_1,[\![q_2,q_3]\!]_\Delta]\!]_\Delta$$

is the Jacobiator of $[\![\cdot\,,\cdot]\!]_\Delta$ in Leibniz form. Hence, the Dorfman connection is flat if and only if the corresponding dull bracket satisfies the Jacobi identity in Leibniz form $[\![q_1,[\![q_2,q_3]\!]_\Delta]\!]_\Delta=[\![[\![q_1,q_2]\!]_\Delta,q_3]\!]_\Delta+[\![q_2,[\![q_1,q_3]\!]_\Delta]\!]_\Delta$ for all $q_1,q_2,q_3\in\Gamma(Q)$. A flat Dorfman connection is called a **Dorfman representation**, if the dual dull bracket is in addition skew-symmetric. The dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$ and the anchor ρ_Q define then a Lie algebroid structure on $\Gamma(Q)$. Conversely, given a Lie algebroid A, then the Lie derivative $\pounds^A\colon\Gamma(A)\times\Gamma(A^*)\to\Gamma(A^*)$ is a Dorfman representation. Hence, Lie algebroids are dual to Dorfman representations.

3.2. **Dorfman 2-representations and split Lie 2-algebroids.** We now define the *Dorfman 2-representations* and show that they are the dual derivations to split Lie 2-algebroids.

A **homological** vector field χ on $\mathcal{M}$ is a derivation of degree 1 of $C^\infty(\mathcal{M})$ such that $\mathcal{Q}^2 = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}]$ vanishes. A homological vector field on a [1]-manifold $\mathcal{M} = E^*[1]$ is the de Rham differential $\mathbf{d}_E$ associated to a Lie algebroid structure on E. This result is due to Vaintrob [33] and was explained in our introduction. A **Lie n-algebroid** is an $[n]$-manifold endowed with a homological vector field (an $\mathbb{N}\mathcal{Q}$-*manifold* of degree n).

A **split Lie n-algebroid** is a split $[n]$-manifold endowed with a homological vector field. Split Lie n-algebroids were studied by Sheng and Zhu [31] and described as vector bundles endowed with a bracket that satisfies the Jacobi identity up to some correction terms, see also [2]. Let us first give in our own words their definition of a split Lie 2-algebroid.

Definition 3.8. *A split Lie 2-algebroid $Q \oplus B^* \to M$ is a pair of an anchored vector bundle* [7] *$(Q \to M, \rho_Q)$ and a vector bundle $B \to M$, together with*

(1) *a vector bundle map $l_1 \colon B^* \to Q$,*
(2) *a skew-symmetric dull bracket*[8] *$[\![\cdot\,,\cdot]\!] \colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(Q)$,*
(3) *a linear connection $\nabla \colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$ and*
(4) *a vector valued 3-form $l_3 \in \Omega^3(Q, B^*)$,*

such that

(i) $\nabla^*_{l_1(\beta_1)}\beta_2 + \nabla^*_{l_1(\beta_2)}\beta_1 = 0$ *for all* $\beta_1, \beta_2 \in \Gamma(B^*)$,
(ii) $[\![q, l_1(\beta)]\!] = l_1(\nabla^*_q \beta)$ *for* $q \in \Gamma(Q)$ *and* $\beta \in \Gamma(B^*)$,
(iii) $\operatorname{Jac}_{[\![\cdot,\cdot]\!]} = -l_1 \circ l_3 \in \Omega^3(Q, Q)$,
(iv) $R_{\nabla^*}(q_1, q_2)\beta = l_3(q_1, q_2, l_1(\beta))$ *for* $q \in \Gamma(Q)$ *and* $\beta \in \Gamma(B^*)$, *and*
(v) $\mathbf{d}_{\nabla^*} l_3 = 0$.

From (iii) follows the identity $\rho_Q \circ l_1 = 0$. Since $\mathbf{d}_{\nabla^*} l_3 = 0$, one could say that a split Lie 2-algebroid is a Lie algebroid "up to homotopy".

As we have seen above, flat Dorfman connections with skew-symmetric dual dull brackets are in duality with Lie algebroids (or equivalently "split Lie 1-algebroids"). As we will see below, we define in fact Dorfman 2-representations as the Lie derivatives that are dual to split Lie 2-algebroids. The notion of Dorfman 2-representation defined below resembles the notion of 2-representation. The meaning of this analogy will become clearer in our study of VB-Courant algebroids.

Definition 3.9. *Let $(Q \to M, \rho_Q)$ be an anchored vector bundle. A (Q, ρ_Q)-Dorfman 2-representation is a quadruple $(\partial_B \colon Q^* \to B, \Delta, \nabla, \omega)$, Δ is a skew-symmetric Dorfman connection*

$$\Delta \colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*),$$

[7] The names that we choose for the vector bundles will become natural in a moment.

[8] To get the definition that was first given in [31], consider the skew symmetric bracket $l_2 \colon \Gamma(Q \oplus B^*) \times \Gamma(Q \oplus B^*) \to \Gamma(Q \oplus B^*)$,

$$l_2((q_1, \beta_1), (q_2, \beta_2)) = ([\![q_1, q_2]\!], \nabla^*_{q_1}\beta_2 - \nabla^*_{q_2}\beta_1) \tag{11}$$

for $q_1, q_2 \in \Gamma(Q)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. Note that this bracket satisfies a Leibniz identity with anchor $\rho_Q \circ \mathrm{pr}_Q \colon Q \oplus B^* \to TM$ and that the Jacobiator of this bracket is given by

$$\operatorname{Jac}_{l_2}((q_1, \beta_1), (q_2, \beta_2), (q_3, \beta_3)) = (-l_1(l_3(q_1, q_2, q_3)), l_3(q_1, q_2, l_1(\beta_3)) + \text{c.p.}$$

∇ is a linear connection

$$\nabla\colon \Gamma(Q)\times\Gamma(B)\to\Gamma(B),$$

and ω is an element of $\Omega^3(Q,B^*)$ such that

(D1) $\partial_B\circ\Delta_q=\nabla_q\circ\partial_B$,
(D2) $\nabla^*_{\partial_B^*\xi_1}\xi_2+\nabla^*_{\partial_B^*\xi_2}\xi_1=0$,
(D3) $\partial_B\circ R(q_1,q_2)=R_\nabla(q_1,q_2)$ *and* $R(q_1,q_2)\circ\partial_B=R_\Delta(q_1,q_2)$ *where* $R(q_1,q_2)=\omega(q_1,q_2,\cdot)^*\in\Gamma(\mathrm{Hom}(B,Q^*))$ *and*
(D4) $\mathbf{d}_{\nabla^*}\omega=0$

for all $\xi_1,\xi_2\in\Gamma(B^*)$, $q,q_1,q_2\in\Gamma(Q)$ and $f\in C^\infty(M)$.

One can see quite easily that the definition of a Dorfman 2-representation is just a rephrasing of the definition of a Lie 2-algebroid. Set $\omega=l_3$ and $\partial_B^*=-l_1$. Then (D1) is (ii) in Definition 3.8 and (D3) is equivalent to $(R_\nabla(q_1,q_2))^*\xi=-R_{\nabla^*}(q_1,q_2)\xi=\omega(q_1,q_2,\partial_B^*\xi)$ for $q_1,q_2\in\Gamma(Q)$ and $\xi\in\Gamma(B^*)$ and $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]_\Delta}=\partial_B^*\omega$.

Note that the vector bundle $Q\oplus B^*$ is anchored by ρ_Q and paired with $Q^*\oplus B$ by the natural pairing and the map $C^\infty(M)\to\Gamma(Q^*\oplus B)$, $f\mapsto(\rho_Q^*\mathbf{d}f,0)$. Hence if we define the Dorfman connection

$$\tilde\Delta\colon \Gamma(Q\oplus B^*)\times\Gamma(Q^*\oplus B)\to\Gamma(Q^*\oplus B)$$

by

$$\tilde\Delta_{(q,\beta)}(\tau,b)=(\Delta_q\tau,\nabla_q b),$$

then $\tilde\Delta$ is the Dorfman connection that is dual to the bracket l_2 defined in (11). Hence, we can think of Dorfman 2-representations as "Lie 2-derivatives", or "Lie derivatives up to homotopy". In other words, as the duals of Lie algebroids are Dorfman representations, the duals of Lie 2-algebroids are Dorfman 2-representations.

3.3. **Split Lie-2-algebroids as split [2]Q-manifolds.** Before we go on with the study of examples, we briefly describe how to construct from the objects in Definitions 3.8 and 3.9 the corresponding homological vector fields on split [2]-manifolds.

Consider a split [2]-manifold $\mathcal{M}=Q^*[1]\oplus B[2]$. Assume that Q is endowed with an anchor ρ_Q and a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$, that it acts on B via a linear connection $\nabla\colon\Gamma(Q)\times\Gamma(B)\to\Gamma(B)$, that ω is an element of $\Omega^3(Q,B^*)$ and that $\partial_B\colon Q^*\to B$ is a vector bundle morphism. Define a vector field $\mathcal{Q}$ of degree 1 on $\mathcal{M}$ by the following formulas:

$$\mathcal{Q}(f)=\rho_Q^*\mathbf{d}f\in\Gamma(Q^*)$$

for $f\in C^\infty(M)$,

$$\mathcal{Q}(\tau)=\mathbf{d}_Q\tau+\partial_B\tau\in\Omega^2(Q)\oplus\Gamma(B)$$

for $\tau\in\Gamma(Q^*)$ and

$$\mathcal{Q}(b)=\mathbf{d}_\nabla b+\langle\omega,b\rangle\in\Omega^1(Q,B)\oplus\Omega^3(Q).$$

for $b\in\Gamma(B)$. Conversely, a relatively easy degree count and study of the graded Leibniz identity for an arbitrary vector field of degree 1 on $\mathcal{M}=Q^*[1]\oplus B[2]$ shows that it must be given as above, defining therefore an anchor ρ_Q, and the structure objects $[\![\cdot\,,\cdot]\!]$, ∇, ω and ∂_B.

We show that $\mathcal{Q}^2=0$ if and only if $(-\partial_B^*\colon B^*\to Q,[\![\cdot\,,\cdot]\!],\nabla,\omega)$ is a split Lie 2-algebroid structure on $B^*\oplus Q$ anchored by ρ_Q. For $f\in C^\infty(M)$ we have

$$\mathcal{Q}^2(f)=\mathbf{d}_Q(\rho_Q^*\mathbf{d}f)+\partial_B(\rho_Q^*\mathbf{d}f)\in\Omega^2(Q)\oplus\Gamma(B).$$

Hence $\mathcal{Q}^2(f)=0$ for all $f\in C^\infty(M)$ if and only if $\partial_B\circ\rho_Q^*=0$ and $\rho_Q[\![q_1,q_2]\!]_\Delta=[\rho_Q(q_1),\rho_Q(q_2)]$ for all $q_1,q_2\in\Gamma(Q)$. Now we assume that these two conditions are satisfied. For $\tau\in\Gamma(Q^*)$ we have

$$\mathcal{Q}^2(\tau)=(\mathbf{d}_Q^2\tau+\langle\omega,\partial_B\tau\rangle)+(\partial_B\mathbf{d}_Q\tau+\mathbf{d}_\nabla(\partial_B\tau))\in\Omega^3(Q)\oplus\Omega^1(Q,B),$$

where $\partial_B\colon\Omega^k(Q)\to\Omega^{k-1}(Q,B)$ the vector bundle morphism defined by

$$\partial_B(\tau_1\wedge\ldots\wedge\tau_k)=\sum_{i=1}^k(-1)^{i+1}\tau_1\wedge\ldots\wedge\hat{i}\wedge\ldots\tau_k\wedge\partial_B\tau_i$$

for all $\tau_1,\tau_2\in\Gamma(Q^*)$. We find $\mathbf{d}_Q^2\tau(q_1,q_2,q_3)=\langle\mathrm{Jac}(q_1,q_2,q_3),\tau\rangle$ and $(\partial_B\mathbf{d}_Q\tau)(q,\beta)=-\langle\partial_B\Delta_q\tau,\beta\rangle$, and so $\mathcal{Q}^2(\tau)=0$ for all $\tau\in\Gamma(Q^*)$ if and only if $\mathrm{Jac}(q_1,q_2,q_3)=-\partial_B^*\omega(q_1,q_2,q_3)$ for all $q_1,q_2,q_3\in\Gamma(Q)$ and $\partial_B\Delta_q\tau=\nabla_q(\partial_B\tau)$ for all $q\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$.

Finally, we find for $b\in\Gamma(B)$:

$$\mathcal{Q}^2(b)=\mathcal{Q}(\mathbf{d}_\nabla b)+\mathbf{d}_Q\langle\omega,b\rangle+\partial_B\langle\omega,b\rangle.$$

The term $\partial_B\langle\omega,b\rangle$ is an element of $\Omega^2(Q,B)$, the term $\mathbf{d}_Q\langle\omega,b\rangle$ is an element of $\Omega^4(Q)$ and $\mathcal{Q}(\mathbf{d}_\nabla b)$ must be in $\Gamma(S^2B)\oplus\Omega^2(Q,B)\oplus\Omega^4(Q)$. A computation yields that its $\Omega^4(Q)$-term is $\langle\omega,\mathbf{d}_\nabla b\rangle$, which is defined by

$$\langle\omega,\mathbf{d}_\nabla b\rangle(q_1,q_2,q_3,q_4)=\sum_{\sigma\in Z_4}(-1)^\sigma\langle\omega(q_{\sigma(1)},q_{\sigma(2)},q_{\sigma(3)}),\nabla_{q_{\sigma(4)}}b\rangle,$$

where Z_4 is the group of cyclic permutations of $\{1,2,3,4\}$. The $\Omega^2(Q,B)$-term is $R_\nabla(\cdot,\cdot)b$ and the $\Gamma(S^2B)$-term is $\nabla_{\partial_B^*}b$ defined by $(\nabla_{\partial_B^*}b)(\beta_1,\beta_2)=\langle\nabla_{\partial_B^*\beta_1}b,\beta_2\rangle+\langle\nabla_{\partial_B^*\beta_2}b,\beta_1\rangle$ for all $\beta_1,\beta_2\in\Gamma(B^*)$. Hence $\mathcal{Q}^2(b)=0$ if and only if $\mathbf{d}_Q\langle\omega,b\rangle+\langle\omega,\mathbf{d}_\nabla b\rangle=0$, which is equivalent to $\mathbf{d}_{\nabla^*}\omega=0$; $\nabla_{\partial_B^*}b=0$, which is equivalent to $\nabla^*_{\partial_B^*\beta_1}\beta_2+\nabla^*_{\partial_B^*\beta_2}\beta_1=0$ for all $\beta_1,\beta_2\in\Gamma(B^*)$; and $R_\nabla(\cdot,\cdot)b=\partial_B\langle\omega,b\rangle$, which is equivalent to $R_{\nabla^*}(q_1,q_2)\beta=\omega(q_1,q_2,\partial_B^*\beta)$ for all $q_1,q_2\in\Gamma(Q)$ and $\beta\in\Gamma(B^*)$.

3.4. **Examples of Dorfman 2-representations and split Lie 2-algebroids.** We describe here four classes of examples of split Lie 2-algebroids. Later we will discuss their geometric meanings. We do not verify in detail the axioms for Dorfman 2-representations of for split Lie 2-algebroids. The computations in order to do this for Examples 3.4.2 and 3.4.3 are long, but straightforward. Note that, alternatively, the next section will provide a geometric proof of the fact that the following objects are Dorfman 2-representations (and split Lie 2-algebroids), since we will find them to be equivalent to special classes of VB-Courant algebroids. Note finally that a fifth fundamental class of examples in discussed in Section 5.

3.4.1. *Dorfman 2-representation associated to a Lie algebroid representation.* Let $(Q\to M,\rho,[\cdot\,,\cdot])$ be a Lie algebroid and $\nabla\colon\Gamma(Q)\times\Gamma(B)\to\Gamma(B)$ a representation of Q on a vector bundle B. Then $(\partial_B=0\colon Q^*\to B,\pounds^Q,\nabla,R=0)$ is a Dorfman 2-representation.

The corresponding Lie 2-algebroid is a semi-direct extension of the Lie algebroid Q (and a special case of the bicrossproduct Lie 2-algebroids defined in §5.1). Here $l_1=0$ and l_2 is given by $l_2(q_1+\beta_1,q_2+\beta_2)=[q_1,q_2]+(\nabla^*_{q_1}\beta_2-\nabla^*_{q_2}\beta_1)$ for $q_1,q_2\in\Gamma(Q)$ and $\beta_1,\beta_2\in\Gamma(B^*)$, and $l_3=0$. Hence $(Q\oplus B^*\to M,\rho=\rho_Q\circ\mathrm{pr}_Q,l_2)$ is simply a Lie algebroid.

3.4.2. *Standard Dorfman 2-representations.* Let $E \to M$ be a vector bundle, set $\partial_E = \mathrm{pr}_E : E \oplus T^*M \to E$, consider a skew-symmetric dull bracket $[\![\cdot\,,\cdot]\!]$ on $\Gamma(TM \oplus E^*)$, with $TM \oplus E^*$ anchored by pr_{TM}, and let $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ be the dual Dorfman connection. This defines as follows a split Lie 2-algebroid structure on the vector bundles $(TM \oplus E^*, \mathrm{pr}_{TM})$ and E^*, or a Dorfman 2-representation of $(TM \oplus E^*, \mathrm{pr}_{TM})$ on $E \oplus T^*M$ and E^*.

Let $\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ be the ordinary linear connection[9] defined by $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$. The vector bundle map $l_1 = -\mathrm{pr}_E^*\colon E^* \to TM \oplus E^*$ is just the opposite of the canonical inclusion. Finally define l_3 by $l_3(v_1, v_2, v_3) = \mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(v_1, v_2, v_3)$ and accordingly $R = R_\Delta \circ \iota_E \in \Omega^2(TM \oplus E^*, \mathrm{Hom}(E, E \oplus T^*M))$. (Note that since $TM \oplus E^*$ is anchored by pr_{TM}, the tangent part of the dull bracket must just be the Lie bracket of vector fields. The Jacobiator $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}$ can hence really be seen as an element of $\Omega^3(TM \oplus E^*, E^*)$.)

A straightforward verification of the axioms shows that l_1, $[\![\cdot\,,\cdot]\!]$, ∇^*, l_3 define a split Lie 2-algebroid. For reasons that will become clearer in §4.4.1, we call *standard* this type of split Lie 2-algebroid and Dorfman 2-representation.

3.4.3. *Adjoint Dorfman 2-representations.* Let $\mathsf{E} \to M$ be a Courant algebroid with anchor ρ_E and bracket $[\![\cdot\,,\cdot]\!]$ and choose a metric linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$, i.e. a linear connection that preserves the pairing. Set $\partial_{TM} = \rho_\mathsf{E}\colon \mathsf{E} \to TM$ and identify E with its dual via the pairing. The map $\Delta\colon \Gamma(\mathsf{E}) \times \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$,

$$\Delta_e e' = [\![e, e']\!] + \nabla_{\rho(e')} e$$

is a Dorfman connection, which we call the *basic Dorfman connection associated to* ∇. The dual skew-symmetric (!) dull bracket is given by $[\![e, e']\!]_\Delta = [\![e, e']\!] - \boldsymbol{\beta}^{-1}\rho^*\langle \nabla_{\cdot} e, e'\rangle$ for all $e, e' \in \Gamma(\mathsf{E})$. The map $\nabla^{\mathrm{bas}}\colon \Gamma(\mathsf{E}) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$,

$$\nabla^{\mathrm{bas}}_e X = [\rho(e), X] + \rho(\nabla_X e)$$

is a linear connection, the *basic connection associated to* ∇.

We now define the *basic curvature* $R^{\mathrm{bas}}_\Delta \in \Omega^2(\mathsf{E}, \mathrm{Hom}(TM, \mathsf{E}))$ by[10]

$$\begin{aligned} R^{\mathrm{bas}}_\Delta(e_1, e_2)X = &- \nabla_X [\![e_1, e_2]\!] + [\![\nabla_X e_1, e_2]\!] + [\![e_1, \nabla_X e_2]\!] \\ &+ \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1}\langle \nabla_{\nabla^{\mathrm{bas}}_{\cdot} X} e_1, e_2\rangle \end{aligned} \tag{12}$$

for all $e_1, e_2 \in \Gamma(\mathsf{E})$ and $X \in \mathfrak{X}(M)$. Then $(\rho, \Delta, \nabla^{\mathrm{bas}}, R^{\mathrm{bas}}_\Delta)$ is a Dorfman 2-representation. Note the similarity of this construction with the one of the adjoint representation up to homotopy (see [8]). The meaning of this similarity will become clear in §4.4.3.

The corresponding *adjoint* split Lie 2-algebroid can be described as follows. The map l_1 is $-\boldsymbol{\beta}^{-1} \circ \rho_\mathsf{E}^*\colon T^*M \to \mathsf{E}$ and $l_2(e_1 + \theta_1, e_2 + \theta_2)$ is

$$([\![e_1, e_2]\!] - \beta^{-1}\rho_\mathsf{E}\langle \nabla_{\cdot} e_1, e_2\rangle) + (\pounds_{\rho(e_1)}\theta_2 - \pounds_{\rho(e_2)}\theta_1 + \langle \rho^*\theta_1, \nabla_{\cdot} e_2\rangle - \langle \rho^*\theta_2, \nabla_{\cdot} e_1\rangle)$$

for $e_1, e_2 \in \Gamma(\mathsf{E})$ and $\theta_1, \theta_2 \in \Omega^1(M)$. The form $l_3 \in \Omega^3(\mathsf{E}, T^*M)$ is given by $l_3(e_1, e_2, e_3) = \langle R^{\mathrm{bas}}_\Delta(e_1, e_2), e_3\rangle$ and corresponds to the tensor Ψ defined in [13, Definition 4.1.2] (the right-hand side of (12)). We will see in [11] that the adjoint

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[9] To see that $\nabla = \mathrm{pr}_E \circ \Delta \circ \iota_E$ is an ordinary connection, recall that since $TM \oplus E^*$ is anchored by pr_{TM}, the map $\mathbf{d}_{E \oplus T^*M} = \mathrm{pr}_{TM}^*\, \mathbf{d}\colon C^\infty(M) \to \Gamma(E \oplus T^*M)$ sends $f \to (0, \mathbf{d}f)$.

[10] We have then $R^{\mathrm{bas}}_\Delta(e_1, e_2)X = -\nabla_X [\![e_1, e_2]\!]_\Delta + [\![\nabla_X e_1, e_2]\!]_\Delta + [\![e_1, \nabla_X e_2]\!]_\Delta + \nabla_{\nabla^{\mathrm{bas}}_{e_2} X} e_1 - \nabla_{\nabla^{\mathrm{bas}}_{e_1} X} e_2 - \beta^{-1}\rho^*\langle R_\nabla(X, \cdot)e_1, e_2\rangle$. Using $-R^*_\nabla = R_{\nabla^*} = R_\nabla$ (where we identify E with its dual using $\langle\cdot\,,\cdot\rangle$), the identity $R^{\mathrm{bas}}_\Delta(e_1, e_2) = -R^{\mathrm{bas}}_\Delta(e_2, e_1)$ is then immediate.

split Lie 2-algebroids are exactly the *split symplectic Lie 2-algebroids*, and correspond hence to splittings of the tangent doubles of Courant algebroids.

3.4.4. *Dorfman 2-representation defined by a 2-representation.* Let $(\partial_B\colon C \to B, \nabla, \nabla, R)$ be a representation up to homotopy of a Lie algebroid A on $B \oplus C$. Then the quadruple $(\partial_B \circ \mathrm{pr}_C\colon C \oplus A^* \to B, \Delta, \nabla, R)$ defined by

$$\begin{aligned} \Delta\colon \Gamma(A \oplus C^*) \times \Gamma(C \oplus A^*) &\to \Gamma(C \oplus A^*) \\ \Delta_{(a,\gamma)}(c, \alpha) &= (\nabla_a c, \pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c\rangle), \end{aligned} \tag{13}$$

$$\nabla\colon \Gamma(A \oplus C^*) \times \Gamma(B) \to \Gamma(B), \qquad \nabla_{(a,\gamma)} b = \nabla_a b \tag{14}$$

with $A \oplus C^*$ anchored by $\rho_A \circ \mathrm{pr}_A$, and $R \in \Omega^2(A \oplus C^*, \mathrm{Hom}(B, C \oplus A^*))$,

$$R((a_1, \gamma_1), (a_2, \gamma_2)) = (R(a_1, a_2), \langle \gamma_2, R(a_1, \cdot)\rangle + \langle \gamma_1, R(\cdot, a_2)\rangle) \tag{15}$$

is a Dorfman 2-representation.

The vector bundle map l_1 is here $l_1 = \iota_{C^*} \circ \partial_B^*$, where $\iota_{C^*}\colon C^* \to A \oplus C^*$ is the canonical inclusion, and the bracket l_2 is given by

$$l_2((a_1, \gamma_1) + \beta_1, (a_2, \gamma_2) + \beta_2) = ([a_1, a_2], \nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1) + (\nabla^*_{a_1}\beta_2 - \nabla^*_{a_2}\beta_1)$$

for $a_1, a_2 \in \Gamma(A)$, $\gamma_1, \gamma_2 \in \Gamma(C^*)$ and $\beta_1, \beta_2 \in \Gamma(B^*)$. The tensor l_3 is finally given by

$$l_3((a_1, \gamma_1), (a_2, \gamma_2), (a_3, \gamma_3)) = \langle R(a_1, a_2), \gamma_3\rangle + \text{c.p.}$$

Note that if we work with the dual A-representation up to homotopy $(\partial_B^*\colon B^* \to C^*, \nabla^*, \nabla^*, -R^*)$, then we get the Lie 2-algebroid defined in [31, Proposition 3.5] as the semi-direct product of a 2- representation and a Lie algebroid. This is then also a special case of the bicrossproduct of a matched pair of 2-representations (see §5.1). We will explain later the more natural choice that we make here.

3.5. **Morphisms of (split) Lie 2-algebroids.** In this section we quickly discuss morphisms of split Lie 2-algebroids.

Definition 3.10. *A morphism $\mu\colon (\mathcal{M}_1, \mathcal{Q}_1) \dashrightarrow (\mathcal{M}_2, \mathcal{Q}_2)$ of Lie 2-algebroids is a morphism $\mu\colon \mathcal{M}_1 \dashrightarrow \mathcal{M}_2$ of the underlying* [2]*-manifolds, such that*

$$\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star\colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1). \tag{16}$$

Assume that the two [2]-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ are split [2]-manifold $\mathcal{M}_1 = Q_1^*[1] \oplus B_1[2]$ and $\mathcal{M}_2 = Q_2^*[1] \oplus B_2[2]$. Then the homological vector fields $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as in §3.3 with two split Lie 2-algebroids; $(\rho_1\colon Q_1 \to TM_1, \partial_1\colon Q_1^* \to B_1, [\![\cdot\,,\cdot]\!]_1, \nabla^1, \omega_1)$ and $(\rho_2\colon Q_2 \to TM_2, \partial_2\colon Q_2^* \to B_2, [\![\cdot\,,\cdot]\!]_2, \nabla^2, \omega_2)$. Further, the morphism $\mu^\star\colon C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ over $\mu_0^*\colon C^\infty(M_2) \to C^\infty(M_1)$ decomposes as $\mu_Q\colon Q_1 \to Q_2$, $\mu_B\colon B_1^* \to B_2^*$ and $\mu_{12}\colon \wedge^2 Q_1 \to B_2^*$, all morphisms over $\mu_0\colon M_1 \to M_2$. We study (16) in these decompositions.

1. The condition $\mu^\star(\mathcal{Q}_2(f)) = \mathcal{Q}_1(\mu^\star(f))$ for all $f \in C^\infty(M_2)$ is $\mu_Q^\star(\rho_2^*\mathbf{d}f) = \rho_1^*\mathbf{d}(\mu_0^* f)$ for all $f \in C^\infty(M_2)$, which is equivalent to

$$T_m\mu_0(\rho_1(q_m)) = \rho_2(\mu_Q(q_m))$$

for all $q_m \in Q_1$. In other words $\mu_Q\colon Q_1 \to Q_2$ over $\mu_0\colon M_1 \to M_2$ is compatible with the anchors $\rho_1\colon Q_1 \to TM_1$ and $\rho_2\colon Q_2 \to TM_2$.

(2) The condition $\mu^\star(\mathcal{Q}_2(\tau)) = \mathcal{Q}_1(\mu^\star(\tau))$ for all $\tau \in \Gamma(Q_2^*)$ reads

$$\mu^\star(\mathbf{d}_2\tau + \partial_2\tau) = \partial_1(\mu_Q^\star\tau) + \mathbf{d}_1(\mu_Q^\star\tau)$$

for all $\tau \in \Gamma(Q_2^*)$. The left-hand side is

$$\underbrace{\mu_Q^\star(\mathbf{d}_2\tau) + \mu_{12}^\star(\partial_2\tau)}_{\in\Omega^2(Q_1)} + \underbrace{\mu_B^\star(\partial_2\tau)}_{\in\Gamma(B_1)}$$

and the right-hand side is

$$\partial_1(\mu_Q^\star\tau) + \mathbf{d}_1(\mu_Q^\star\tau) \in \Gamma(B_1) \oplus \Omega^2(Q_1).$$

Hence, $\mu^\star \circ \mathcal{Q}_2 = \mathcal{Q}_1 \circ \mu^\star$ on degree 1 functions if and only if $\mu_Q \circ \partial_1^* = \partial_2^* \circ \mu_B$ and $\mu_Q^\star(\mathbf{d}_2\tau) + \mu_{12}^\star(\partial_2\tau) = \mathbf{d}_1(\mu_Q^\star\tau)$ for all $\tau \in \Gamma(Q_2^*)$.

(3) Finally we find that $\mu^\star(\mathcal{Q}_2(b)) = \mathcal{Q}_1(\mu^\star(b))$ for all $b \in \Gamma(B_2)$ if and only if

$$\mu^\star(\mathbf{d}_{\nabla^2}b) = \mathbf{d}_{\nabla^1}(\mu_B^\star(b)) + \partial_1(\mu_{12}^\star(b)) \; \in \Omega^1(Q_1, B_1)$$

for all $b \in \Gamma(B_2)$ and

$$\mu_Q^\star\omega_2 = \mu_B \circ \omega_1 - \mathbf{d}_{\mu_0^\star\nabla^2}\mu_{12} \; \in \Omega^3(Q_1, \mu_0^* B_2^*).$$

In the equalities above we have used the following constructions. The form $\mu^\star(\mathbf{d}_{\nabla^2}b) \in \Omega^1(Q_1, B_1)$ is defined by $(\mu^\star(\mathbf{d}_{\nabla^2}b))(q_m) = {\mu_B}_m^*(\nabla^2_{\mu_Q(q_m)}b) \in B_1(m)$ for all $q_m \in Q_1$. Recall that μ_{12} can be seen as an element of $\Omega^2(Q_1, \mu_0^* B_2^*)$. The tensors $\mu_Q^\star\omega_2 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ and $\mu_B \circ \omega_1 \in \Omega^2(Q_1, \mu_0^* B_2^*)$ can be defined as follows:

$$(\mu_Q^\star\omega_2)(q_1(m), q_2(m), q_3(m)) = \omega_2(\mu_Q(q_1(m)), \mu_Q(q_2(m)), \mu_Q(q_3(m)))$$

in $B_2^*(\mu_0(m))$, and

$$(\mu_B \circ \omega_1)(q_1(m), q_2(m), q_3(m)) = \mu_B(\omega_1)(q_1(m), q_2(m), q_3(m))) \in B_2^*(\mu_0(m))$$

for all $q_1, q_2, q_3 \in \Gamma(Q_1)$. The linear connection $\mu_Q^\star\nabla^2 \colon \Gamma(Q_1) \times \Gamma(\mu_0^* B_2^*) \to \Gamma(\mu_0^* B_2^*)$ is defined by

$$(\mu_Q^\star\nabla^2)_q(\mu_0^!\beta)(m) = {\nabla^2}^*_{\mu_Q(q(m))}\beta \in B_2^*(\mu_0(m))$$

for all $q \in \Gamma(Q_1)$ and $\beta \in \Gamma(B_2^*)$.

Definition 3.11. *We call a triple (μ_Q, μ_B, μ_{12}) over μ_0 as above a* ***morphism of split Lie 2-algebroids****.*

In particular, if $\mathcal{M}_1 = \mathcal{M}_2$, $\mu_0 = \mathrm{Id}_M \colon M \to M$, $\mu_Q = \mathrm{Id}_Q \colon Q \to Q$ and $\mu_B = \mathrm{Id}_{B^*} \colon B^* \to B^*$, then $\mu_{12} \in \Omega^2(Q, B^*)$ is just a change of splitting. The five conditions above simplify to

(1) The dull brackets are related by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^*\mu_{12}(q, q')$.
(2) The connections are related by $\nabla^2_q b = \nabla^1_q b - \partial_B\langle\mu_{12}(q, \cdot), b\rangle$.
(3) The curvature terms are related by $\omega_1 - \omega_2 = \mathbf{d}_{1,\nabla^2}\mu_{12}$.

The operator $\mathbf{d}_{1,\nabla^2} \colon \Omega^\bullet(Q, B^*) \to \Omega^{\bullet+1}(Q, B^*)$ is defined by the dull bracket $[\![\cdot\,, \cdot]\!]_1$ and the connection ${\nabla^2}^*$.

4. VB-Courant algebroids and Lie 2-algebroids

In this section we describe and prove in detail the equivalence between VB-Courant algebroids and Lie 2-algebroids. In short, a homological vector field on a [2]-manifold defines an anchor and a Courant bracket on the corresponding metric double vector bundle. This Courant bracket and this anchor are automatically compatible with the metric and define so a linear Courant algebroid structure on the double vector bundle. Note that a correspondence of Lie 2-algebroids and VB-Courant algebroids has already been discussed by Li-Bland [13]. Our goal is to make this result more constructive by deducing it from the results in [10], and to illustrate it with several (partly new) examples.

4.1. **Definition and observations.** We will work with the following definition of a VB-Courant algebroid, which is due to Li-Bland [13].

Definition 4.1. *A VB-Courant algebroid is a metric double vector bundle*

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle q_B} \\ Q & \xrightarrow[q_Q]{} & M \end{array}$$

with core Q^* such that $\mathbb{E} \to B$ is a Courant algebroid and the following conditions are satisfied.

(1) *The anchor map $\Theta\colon \mathbb{E} \to TB$ is linear. That is,*

(17)
$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_Q}\downarrow & C \searrow & \downarrow{\scriptstyle q_B} \\ Q & \xrightarrow[q_Q]{} & M \end{array} \quad \xrightarrow{\Theta} \quad \begin{array}{ccc} TB & \xrightarrow{p_B} & B \\ {\scriptstyle Tq_B}\downarrow & B \searrow & \downarrow{\scriptstyle q_B} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

is a morphism of double vector bundles.

(2) *The Courant bracket is linear. That is*

$$[\![\Gamma^l_B(\mathbb{E}), \Gamma^l_B(\mathbb{E})]\!] \subseteq \Gamma^l_B(\mathbb{E}), \quad [\![\Gamma^l_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E}), \quad [\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0.$$

We make the following observations. Let $\rho_Q\colon Q \to TM$ be the side map of the anchor, i.e. if $\pi_Q(\chi) = q$ for $\chi \in \mathbb{E}$, then $Tq_B(\Theta(\chi)) = \rho_Q(q)$. In other words, if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$ then $\Theta(\chi)$ is linear over $\rho_Q(q)$. Let $\partial_B\colon Q^* \to B$ be the core map defined as follows by the anchor Θ:

$$\Theta(\sigma^\dagger) = (\partial_B \sigma)^\uparrow \tag{18}$$

for all $\sigma \in \Gamma(Q^*)$ (∂_B is sometimes called the "core-anchor"). Then the operator $\mathcal{D} = \Theta^* \mathbf{d}\colon C^\infty(B) \to \Gamma_B(\mathbb{E})$ satisfies $\mathcal{D}(q_B^* f) = (\rho_Q^* \mathbf{d} f)^\dagger$ for all $f \in C^\infty(M)$ and (8) yields immediately

$$\partial_B \circ \rho_Q^* = 0, \qquad \text{which is equivalent to} \qquad \rho_Q \circ \partial_B^* = 0. \tag{19}$$

Recall finally that if $\chi \in \Gamma^l_B(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, then $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for all $\tau \in \Gamma(Q^*)$.

4.2. **The fat Courant algebroid.** Recall that the fat bundle $\widehat{\mathbb{E}} \to M$ is the vector bundle which sheaf of sections is the sheaf of $C^\infty(M)$-modules $\Gamma^l_B(\mathbb{E})$, the linear sections of $\mathbb{E}$ over B. Gracia-Saz and Mehta show in [8] that if $\mathbb{E}$ is endowed with a linear Lie algebroid structure over B, then $\widehat{\mathbb{E}} \to M$ inherits a Lie algebroid structure, which is called the "fat Lie algebroid". For completeness, we describe here quickly the counterpart of this in the case of a linear Courant algebroid structure on $\mathbb{E} \to B$.

Note that the restriction of the pairing on $\mathbb{E}$ to linear sections of $\mathbb{E}$ defines a nondegenerate pairing on $\widehat{\mathbb{E}}$ with values in B^*. Since the Courant bracket of linear sections is again linear, we get the following theorem.

Theorem 4.2. *The vector bundle $\widehat{\mathbb{E}}$ inherits a Courant algebroid structure with pairing in B^*.*

We will come back to this structure in Corollary 4.9. Recall that for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, the core-linear section $\widetilde{\phi}$ of $\mathbb{E} \to B$ is defined by $\widetilde{\phi}(b_m) = 0_{b_m} +_B \overline{\phi(b_m)}$. Note that $\widehat{\mathbb{E}}$ is also naturally paired with Q^*: $\langle \chi(m), \sigma(m)\rangle = \langle \pi_Q(\chi(m)), \sigma(m)\rangle$ for all $\chi \in \Gamma^l_B(\mathbb{E}) = \Gamma(\widehat{\mathbb{E}})$ and $\sigma \in \Gamma(Q^*)$. This pairing is degenerate since it restricts to 0 on $\operatorname{Hom}(B, Q^*) \times_M Q^*$. The following proposition can easily be proved.

Proposition 4.3. (1) *The map*

$$\Delta\colon \Gamma(\widehat{\mathbb{E}}) \times \Gamma(Q^*) \to \Gamma(Q^*) \quad \textit{defined by} \quad (\Delta_\chi \tau)^\dagger = [\![\chi, \tau^\dagger]\!]$$

is a flat Dorfman connection, where $\widehat{\mathbb{E}}$ is endowed with the anchor $\rho_Q \circ \pi_Q$ and paired with Q^* as above.

(2) *The map*

$$\nabla^*\colon \Gamma(\widehat{\mathbb{E}}) \times \Gamma(B^*) \to \Gamma(B^*) \quad \textit{defined by} \quad \ell_{\nabla^*_\chi \beta} = \Theta(\chi)(\ell_\beta)$$

for all $\beta \in \Gamma(B^*)$ is a flat connection.

We dualise the connection ∇^* to a flat connection $\nabla\colon \Gamma(\widehat{\mathbb{E}}) \times \Gamma(B) \to \Gamma(B)$.

Proposition 4.4. *The following hold for Δ and ∇:*

(1) $\partial_B \circ \Delta = \nabla \circ \partial_B$ *and*
(2) $\left[\!\left[\chi, \widetilde{\phi}\right]\!\right]_{\widehat{\mathbb{E}}} = \widetilde{\Delta_\chi \circ \phi - \phi} \circ \nabla_\chi$

for $\chi \in \Gamma(\widehat{\mathbb{E}})$ and $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$.

Proof. Choose $\chi \in \Gamma^l_B(\mathbb{E})$ and $\tau \in \Gamma(Q^*)$. Then

$$(\partial_B \circ \Delta_\chi \tau)^\uparrow = \Theta(\Delta_\chi \tau^\dagger) = \Theta\left([\![\chi, \tau^\dagger]\!]\right) = \left[\Theta(\chi), (\partial_B \tau)^\uparrow\right] = (\nabla_\chi(\partial_B \tau))^\uparrow.$$

The second equation is easy to check by writing $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \cdot \tau_i^\dagger$ with $\beta_i \in \Gamma(B^*)$ and $\tau_i \in \Gamma(Q^*)$. □

Lemma 4.5. *For $\phi, \psi \in \Gamma(\operatorname{Hom}(B, Q^*))$ and $\tau \in \Gamma(Q^*)$, we have*

(1) $\left[\!\left[\tau^\dagger, \widetilde{\phi}\right]\!\right] = (\phi(\partial_B \tau))^\dagger = -\left[\!\left[\widetilde{\phi}, \tau^\dagger\right]\!\right]$ *and*
(2) $\left[\!\left[\widetilde{\phi}, \widetilde{\psi}\right]\!\right] = \psi \circ \partial_B \circ \widetilde{\phi - \phi} \circ \partial_B \circ \psi.$

Remark 4.6. Note that the second equality is the induced Lie algebra bundle structure induced on $\operatorname{Hom}(B, Q^*)$ by ∂_B.

Proof. We write $\phi=\sum_{i=1}^n \beta_i\cdot\tau_i$ and $\psi=\sum_{j=1}^n \beta_j'\cdot\tau_j$ with $\beta_1,\ldots,\beta_n,\beta_1',\ldots,\beta_n'\in\Gamma(B^*)$ and $\tau_1,\ldots,\tau_n\in\Gamma(Q^*)$. Hence, we have $\widetilde{\phi}=\sum_{i=1}^n \ell_{\beta_i}\tau_i^\dagger$ and $\widetilde{\psi}=\sum_{j=1}^n \ell_{\beta_j'}\tau_j^\dagger$. First we compute

$$\left[\!\!\left[\tau^\dagger,\sum_{i=1}^n \ell_{\beta_i}\tau_i^\dagger\right]\!\!\right]=\sum_{i=1}^n(\partial_B\tau)^\uparrow(\ell_{\beta_i})\tau_i^\dagger=\sum_{i=1}^n q_B^*\langle\partial_B\tau,\beta_i\rangle\tau_i^\dagger=\left(\sum_{i=1}^n\langle\partial_B\tau,\beta_i\rangle\tau_i\right)^\dagger$$

and we get (1). Since $\langle\tau^\dagger,\widetilde{\phi}\rangle=0$, the second equality follows. Then we have

$$\begin{aligned}\left[\!\!\left[\sum_{i=1}^n \ell_{\beta_i}\tau_i^\dagger,\sum_{j=1}^n \ell_{\beta_j'}\tau_j^\dagger\right]\!\!\right]&=\sum_{i=1}^n\sum_{j=1}^n \ell_{\beta_i}(\partial_B\tau_i)^\uparrow(\ell_{\beta_j'})\tau_j^\dagger-\ell_{\beta_j'}(\partial_B\tau_j)^\uparrow(\ell_{\beta_i})\tau_i^\dagger\\&=\left(\sum_{i=1}^n\sum_{j=1}^n\langle\partial_B\tau_i,\beta_j'\rangle\cdot\beta_i\cdot\tau_j-\langle\partial_B\tau_j,\beta_i\rangle\cdot\beta_j'\cdot\tau_i\right)^\dagger,\end{aligned}$$

which leads to (2). □

4.3. **Dorfman 2-representations and Lagrangian decompositions of VB-algebroids.** In this section, we study in detail the structure of a VB-Courant algebroid, using Lagrangian decompositions of the underlying metric double vector bundle. Our goal is the following theorem. Note the similarity of this result with Theorem 2.2 in the VB-algebroid case.

Theorem 4.7. *Let $(\mathbb{E};Q,B;M)$ be a VB-Courant algebroid and choose a Lagrangian splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$. Then there exists a Dorfman 2-representation (Δ,∇,R) of (Q,ρ_Q) on the core-anchor $\partial_B\colon Q^*\to B$ such that*

$$\begin{aligned}\Theta(\sigma_Q(q))&=\widehat{\nabla_q}\in\mathfrak{X}(B),\\ [\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]&=\sigma_Q([\![q_1,q_2]\!]_\Delta)-\widetilde{R(q_1,q_2)},\\ [\![\sigma_Q(q),\tau^\dagger]\!]&=(\Delta_q\tau)^\dagger\end{aligned}\tag{20}$$

for all $q,q_1,q_2\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$, where $[\![\cdot\,,\cdot]\!]_\Delta\colon\Gamma(Q)\times\Gamma(Q)\to\Gamma(Q)$ is the dull bracket that is dual to Δ.

Conversely, a Lagrangian splitting $\Sigma\colon Q\times B^*\to\mathbb{E}$ of the metric double vector bundle $\mathbb{E}$ together with a Dorfman 2-representation define a linear Courant algebroid structure on $\mathbb{E}$ by (20).

First we will construct the objects $[\![\cdot\,,\cdot]\!]_\Delta,\Delta,\nabla,R$ as in the theorem, and then we will prove in the appendix that they satisfy the axioms of a Dorfman 2-representation.

4.3.1. *Construction of the Dorfman 2-representation and outline of the proof.* The objects $[\![\cdot\,,\cdot]\!]_\Delta$, ∇, Δ and R are defined as in the theorem. Let us be more precise.

First recall that, by definition, the Courant bracket of two linear sections of $\mathbb{E}\to B$ is again linear. Hence, we can denote by $[\![q_1,q_2]\!]_\sigma$ the section of Q such that

$$\pi_Q\circ[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]=[\![q_1,q_2]\!]_\sigma\circ q_B.\tag{21}$$

Since for each $q\in\Gamma(Q)$, the anchor $\Theta(\sigma_Q(q))$ is a linear vector field on B over $\rho_Q(q)\in\mathfrak{X}(M)$, there exists a derivation $D_q\colon\Gamma(B^*)\to\Gamma(B^*)$ over $\rho_Q(q)$ such that

$\Theta(\sigma_Q(q))(\ell_\beta) = \ell_{D_q\beta}$ for all $\beta \in \Gamma(B^*)$, and $\Theta(\sigma_Q(q))(q_B^* f) = q_B^*(\rho_Q(q)(f))$ for all $f \in C^\infty(M)$. This defines a linear Q-connection $\nabla\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$:

$$\nabla_q b = D_q^* b$$

for all $b \in \Gamma(B)$. Then by definition, $\Theta(\sigma_Q(q)) = \widehat{\nabla_q} \in \mathfrak{X}^l(B)$.

For $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$, the bracket $[\![\sigma_Q(q), \tau^\dagger]\!]$ is a core section. It is easy to check that the map $\Delta\colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ defined by

$$[\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q \tau)^\dagger$$

is a Dorfman connection.[11]

The difference of the two linear sections $[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] - \sigma_Q([\![q_1, q_2]\!]_\sigma)$ is again a linear section, which projects to 0 under π_Q. Hence, there exists a vector bundle morphism $R(q_1, q_2)\colon B \to Q^*$ such that $\sigma_Q([\![q_1, q_2]\!]_\sigma) - [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \widetilde{R(q_1, q_2)}$. This defines a map $R\colon \Gamma(Q) \times \Gamma(Q) \to \Gamma(\mathrm{Hom}(B, Q^*))$. We show in the appendix that $(\partial_B, \nabla, \Delta, R)$ is a Dorfman 2-representation, and that $[\![\cdot\,,\cdot]\!]_\Delta = [\![\cdot\,,\cdot]\!]_\sigma$.

Conversely, choose a Lagrangian splitting $\Sigma\colon Q \times_M B$ of a metric double vector bundle $(\mathbb{E}, Q; B, M)$ with core Q^* and let $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ be the subset $\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma(\mathbb{E})$. Choose a Dorfman 2-representation $(\partial_B\colon Q^* \to B, \nabla, \Delta, R)$ of (Q, ρ_Q). Define then a vector bundle map $\Theta\colon \mathbb{E} \to TB$ over the identity on B by $\Theta(\sigma_Q(q)) = \widehat{\nabla_q}$ and $\Theta(\tau^\dagger) = (\partial_B\tau)^\dagger$ and a bracket $[\![\cdot\,,\cdot]\!]$ on $\mathcal{S}$ by

$$[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = \sigma_Q([\![q_1, q_2]\!]_\Delta) - \widetilde{R(q_1, q_2)}, \quad [\![\sigma_Q(q), \tau^\dagger]\!] = (\Delta_q\tau)^\dagger,$$
$$[\![\tau^\dagger, \sigma_Q(q)]\!] = \left(-\Delta_q\tau + \rho_Q^*\mathbf{d}\langle \tau, q\rangle\right)^\dagger, \quad [\![\tau_1^\dagger, \tau_2^\dagger]\!] = 0.$$

We show in the appendix that this bracket, the pairing and the anchor satisfy the conditions of Lemma 3.3, and so that $(\mathbb{E}, B; Q, M)$ with this structure is a VB-Courant algebroid.

4.3.2. *Change of Lagrangian decomposition.* Next we study how the Dorfman 2-representation $(\partial_B\colon Q^* \to B, \nabla, \Delta, R)$ associated to a Lagrangian decomposition of a VB-Courant algebroid changes when the Lagrangian decomposition changes.

The proof of the following proposition is straightforward and left to the reader. Compare this result with the equations subsequent to Definition 3.11, that describe a change of splittings of split Lie 2-algebroid.

Proposition 4.8. *Let $\Sigma^1, \Sigma^2\colon B \times_M Q \to \mathbb{E}$ be two Lagrangian splittings and let $\phi \in \Gamma(Q^* \otimes Q^* \otimes B^*)$ be the change of lift.*

1. *The Dorfman connections are related by $\Delta_q^2\tau = \Delta_q^1\tau - \phi(q)(\partial_B\tau)$*
2. *and the dull brackets consequently by $[\![q, q']\!]_2 = [\![q, q']\!]_1 + \partial_B^*\phi(q)^*(q')$.*
3. *The connections are related by $\nabla_q^2 = \nabla_q^1 - \partial_B \circ \phi(q)$.*
4. *The curvature terms are related by $\omega_{R^1} - \omega_{R^2} = \mathbf{d}_{\nabla^{2*}}\phi$, where the operator $\mathbf{d}_{\nabla^{2*}}$ is defined with the dull bracket $[\![\cdot\,,\cdot]\!]_1$ on $\Gamma(Q)$ and $\omega_{R_i}(q_1, q_2, q_3) = R_i(q_1, q_2)^* q_3$.*

As an application, we get the following corollary of Theorem 4.7 and Theorem 4.2. Given $\Delta\colon \Gamma(Q) \times \Gamma(Q^*) \to \Gamma(Q^*)$ and $\nabla\colon \Gamma(Q) \times \Gamma(B) \to \Gamma(B)$, we define the derivations $\Diamond\colon \Gamma(Q) \times \Gamma(\mathrm{Hom}(B, Q^*)) \to \Gamma(\mathrm{Hom}(B, Q^*))$ by $(\Diamond_q\phi)(b) = \Delta_q(\phi(b)) - \phi(\nabla_q b)$.

[11] Note that Condition $(C3)$ then implies that $[\![\tau^\dagger, \sigma_Q(q)]\!] = \left(-\Delta_q\tau + \rho_Q^*\mathbf{d}\langle \tau, q\rangle\right)^\dagger$.

Corollary 4.9. *Let* $(Q\oplus B^*\to M,\rho_Q,l_1,[\![\cdot\,,\cdot]\!],\nabla,l_3)$ *be a split Lie 2-algebroid and* $(\partial_B=-l_1^*,\Delta,\nabla,R)$ *the dual Dorfman 2-representation. Then the vector bundle* $\mathsf{E}:=Q\oplus\operatorname{Hom}(B,Q^*)$ *is a Courant algebroid with pairing in* B^* *given by* $\langle(q_1,\phi_1),(q_2,\phi_2)\rangle=\phi_1^*(q_2)+\phi_2^*(q_1)$*, with the anchor* $\tilde\rho\colon\mathsf{E}\to\widehat{\operatorname{Der}(B)}$*,* $\tilde\rho(q,\phi)^*=\nabla_q^*+\phi^*\circ l_1$ *over* $\rho(q)$ *and the bracket given by*

$$[\![(q_1,\phi_1),(q_2,\phi_2)]\!]=\Big([\![q_1,q_2]\!]_\Delta+l_1^*(\phi_1^*(q_2)),\Diamond_{q_1}\phi_2-\Diamond_{q_2}\phi_1+\nabla_\cdot^*(\phi_1^*(q_2))\\
+\phi_2\circ l_1^*\circ\phi_1-\phi_1\circ l_1^*\circ\phi_2+R(q_1,q_2)\Big).$$

The map $\mathcal{D}\colon\Gamma(B^*)\to\Gamma(\mathsf{E})$ *sends* q *to* $(l_1(q),\nabla_\cdot^*q)$*. The bracket does not depend on the choice of splitting.*

4.4. **Examples of VB-algebroids and the corresponding Dorfman 2-representations.** We give here some examples of VB-Courant algebroids, and we compute the corresponding classes of split Lie 2-algebroids. We find the split Lie 2-algebroids described in Section 3.4. In each of the examples below, it is easy to check that the Courant algebroid structure is linear. Hence, it is easy to check geometrically that the objects described in 3.4 are indeed split Lie 2-algebroids. This is why we omitted the detailed computations in that section.

4.4.1. *The standard Courant algebroid over a vector bundle.* We have discussed this example in great detail in [9], but not in the language of Dorfman 2-representations and Lie 2-algebroids. Note further that, in [9], we worked with general, not necessarily Lagrangian, linear splittings.

Let $q_E\colon E\to M$ be a vector bundle and consider the VB-Courant algebroid

$$\begin{array}{ccc} TE\oplus T^*E & \xrightarrow{\Phi_E:=(q_{E*},r_E)} & TM\oplus E^* \\ {\scriptstyle\pi_E}\big\downarrow & & \big\downarrow \\ E & \xrightarrow[q_E]{} & M \end{array}$$

with base E and side $TM\oplus E^*\to M$, and with core $E\oplus T^*M\to M$, or in other words the standard (VB-)Courant algebroid over a vector bundle $q_E\colon E\to M$. Recall that $TE\oplus T^*E$ has a natural linear metric (see [10]). Linear splittings of $TE\oplus T^*E$ are in bijection with dull brackets on sections of $TM\oplus E^*$ [9], and so also with Dorfman connections $\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$, and Lagrangian splittings of $TE\oplus T^*E$ are in bijection with skew-symmetric dull brackets on sections of $TM\oplus E^*$ [10].

The anchor $\Theta=\operatorname{pr}_{TE}\colon TE\oplus T^*E\to TE$ restricts to the map $\partial_E=\operatorname{pr}_E\colon E\oplus T^*M\to E$ on the cores, and defines an anchor $\rho_{TM\oplus E^*}=\operatorname{pr}_{TM}\colon TM\oplus E^*\to TM$ on the side. In other words, the anchor of $(e,\theta)^\dagger$ is $e^\uparrow\in\mathfrak{X}^c(E)$ and if $\widetilde{(X,\epsilon)}$ is a linear section of $TE\oplus T^*E\to E$ over $(X,\epsilon)\in\Gamma(TM\oplus E^*)$, the anchor $\Theta(\widetilde{(X,\epsilon)})\in\mathfrak{X}^l(E)$ is linear over X.

Let $\iota_E\colon E\to E\oplus T^*M$ be the canonical inclusion. In [9] we prove the following result (for general linear splittings).

Theorem 4.10. *Choose* $q,q_1,q_2\in\Gamma(TM\oplus E^*)$ *and* $\tau,\tau_1,\tau_2\in\Gamma(E\oplus T^*M)$*. The Courant-Dorfman bracket on sections of* $TE\oplus T^*E\to E$ *is given by*

(3) $\left[\!\left[\sigma(q),\tau^\dagger\right]\!\right]=(\Delta_q\tau)^\dagger$,

(4) $[\![\sigma(q_1), \sigma(q_2)]\!] = \sigma([\![q_1, q_2]\!]_\Delta) - R_\Delta\widetilde{(q_1, q_2)} \circ \iota_E$.

The anchor ρ is described by

(5) $\Theta(\sigma(q)) = \widehat{\nabla_q^*} \in \mathfrak{X}(E)$,

where $\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E)$ is defined by $\nabla_q = \mathrm{pr}_E \circ \Delta_q \circ \iota_E$ for all $q \in \Gamma(TM \oplus E^*)$.

Hence, if we choose a Lagrangian splitting of $TE \oplus T^*E$, we find the Dorfman 2-representation of Example 3.4.2.

4.4.2. *VB-Courant algebroid defined by a VB-Lie algebroid.* More generally, let

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

(with core C) be endowed with a VB-Lie algebroid structure $(D \to B, A \to M)$. Then the pair $(D, D\,\overset{*}{|}\,B)$ of vector bundles over B is a Lie bialgebroid, with $D\,\overset{*}{|}\,B$ endowed with the trivial Lie algebroid structure. We get a linear Courant algebroid $D \oplus_B (D\,\overset{*}{|}\,B)$ over B with side $A \oplus C^*$

$$\begin{array}{ccc} D \oplus_B (D\,\overset{*}{|}\,B) & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \oplus C^* & \longrightarrow & M \end{array}$$

and core $C \oplus A^*$. We check that the Courant algebroid structure is linear. Let $\Sigma\colon A \times_M B \to D$ be a linear splitting of D. Recall from Appendix B that we can define a linear splitting of $D\,\overset{*}{|}\,B$ by $\Sigma^\star\colon B \times_M C^* \to D\,\overset{*}{|}\,B$, $\langle \Sigma^\star(b_m, \gamma_m), \Sigma(a_m, b_m)\rangle = 0$ and $\langle \Sigma^\star(b_m, \gamma_m), c^\dagger(b_m)\rangle = \langle \gamma_m, c(m)\rangle$ for all $b_m \in B$, $a_m \in A$, $\gamma_m \in C^*$ and $c \in \Gamma(C)$. The linear splitting $\tilde{\Sigma}\colon B \times_M (A \oplus C^*) \to D \oplus_B (D\,\overset{*}{|}\,B)$, $\tilde{\Sigma}(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\star(b_m, \gamma_m))$ is then a Lagrangian splitting. A computation shows that the Courant bracket on $\Gamma_B(D \oplus_B (D\,\overset{*}{|}\,B))$ is given by

$$\begin{aligned}
&[\![\tilde{\sigma}_{A\oplus C^*}(a_1, \gamma_1), \tilde{\sigma}_{A\oplus C^*}(a_2, \gamma_2)]\!] \\
&= ([\sigma_A(a_1), \sigma_A(a_2)], \pounds_{\sigma_A(a_1)}\sigma^\star_{C^*}(\gamma_2) - \mathbf{i}_{\sigma_A(a_2)}\mathbf{d}\sigma^\star_{C^*}(\gamma_1)) \\
&= \Big(\sigma_A([a_1, a_2]) - \widetilde{R(a_1, a_2)}, \sigma^\star_{C^*}(\nabla^*_{a_1}\gamma_2 - \nabla^*_{a_2}\gamma_1) - \widetilde{\langle \gamma_2, R(a_1, \cdot)\rangle} + \widetilde{\langle \gamma_1, R(a_2, \cdot)\rangle}\Big) \\
&\big[\!\big[\tilde{\sigma}_{A\oplus C^*}(a, \gamma), (c, \alpha)^\dagger\big]\!\big] = \big(\nabla_a c^\dagger, (\pounds_a \alpha + \langle \nabla^*_\cdot \gamma, c\rangle)^\dagger\big) \\
&\big[\!\big[(c_1, \alpha_1)^\dagger, (c_2, \alpha_2)^\dagger\big]\!\big] = 0,
\end{aligned}$$

and the anchor of $D \oplus_B (D\,\overset{*}{|}\,B)$ is defined by

$$\Theta(\tilde{\sigma}_{A\oplus C^*}(a, \gamma)) = \Theta(\sigma_A(a)) = \widehat{\nabla_a} \in \mathfrak{X}^l(B), \quad \Theta((c, \alpha)^\dagger) = (\partial_B c)^\uparrow \in \mathfrak{X}^c(B),$$

where $(\partial_B\colon C \to B, \nabla\colon \Gamma(A) \times \Gamma(B) \to \Gamma(B), \nabla\colon \Gamma(A) \times \Gamma(C) \to \Gamma(C), R)$ is the 2-representation of A associated to the splitting $\Sigma\colon A \times_M B \to D$ of the VB-algebroid $(D \to B, A \to M)$. Hence, we have found the split Lie 2-algebroid described in Example 3.4.4.

4.4.3. *The tangent Courant algebroid.* We consider here a Courant algebroid $(\mathsf{E},\rho_{\mathsf{E}},[\![\cdot\,,\cdot]\!],\langle\cdot\,,\cdot\rangle)$. In this example, E will always be anchored by the Courant algebroid anchor map ρ_{E} and paired with itself by $\langle\cdot\,,\cdot\rangle$ and $\mathcal{D}=\boldsymbol{\beta}^{-1}\circ\rho_{\mathsf{E}}^*\circ\mathbf{d}\colon C^\infty(M)\to\Gamma(\mathsf{E})$. Note that $[\![\cdot\,,\cdot]\!]$ is not a dull bracket.

We show that, after the choice of a metric connection on E and so of a Lagrangian splitting $\Sigma^\nabla\colon TM\times_M\mathsf{E}\to T\mathsf{E}$ (see Examples 2.6 and 2.1), the VB-Courant algebroid structure on $(T\mathsf{E}\to TM,\mathsf{E}\to M)$ is equivalent to the Dorfman 2-representation defined by ∇ as in Example 3.4.3.

Theorem 4.11. *Choose a linear connection $\nabla\colon\mathfrak{X}(M)\times\Gamma(\mathsf{E})\to\Gamma(\mathsf{E})$ that preserves the pairing on E. The Courant algebroid structure on $T\mathsf{E}\to TM$ can be described as follows:*

(1) *The pairing is given by*

$$\langle e_1^\dagger,e_2^\dagger\rangle=0,\quad \langle\sigma_{\mathsf{E}}^\nabla(e_1),e_2^\dagger\rangle=p_M^*\langle e_1,e_2\rangle,\qquad \textit{and } \langle\sigma_{\mathsf{E}}^\nabla(e_1),\sigma_{\mathsf{E}}^\nabla(e_2)\rangle=0,$$

(2) *the anchor is given by $\Theta(\sigma_{\mathsf{E}}^\nabla(e))=\widehat{\nabla_e^{\rm bas}}$ and $\Theta(e^\dagger)=(\rho_{\mathsf{E}}(e))^\uparrow$,*

(3) *and the bracket is given by*

$$\left[\!\!\left[e_1^\dagger,e_2^\dagger\right]\!\!\right]=0,\qquad \left[\!\!\left[\sigma_{\mathsf{E}}^\nabla(e_1),e_2^\dagger\right]\!\!\right]=(\Delta_{e_1}e_2)^\dagger$$

and

$$[\![\sigma_{\mathsf{E}}^\nabla(e_1),\sigma_{\mathsf{E}}^\nabla(e_2)]\!]=\sigma_{\mathsf{E}}^\nabla([\![e_1,e_2]\!]_\Delta)-\widetilde{R_\Delta^{\rm bas}(e_1,e_2)}$$

for all $e,e_1,e_2\in\Gamma(\mathsf{E})$.

Proof. We use the characterisation of the tangent Courant algebroid in [3] (see also [13]): the pairing has already been discussed in Example 2.6. It is given by $\langle Te_1,Te_2\rangle=\ell_{\mathbf{d}\langle e_1,e_2\rangle}$ and $\langle Te_1,e_2^\dagger\rangle=p_M^*\langle e_1,e_2\rangle$. The anchor is given by $\Theta(Te)=\widehat{\pounds_{\rho_{\mathsf{E}}(e)}}\in\mathfrak{X}(TM)$ and $\Theta(e^\dagger)=(\rho_{\mathsf{E}}(e))^\uparrow\in\mathfrak{X}(TM)$. The bracket is given by $[\![Te_1,Te_2]\!]=T[\![e_1,e_2]\!]$ and $[\![Te_1,e_2^\dagger]\!]=[\![e_1,e_2]\!]^\dagger$ for all $e,e_1,e_2\in\Gamma(\mathsf{E})$.

(1) is easy to check (see also Example 2.6 and [10]). We check (2), i.e. that the anchor satisfies $\Theta(\sigma_{\mathsf{E}}^\nabla(e))=\widehat{\nabla_e^{\rm bas}}$. We have for $\theta\in\Omega^1(M)$ and $v_m\in TM$: $\Theta(\sigma_{\mathsf{E}}^\nabla(e)(v_m))(\ell_\theta)=\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}(v_m)-\langle\theta_m,\rho_{\mathsf{E}}(\nabla_{v_m}e)\rangle=\ell_{\nabla_e^{\rm bas *}\theta}(v_m)$ and for $f\in C^\infty(M)$: $\Theta(\sigma_{\mathsf{E}}^\nabla(e))(p_M^*f)=p_M^*(\rho_{\mathsf{E}}(e)f)$. This proves the equality.

Then we compute the brackets of our linear and core sections. Choose sections ϕ,ϕ' of $\operatorname{Hom}(TM,\mathsf{E})$. Then $\left[\!\!\left[Te,\widetilde{\phi}\right]\!\!\right]=\widetilde{\pounds_e\phi}$, with $\pounds_e\phi\in\Gamma(\operatorname{Hom}(TM,\mathsf{E}))$ defined by $(\pounds_e\phi)(X)=[\![e,\phi(X)]\!]-\phi([\rho_{\mathsf{E}}(e),X])$ for all $X\in\mathfrak{X}(M)$. The equality $\left[\!\!\left[\widetilde{\phi},Te\right]\!\!\right]=-\widetilde{\pounds_e\phi}+\mathcal{D}\ell_{\langle\phi(\cdot),e\rangle}$ follows. For $\theta\in\Omega^1(M)$, we compute $\left\langle\mathcal{D}\ell_\theta,e^\dagger\right\rangle=\Theta(e^\dagger)(\ell_\theta)=p_M^*\langle\rho_{\mathsf{E}}(e),\theta\rangle$. Thus, $\mathcal{D}\ell_\theta=T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta)+\widetilde{\psi}$ for a section $\psi\in\Gamma(\operatorname{Hom}(TM,\mathsf{E}))$ to be determined. Since $\langle\mathcal{D}\ell_\theta,Te\rangle=\Theta(Te)(\ell_\theta)=\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, the bracket $\langle T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\theta)+\widetilde{\psi},Te\rangle=\ell_{\mathbf{d}\langle\theta,\rho_{\mathsf{E}}(e)\rangle+\langle\psi(\cdot),e\rangle}$ must equal $\ell_{\pounds_{\rho_{\mathsf{E}}(e)}\theta}$, and we find $\langle\psi(\cdot),e\rangle=\mathbf{i}_{\rho_{\mathsf{E}}(e)}\mathbf{d}\theta$. Because $e\in\Gamma(\mathsf{E})$ was arbitrary we find $\psi(X)=-\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\theta$ for $X\in\mathfrak{X}(M)$. We get in particular

$$\left[\!\!\left[\widetilde{\phi},Te\right]\!\!\right]=-\widetilde{\pounds_e\phi}+T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\phi(\cdot),e\rangle)-\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{i}_X\mathbf{d}}\langle\phi(\cdot),e\rangle.$$

The formula $\left[\!\!\left[\widetilde{\phi},\widetilde{\phi'}\right]\!\!\right] = \phi'\circ\rho_{\mathsf{E}}\circ\widetilde{\phi - \phi}\circ\rho_{\mathsf{E}}\circ\phi'$ can easily be checked, as well as $\left[\!\!\left[\widetilde{\phi},e^\dagger\right]\!\!\right] = -\left[\!\!\left[e^\dagger,\widetilde{\phi}\right]\!\!\right] = -(\phi(\rho_{\mathsf{E}}(e)))^\dagger$. Using this, we find now easily that

$$\begin{aligned}
&[\![\sigma_{\mathsf{E}}^\nabla(e_1),\sigma_{\mathsf{E}}^\nabla(e_2)]\!] = \left[\!\!\left[Te_1 - \widetilde{\nabla.e_1}, Te_2 - \widetilde{\nabla.e_2}\right]\!\!\right]\\
&= T\,[\![e_1,e_2]\!] - \widetilde{\pounds_{e_1}\nabla.e_2} + \widetilde{\pounds_{e_2}\nabla.e_1} - T(\boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\langle\nabla.e_1,e_2\rangle)\\
&\quad + \boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{d}\langle\nabla.e_1,e_2\rangle} + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla.e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla.e_2)}e_1}\\
&= T\,[\![e_1,e_2]\!]_\Delta - \widetilde{\pounds_{e_1}\nabla.e_2} + \widetilde{\pounds_{e_2}\nabla.e_1} + \boldsymbol{\beta}^{-1}\rho_{\mathsf{E}}^*\widetilde{\mathbf{d}\langle\nabla.e_1,e_2\rangle}\\
&\quad + \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla.e_1)}e_2} - \widetilde{\nabla_{\rho_{\mathsf{E}}(\nabla.e_2)}e_1}.
\end{aligned}$$

Since for all $X\in\mathfrak{X}(M)$, we have

$$\begin{aligned}
&-(\pounds_{e_1}\nabla.e_2)(X) + (\pounds_{e_2}\nabla.e_1)(X) + \beta^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\langle\nabla.e_1,e_2\rangle\\
&= -[\![e_1,\nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 + [\![e_2,\nabla_X e_1]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\mathbf{i}_X\mathbf{d}\langle\nabla.e_1,e_2\rangle\\
&= -[\![e_1,\nabla_X e_2]\!] + \nabla_{[\rho_{\mathsf{E}}(e_1),X]}e_2 - [\![\nabla_X e_1,e_2]\!] - \nabla_{[\rho_{\mathsf{E}}(e_2),X]}e_1 + \beta^{-1}\rho_{\mathsf{E}}^*\pounds_X\langle\nabla.e_1,e_2\rangle,
\end{aligned}$$

we find that $[\![\sigma_{\mathsf{E}}^\nabla(e_1),\sigma_{\mathsf{E}}^\nabla(e_2)]\!] = T[\![e_1,e_2]\!]_\Delta - \widetilde{R_\Delta^{\text{bas}}(e_1,e_2)}$. Finally we compute $\left[\!\!\left[\sigma_{\mathsf{E}}^\nabla(e_1),e_2^\dagger\right]\!\!\right] = \left[\!\!\left[Te_1 - \widetilde{\nabla.e_1},e_2^\dagger\right]\!\!\right] = [\![e_1,e_2]\!]^\dagger + \nabla_{\rho_{\mathsf{E}}(e_2)}e_1^\dagger = \Delta_{e_1}e_2^\dagger$. □

4.5. **Categorical equivalence of Lie 2-algebroids and VB-Courant algebroids.** In this section we quickly describe morphisms of VB-Courant algebroids. Then we find an equivalence between the category of VB-Courant algebroids and the category of Lie 2-algebroids.

4.5.1. *Morphisms of VB-Courant algebroids.* Recall from §2.2 that a morphism $\Omega\colon\mathbb{E}_1 \dashrightarrow \mathbb{E}_2$ of metric double vector bundles is an isotropic relation $\Omega\subseteq\overline{\mathbb{E}_1}\times\mathbb{E}_2$ that is the dual of a morphism $\mathbb{E}_1 \ast Q_1\to\mathbb{E}_2\ast Q_2$. Assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ have linear Courant algebroid structures. Then Ω is a morphism of VB-Courant algebroid if it is a Dirac structure (with support) in $\overline{\mathbb{E}_1}\times\mathbb{E}_2$.

Choose two Lagrangian splittings $\Sigma^1\colon Q_1\times B_1\to\mathbb{E}_1$ and $\Sigma^2\colon Q_2\times B_2\to\mathbb{E}_2$. Then there exists four structure maps $\omega_0\colon M_1\to M_2$, $\omega_Q\colon Q_1\to Q_2$, $\omega_B\colon B_1^*\to B_2^*$ and $\omega_{12}\in\Omega^2(Q_1,\omega_0^*B_2^*)$ that define completely Ω. More precisely, Ω is spanned over $\operatorname{Graph}(\omega_Q\colon Q_1\to Q_2)$ by sections $\tilde{b}\colon \operatorname{Graph}(\omega_Q)\to\Omega$,

$$\tilde{b}(q_m,\omega_Q(q_m)) = \left(\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m), \sigma_{B_2}(b)(\omega_Q(q_m))\right)$$

for all $b\in\Gamma_{M_2}(B_2)$, and $\tau^\times\colon \operatorname{Graph}(\omega_Q)\to\Omega$,

$$\tau^\times(q_m,\omega_Q(q_m)) = \left((\omega_Q^\star\tau)^\dagger(q_m),\tau^\dagger(\omega_Q(q_m))\right)$$

for all $\tau\in\Gamma_{M_2}(Q_2^*)$. Note that Ω projects under $\pi_{B_1}\times\pi_{B_2}$ to $R_{\omega_B^*}\subseteq B_1\times B_2$. If $q\in\Gamma(Q_1)$ then $\omega_Q^! q\in\Gamma_{M_1}(\omega_0^*Q_2)$ can be written as $\sum_i f_i\omega_0^! q_i$ with $f_i\in C^\infty(M_1)$ and $q_i\in\Gamma_{M_2}(Q_2)$. The pair $\left(\sigma_{B_1}(\omega_B^\star b)(q_m) + \widetilde{\omega_{12}^\star(b)}(q_m), \sigma_{B_2}(b)(\omega_Q(q_m))\right)$ can be written as

$$\left(\left(\sigma_{Q_1}(q) + \langle\omega_{12}(q,\cdot),b(\omega_0(m))\rangle^\dagger\right)(\omega_B^\star b(m)), \sum_i f_i(m)\sigma_{Q_2}(q_i)(b(\omega_0(m)))\right).$$

Hence, Ω is spanned by the restrictions to $R_{\omega_B^*}$ of sections

$$\left(\sigma_{Q_1}(q)\circ \mathrm{pr}_1 + \langle \omega_{12}(q,\cdot), \mathrm{pr}_2\rangle^\dagger \circ \mathrm{pr}_1, \sum_i (f_i\circ q_{B_1}\circ \mathrm{pr}_1)\cdot (\sigma_{Q_2}(q_i)\circ \mathrm{pr}_2)\right) \tag{22}$$

for all $q\in\Gamma_{M_1}(Q_1)$ and

$$\left((\omega_Q^\star \tau)^\dagger\circ \mathrm{pr}_1, \tau^\dagger\circ \mathrm{pr}_2\right) \tag{23}$$

for all $\tau\in\Gamma(Q_2^*)$.

Checking all the conditions in Lemma 3.4 on the two types of sections (22) and (23) yields that $\Omega\to R_{\omega_B^*}$ is a Dirac structure with support if and only if

(1) $\omega_Q\colon Q_1\to Q_2$ over $\omega_0\colon M_1\to M_2$ is compatible with the anchors $\rho_1\colon Q_1\to TM_1$ and $\rho_2\colon Q_2\to TM_2$:
$$T_m\omega_0(\rho_1(q_m)) = \rho_2(\omega_Q(q_m))$$
for all $q_m\in Q_1$,

(2) $\partial_1\circ\omega_Q^\star = \omega_B^\star\circ\partial_2$ as maps from $\Gamma(Q_2^*)$ to $\Gamma(B_1)$, or equivalently $\omega_Q\circ\partial_1^* = \partial_2^*\circ\omega_B$,

(3) ω_Q preserves the dull brackets up to $\partial_2^*\omega_{12}$: i.e. $\omega_Q^\star(\mathbf{d}_2\tau)+\omega_{12}^\star(\partial_2\tau) = \mathbf{d}_1(\omega_Q^\star\tau)$ for all $\tau\in\Gamma(Q_2^*)$.

(4) ω_B and ω_Q intertwines the connections ∇^1 and ∇^2 up to $\partial_1\circ\omega_{12}$:
$$\omega_B^\star((\omega_Q^\star\nabla^2)_q b) = \nabla^1_q(\omega_B^\star(b)) - \partial_1\circ\langle\omega_{12}(q,\cdot), b\rangle\in\Gamma(B_1)$$
for all $q_m\in Q_1$ and $b\in\Gamma(B^2)$, and

(5) $\omega_Q^\star\omega_{R_2} - \omega_B\circ\omega_{R_1} = -\mathbf{d}_{(\omega_Q^\star\nabla^2)}\omega_{12}\in\Omega^3(Q_1,\omega_0^*B_2^*)$.

We find so that Ω is a morphism of VB-Courant algebroid if and only if it induces a morphism of split Lie 2-algebroids after any choice of Lagrangian decompositions of $\mathbb{E}_1$ and $\mathbb{E}_2$.

4.5.2. *Equivalence of categories.* The functors §2.2 between the category of metric double vector bundles and the category of [2]-manifolds restrict to functors between the category of VB-Courant algebroids and the category of Lie [2]-algebroids.

Theorem 4.12. *The category of Lie 2-algebroids is equivalent to the category of VB-Courant algebroids.*

Proof. Let $(\mathcal{M},\mathcal{Q})$ be a Lie 2-algebroid and consider the double vector bundle $\mathbb{E}_{\mathcal{M}}$ corresponding to $\mathcal{M}$. Choose a splitting $\mathcal{M}\simeq Q^*[1]\oplus B[2]$ of $\mathcal{M}$ and consider the corresponding Lagrangian splitting Σ of $\mathbb{E}_{\mathcal{M}}$.

As we have seen in §3.2, the split Lie 2-algebroid $(Q^*[1]\oplus B[2],\mathcal{Q})$ is equivalent to a Dorfman 2-representation. By Theorem 4.7, this Dorfman 2-representation defines a VB-Courant algebroid structure on the decomposition of $\mathbb{E}_{\mathcal{M}}$ and so by isomorphism on $\mathbb{E}_{\mathcal{M}}$. Further, by Proposition 4.8, the Courant algebroid structure on $\mathbb{E}_{\mathcal{M}}$ does not depend on the choice of splitting of $\mathcal{M}$, since a different choice of splitting will induce a change of Lagrangian splitting of $\mathbb{E}_{\mathcal{M}}$. This shows that the functor $\mathcal{G}$ restricts to a functor $\mathcal{G}_Q$ from the category of Lie 2-algebroids to the category of VB-Courant algebroids.

Sections 3.5 and 4.5.1 show that morphisms of split Lie 2-algebroids are sent by $\mathcal{G}$ to morphisms of decomposed VB-Courant algebroids.

The functor $\mathcal{F}$ restricts in a similar manner to a functor $\mathcal{F}_{\mathrm{VBC}}$ from the category of VB-Courant algebroids to the category of Lie 2-algebroids. The natural transformations found in the proof of Theorem 2.7 restrict to natural transformations $\mathcal{F}_{\mathrm{VBC}}\mathcal{G}_Q \simeq \mathrm{Id}$ and $\mathcal{G}_Q\mathcal{F}_{\mathrm{VBC}} \simeq \mathrm{Id}$. □

5. Application: VB-bialgebroids and bicrossproducts of matched pairs of 2-representations

In this section we show that the bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid and we explain geometrically this result.

5.1. **The bicrossproduct of a matched pair of 2-representations.** We construct a split Lie 2-algebroid $(A \oplus B) \oplus C$ induced by a matched pair of 2-representations as in Definition 2.4. The vector bundle $A \oplus B \to M$ is anchored by $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B$ and paired with $A^* \oplus B^*$ as follows:

$$\langle (a, b), (\alpha, \beta)\rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The morphism $A^* \oplus B^* \to C^*$ is $\partial_A^* \circ \mathrm{pr}_{A^*} + \partial_B^* \circ \mathrm{pr}_{B^*}$. The $A \oplus B$-Dorfman connection on $A^* \oplus B^*$ is defined by

$$\Delta_{(a,b)}(\alpha, \beta) = (\nabla_b^*\alpha + \pounds_a\alpha - \langle \nabla.b, \beta\rangle, \nabla_a^*\beta + \pounds_b\beta - \langle \nabla.a, \alpha\rangle).$$

The dual dull bracket on $\Gamma(A \oplus B)$ is

$$[\![(a, b), (a', b')]\!] = ([a, a'] + \nabla_b a' - \nabla_{b'} a, [b, b'] + \nabla_a b' - \nabla_{a'} b). \tag{24}$$

The $A \oplus B$-connection on C^* is simply given by $\nabla^*_{(a,b)}\gamma = \nabla^*_a\gamma + \nabla^*_b\gamma$ and the dual connection is $\nabla\colon \Gamma(A \oplus B) \times \Gamma(C) \to \Gamma(C)$,

$$\nabla_{(a,b)}c = \nabla_a c + \nabla_b c. \tag{25}$$

Finally, the form $\omega = l_3 \in \Omega^3(A \oplus B, C)$ is given by

$$\begin{aligned} \omega_R((a_1, b_1), (a_2, b_2), (a_3, b_3)) = {} & R(a_1, a_2)b_3 + R(a_2, a_3)b_1 + R(a_3, a_1)b_2 \\ & - R(b_1, b_2)a_3 - R(b_2, b_3)a_1 - R(b_3, b_1)a_2. \end{aligned} \tag{26}$$

The quadruple $(\partial_A^* + \partial_B^*\colon A^* \oplus B^* \to C^*, \nabla, \Delta, R)$ is a Dorfman 2-representation. Equivalently the vector bundle $(A \oplus B) \oplus C \to M$ with the anchor $\rho_A \circ \mathrm{pr}_A + \rho_B \circ \mathrm{pr}_B\colon A \oplus B \to TM$, $l_1 = (-\partial_A; \partial_B)\colon C \to A \oplus B$, $l_3 = \omega_R$ and the skew-symmetric dull bracket (24) define a split Lie 2-algebroid. Moreover, we prove the following theorem:

Theorem 5.1. *The bicrossproduct of a matched pair of 2-representations is a split Lie 2-algebroid with the structure given above. Conversely if $(A \oplus B) \oplus C$ has a split Lie 2-algebroid structure such that*

1. $[\![(a_1, 0), (a_2, 0)]\!] = ([a_1, a_2], 0)$ *with a section* $[a_1, a_2] \in \Gamma(A)$ *for all* $a_1, a_2 \in \Gamma(A)$ *and in the same manner* $[\![(0, b_1), (0, b_2)]\!] = (0, [b_1, b_2])$ *with a section* $[b_1, b_2] \in \Gamma(B)$ *for all* $b_1, b_2 \in \Gamma(B)$*, and*
2. $l_3((a_1, 0), (a_2, 0), (a_3, 0)) = 0$ *and* $l_3((0, b_1), (0, b_2), (0, b_3)) = 0$ *for all* a_1, a_2, a_3 *in* $\Gamma(A)$ *and* b_1, b_2, b_3 *in* $\Gamma(B)$*,*

then A and B are Lie subalgebroids of $(A \oplus B) \oplus C$ and $(A \oplus B) \oplus C$ is the bicrossproduct of a matched pair of 2-representations of A on $B \oplus C$ and of B on $A \oplus C$. The 2-representation of A is given by

$$\begin{aligned} &\partial_B(c) = \mathrm{pr}_B(l_1(c)),\ \nabla_a b = \mathrm{pr}_B[\![(a,0),(0,b)]\!],\ \nabla_a c = \nabla_{(a,0)}c, \\ &R_{AB}(a_1,a_2)b = l_3(a_1,a_2,b) \end{aligned} \tag{27}$$

and the B-representation is given by

$$\begin{aligned} &\partial_A(c) = -\mathrm{pr}_A(l_1(c)),\ \nabla_b a = \mathrm{pr}_A[\![(0,b),(a,0)]\!],\ \nabla_b c = \nabla_{(0,b)}c, \\ &R_{BA}(b_1,b_2)a = -l_3(b_1,b_2,a). \end{aligned} \tag{28}$$

Proof. Assume first that $(A \oplus B) \oplus C$ is a split Lie 2-algebroid with (1) and (2). The bracket $[\cdot\,,\cdot]\colon \Gamma(A)\times\Gamma(A)\to\Gamma(A)$ defined by $[\![(a_1,0),(a_2,0)]\!] = ([a_1,a_2],0)$ is obviously skew-symmetric and $\mathbb{R}$-bilinear. Define an anchor ρ_A on A by $\rho_A(a) = \rho_{A\oplus B}(a,0)$. Then we get immediately

$$([a_1, fa_2],0) = [\![(a_1,0), f(a_2,0)]\!] = f([a_1,a_2],0) + \rho_{A\oplus B}(a_1,0)(f)(a_2,0),$$

which shows that $[a_1, fa_2] = f[a_1,a_2] + \rho_A(a_1)(f)a_2$ for all $a_1,a_2 \in \Gamma(A)$. Further, we find $\mathrm{Jac}_{[\cdot\,,\cdot]}(a_1,a_2,a_3) = \mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}((a_1,0),(a_2,0),(a_3,0))) = -(\mathrm{pr}_A \circ l_1 \circ l_3)((a_1,0),(a_2,0),(a_3,0))) = 0$ since l_3 vanishes on sections of A. Hence A is a wide subalgebroid of the split Lie 2-algebroid. In a similar manner, we find a Lie algebroid structure on B. Next we prove that (27) defines a 2-representation of A. Using (ii) in Definition 3.8 we find for $a\in\Gamma(A)$ and $c\in\Gamma(C)$ that

$$\begin{aligned} \partial_B(\nabla_a c) &= (\mathrm{pr}_B \circ l_1)(\nabla_{(a,0)}c) \overset{\text{(ii)}}{=} \mathrm{pr}_B[\![(a,0), l_1(c)]\!] \\ &= \mathrm{pr}_B[\![(a,0),(0,\mathrm{pr}_B(l_1(c)))]\!] = \nabla_a(\partial_B c). \end{aligned}$$

In the third equation we have used Condition (1) and in the last equation the definitions of ∂_B and $\nabla_a\colon \Gamma(B)\to\Gamma(B)$. In the following, we will write for simplicity a for $(a,0)\in\Gamma(A\oplus B)$, etc. We get easily

$$R_{AB}(a_1,a_2)\partial_B c = l_3(a_1,a_2,\mathrm{pr}_B(l_1(c))) = l_3(a_1,a_2,l_1(c)) \overset{\text{(iv)}}{=} R_\nabla(a_1,a_2)c$$

and

$$\partial_B R_{AB}(a_1,a_2)b = (\mathrm{pr}_B \circ l_1 \circ l_3)(a_1,a_2,b) \overset{\text{(iii)}}{=} -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1,a_2,b))$$

for all $a_1,a_2\in\Gamma(A)$, $b\in\Gamma(B)$ and $c\in\Gamma(C)$. By Condition (1) and the definition of $\nabla_a\colon\Gamma(B)\to\Gamma(B)$, we find

$$\begin{aligned} R_\nabla(a_1,a_2)b &= \mathrm{pr}_B[\![a_1,[\![a_2,b]\!]]\!] - \mathrm{pr}_B[\![a_2[\![a_1,b]\!]]\!] - \mathrm{pr}_B[\![[\![a_1,a_2]\!],b]\!] \\ &= -\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1,a_2,b)). \end{aligned}$$

Hence, $\partial_B R_{AB}(a_1,a_2)b = R_\nabla(a_1,a_2)b$. Finally, an easy computation along the same lines shows that

$$\langle(\mathbf{d}_{\nabla^{\mathrm{Hom}}}R_{AB})(a_1,a_2,a_3),b\rangle = (\mathbf{d}_\nabla l_3)(a_1,a_2,a_3,b) \tag{29}$$

for $a_1,a_2,a_3\in\Gamma(A)$ and $b\in\Gamma(B)$. Since $\mathbf{d}_\nabla l_3 = 0$, we find $\mathbf{d}_{\nabla^{\mathrm{Hom}}}R_{AB} = 0$. In a similar manner, we prove that (28) defines a 2-representation of B. Further, by construction of the 2-representations, the split Lie 2-algebroid structure on $(A\oplus B)\oplus C)$ must be defined as in (24), (25) and (26), with the anchor $\rho_A\circ\mathrm{pr}_A + \rho_B\circ\mathrm{pr}_B$ and $l_1 = (-\partial_A,\partial_B)$. Hence, to conclude the proof, it only remains to check that the split Lie 2-algebroid conditions for these objects are equivalent to the seven conditions in Definition 2.4 for the two 2-representations.

First, we find immediately that (M1) is equivalent to (i). Then we find by construction

$$[a, \partial_A c] + \nabla_{\partial_B c} a = -[a, \mathrm{pr}_A(l_1(c))] + \nabla_{\mathrm{pr}_B(l_1(c))} a = \mathrm{pr}_A [\![l_1(c), a]\!] = -\mathrm{pr}_A [\![a, l_1(c)]\!].$$

Hence, we find (M2) if and only if $\mathrm{pr}_A [\![a, l_1(c)]\!] = \mathrm{pr}_A \circ l_1(\nabla_a c)$. But since

$$\begin{aligned}[\![a, l_1 c]\!] &= (\mathrm{pr}_A [\![a, l_1(c)]\!], \nabla_a \mathrm{pr}_B l_1(c)) = (\mathrm{pr}_A [\![a, l_1(c)]\!], \nabla_a \partial_B(c)) \\ &= (\mathrm{pr}_A [\![a, l_1(c)]\!], \partial_B \nabla_a c) = (\mathrm{pr}_A [\![a, l_1(c)]\!], \mathrm{pr}_B(l_1(\nabla_a c))),\end{aligned}$$

we have $\mathrm{pr}_A [\![a, l_1(c)]\!] = \mathrm{pr}_A \circ l_1(\nabla_a c)$ if and only if $[\![a, l_1 c]\!] = l_1(\nabla_a c)$. Hence (M2) is satisfied if and only if $[\![a, l_1 c]\!] = l_1(\nabla_a c)$ for all $a \in \Gamma(A)$ and $c \in \Gamma(C)$. In a similar manner, we find that (M3) is equivalent to $[\![b, l_1 c]\!] = l_1(\nabla_b c)$ for all $b \in \Gamma(B)$ and $c \in \Gamma(C)$. This shows that (M2) and (M3) together are equivalent to (ii).

Next, a simple computation shows that (M4) is equivalent to $R_\nabla(b, a)c = l_3(b, a, l_1(c))$. Since $R_\nabla(a, a')c = R_{AB}(a, a')\partial_B c = l_3(a, a', \mathrm{pr}_B(l_1(c))) = l_3(a, a', l_1(c))$ and $R_\nabla(b, b')c = l_3(b, b', l_1(c))$, we get that (M4) is equivalent to (iv).

Two straightwforward computations show that (M5) is equivalent to

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = -\mathrm{pr}_A(l_1 l_3(a_1, a_2, b))$$

and that (M6) is equivalent to

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, a)) = -\mathrm{pr}_B(l_1 l_3(b_1, b_2, a)).$$

But since $\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = -R_\nabla(a_1, a_2)b$ by construction and $R_\nabla(a_1, a_2)b = \partial_B R_{AB}(a_1, a_2)b = \mathrm{pr}_B(l_1 l_3(a_1, a_2, b))$, we find

$$\mathrm{pr}_B(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, b)) = -\mathrm{pr}_B(l_1 l_3(a_1, a_2, b)),$$

and in a similar manner

$$\mathrm{pr}_A(\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, a)) = -\mathrm{pr}_A(l_1 l_3(b_1, b_2, a)).$$

Since $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(a_1, a_2, a_3) = 0$, $\mathrm{Jac}_{[\![\cdot\,,\cdot]\!]}(b_1, b_2, b_3) = 0$, and l_3 vanishes on sections of A, and respectively on sections of B, we conclude that (M5) and (M6) together are equivalent to (iii).

Finally, a slightly longer, but still straightforward computation shows that

$$(\mathbf{d}_{\nabla^B} R_{AB})(b_1, b_2)(a_1, a_2) - (\mathbf{d}_{\nabla^A} R_{BA})(a_1, a_2)(b_1, b_2) = (\mathbf{d}_\nabla l_3)(a_1, a_2, b_1, b_2)$$

for all $a_1, a_2 \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(B)$. This, (29), the corresponding identity for R_{BA}, and the vanishing of l_3 on sections of A, and respectively on sections of B, show that (M7) is equivalent to (v). □

If $C = 0$, then $R_{AB} = 0$, $R_{BA} = 0$, $\partial_A = 0$ and $\partial_B = 0$ and the matched pair of 2-representations is just a matched pair of Lie algebroids. The double is then concentrated in degree 0, with $l_3 = 0$ and l_2 is the bicrossproduct Lie algebroid structure on $A \oplus B$ with anchor $\rho_A + \rho_B$ [16, 25]. Hence, in that case the split Lie 2-algebroid is just the bicrossproduct of a matched pair of representations and the dual (flat) Dorfman connection is the corresponding Lie derivative. The Lie 2-algebroid is in that case a genuine Lie 1-algebroid.

In a current work in progress we define the notion of matched pair of higher representations up to homotopy, and we show that the induced doubles are split Lie n-algebroids.

In the case where B has a trivial Lie algebroid structure and acts trivially up to homotopy on $\partial_A = 0\colon C \to A$, the double is the semi-direct product Lie 2-algebroid found in [31, Proposition 3.5] (see §3.4.4).

5.2. **VB-bialgebroids and double Lie algebroids.** Consider a double vector bundle $(D; A, B; M)$ with core C and a VB-Lie algebroid structure on each of its sides. Recall from §2.1.1 that $(D; A, B, M)$ is a double Lie algebroid if and only if, for any linear splitting of D, the two induced 2-representations (denoted as in §2.1.1) form a matched pair. By definition of a double Lie algebroid, $(D \ast A, D \ast B)$ is then a Lie bialgebroid over C^* [21], and so the double vector bundle

$$\begin{array}{ccc} D \ast A \oplus D \ast B & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A \oplus B & \longrightarrow & M \end{array}$$

with core $B^* \oplus A^*$ has the structure of a VB-Courant algebroid with base C^* and side $A \oplus B$. Note that we call the pair $(D \ast A, D \ast B)$ a **VB-bialgebroid over** C^*. Conversely, a VB-Courant algebroid $(\mathbb{E}; Q, B; M)$ with two transverse VB-Dirac structures $(D_1; Q_1, B; M)$ and $(D_2; Q_2, B; M)$ defines a VB-bialgebroid (D_1, D_2) over B. It is not difficult to see that a VB-bialgebroid[12] $(D_A \to X, A \to M)$, $(D_B \to X, B \to M)$ is equivalent to a double Lie algebroid structure on $(D_A \ast A; B, A; M) \simeq (D_B \ast B; B, A; M)$ with core X^*.

Consider again a double Lie algebroid $(D; A, B; M)$, together with a linear splitting $\Sigma\colon A \times_M B \to D$. Then

$$\tilde{\Sigma}\colon (A \oplus B) \times_M C^* \to D \ast A \oplus D \ast B$$

defined by $\tilde{\Sigma}((a(m), b(m)), \gamma_m) = (\sigma_A^\star(a)(\gamma_m), \sigma_B^\star(b)(\gamma_m))$, where $\sigma_A^\star\colon \Gamma(A) \to \Gamma^l_{C^*}(D \ast A)$ and $\sigma_B^\star\colon \Gamma(B) \to \Gamma^l_{C^*}(D \ast B)$ are defined as in Appendix B, is a linear Lagrangian splitting of $D \ast A \oplus D \ast B$ (see (30)). Recall from §2.1.1 that the splitting $\Sigma^\star\colon A \times_M C^* \to D \ast A$ of the VB-algebroid $(D \ast A \to C^*, A \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ of A on the complex $\partial_B^*\colon B^* \to C^*$. In the same manner, the splitting $\Sigma^\star\colon B \times_M C^* \to D \ast B$ of the VB-algebroid $(D \ast B \to C^*, B \to M)$ corresponds to the 2-representation $(\nabla^{C^*}, \nabla^{A^*}, -R^*)$ of B on the complex $\partial_A^*\colon A^* \to C^*$.

We check that the split Lie 2-algebroid corresponding to the linear splitting $\tilde{\Sigma}$ of $D \ast A \oplus D \ast B$ is the bicrossproduct of the matched pair of 2-representations. The equalities in (30) imply that we have to consider $A \oplus B$ as paired with $A^* \oplus B^*$ in the non standard way:

$$\langle (a, b), (\alpha, \beta) \rangle = \alpha(a) - \beta(b)$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$. The anchor of $\tilde{\sigma}(a, b) = (\sigma^\star(a), \sigma^\star(b))$ is $\widehat{\nabla_a^*} + \widehat{\nabla_b^*} \in \mathfrak{X}^l(C^*)$, and the anchor of $(\alpha, \beta)^\dagger = (\beta^\dagger, \alpha^\dagger) \in \Gamma^c_{C^*}(D \ast A \oplus D \ast B)$ is $(\partial_B^* \beta + \partial_A^* \alpha)^\uparrow \in \mathfrak{X}^c(C^*)$. The Courant bracket $[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!]$ is

$$\left([\sigma_A^\star(a), \beta^\dagger] + \pounds_{\sigma_B^\star(b)} \beta^\dagger - \mathbf{i}_{\alpha^\dagger} \mathbf{d}_{D \ast B} \sigma_A^\star(a), [\sigma_B^\star(b), \alpha^\dagger] + \pounds_{\sigma_A^\star(a)} \alpha^\dagger - \mathbf{i}_{\beta^\dagger} \mathbf{d}_{D \ast A} \sigma_B^\star(b) \right),$$

where $\mathbf{d}_{D \ast A}\colon \Gamma_{C^*}(\bigwedge^\bullet D \ast B) \to \Gamma_{C^*}(\bigwedge^{\bullet+1} D \ast B)$ is defined as usual by the Lie algebroid $D \ast A$, and similarly for $D \ast B$ (bear in mind that some non standard signs

[12] D_A has necessarily core B^* and D_B has core A^*.

arise from the signs in (30)). The derivation $\pounds\colon \Gamma(D \ast A) \times \Gamma(D \ast B) \to \Gamma(D \ast B)$ is described by

$$\pounds_{\beta^\dagger}\alpha^\dagger = 0, \quad \pounds_{\beta^\dagger}\sigma_B^\star(b) = -\langle b, \nabla^*_\cdot \beta\rangle^\dagger, \quad \pounds_{\sigma_A^\star(a)}\alpha^\dagger = \pounds_a\alpha^\dagger,$$
$$\pounds_{\sigma_A^\star(a)}\sigma_B^\star(b) = \sigma_B^\star(\nabla_a b) + \widetilde{R(a,\cdot)}b,$$

in [7, Lemma 4.8]. Similar formulae hold for $\pounds\colon \Gamma(D \ast B) \times \Gamma(D \ast A) \to \Gamma(D \ast A)$. We get

$$[\![(\sigma_A^\star(a), \sigma_B^\star(b)), (\beta^\dagger, \alpha^\dagger)]\!] = \left((\nabla^*_a\beta + \pounds_b\beta - \langle \nabla_\cdot a, \alpha\rangle)^\dagger, (\nabla^*_b\alpha + \pounds_a\alpha - \langle\nabla_\cdot b, \beta\rangle)^\dagger\right).$$

In the same manner, we get

$$\begin{aligned}
&[\![(\sigma_A^\star(a_1), \sigma_B^\star(b_1)), (\sigma_A^\star(a_2), \sigma_B^\star(b_2))]\!] \\
&= (\sigma_A^\star([a, a'] + \nabla_b a' - \nabla_{b'} a), \sigma_B^\star([b, b'] + \nabla_a b' - \nabla_{a'} b)) \\
&\quad + \left(-\widetilde{R(a_1, a_2)} + \widetilde{R(b_1, \cdot)}a_2 - \widetilde{R(b_2, \cdot)}a_1, -\widetilde{R(b_1, b_2)} + \widetilde{R(a_1, \cdot)}b_2 - \widetilde{R(a_2, \cdot)}b_1\right).
\end{aligned}$$

Hence we have the following result. Recall that we have found above that double Lie algebroids are equivalent to VB-Courant algebroids with two transverse VB-Dirac structures.

Theorem 5.2. *The correspondence established in Theorem 4.7, between decomposed VB-Courant algebroids and split Lie 2-algebroids, restricts to a correspondence between decomposed double Lie algebroids and split Lie 2-algebroids that are the bicrossproducts of matched pairs of 2-representations.*

In other words, decomposed VB-bialgebroids are equivalent to matched pairs of 2-representations.

Recall that if the vector bundle C is trivial, the matched pair of 2-representations is just a matched pair of the Lie algebroids A and B. The corresponding double Lie algebroid is the decomposed double Lie algebroid $(A \times_M B, A, B, M)$ found in [21]. The corresponding VB-Courant algebroid is

$$\begin{array}{ccc}
A \times_M B^* \oplus A^* \times_M B & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow \\
A \oplus B & \longrightarrow & M
\end{array}$$

with core $B^* \oplus A^*$. In that case there is a natural Lagrangian splitting and the corresponding Lie 2-algebroid is just the bicrossproduct Lie algebroid structure defined on $A \oplus B$ by the matched pair, see also the end of §5. This shows that the two notions of double of a matched pair of Lie algebroids; the bicrossproduct Lie algebroid in [25] and the double Lie algebroid in [21] are just the ℕ-geometric and the classical descriptions of the same object, and special cases of Theorem 5.2.

5.3. **Example: the two definitions of Lie bialgebroids.** Recall that a Lie bialgebroid (A, A^*) is a pair of Lie algebroids $(A \to M, \rho, [\cdot\,,\cdot])$ and $(A^* \to M, \rho_\star, [\cdot\,,\cdot]_\star)$ in duality such that $A \oplus A^* \to M$ with the anchor $\rho + \rho_\star$, the pairing

$$\langle (a_1, \alpha_1), (a_2, \alpha_2)\rangle = \alpha_1(a_2) + \alpha_2(a_1)$$

and the bracket

$$[\![(a_1, \alpha_1), (a_2, \alpha_2)]\!] = ([a_1, a_2] + \pounds_{\alpha_1} a_2 - \mathbf{i}_{\alpha_2}\mathbf{d}_{A^*} a_1, [\alpha_1, \alpha_2]_\star + \pounds_{a_1}\alpha_2 - \mathbf{i}_{a_2}\mathbf{d}_A \alpha_1)$$

is a Courant algebroid. Lie bialgebroids were originally defined in a different manner [22], and the definition above is at the origin of the abstract definition of Courant algebroids [15]. This Courant algebroid is sometimes called the bicrossproduct of the Lie bialgebroid, or the double of the Lie bialgebroid.

Mackenzie came up in [21] with an alternative notion of double of a Lie bialgebroid. Given a Lie bialgebroid as above, the double vector bundle

$$\begin{array}{ccc} T^*A \simeq T^*A^* & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

is a double Lie algebroid with the following structures. The Lie algebroid structure on A defines a linear Poisson structure on A^*, and so a linear Lie algebroid structure on $T^*A^* \to A^*$. In the same manner, the Lie algebroid structure on A^* defines a linear Poisson structure on A, and so a linear Lie algebroid structure on $T^*A \to A$ (see [7] for more details and for the matched pairs of 2-representations associated to a choice of linear splitting). The VB-Courant algebroid defined by this double Lie algebroid is $T^*A \overset{*}{} A \oplus T^*A \overset{*}{} A^*$ which is isomorphic to

$$\begin{array}{ccc} TA \oplus TA^* & \longrightarrow & TM \\ \downarrow & & \downarrow \\ A \oplus A^* & \longrightarrow & M \end{array}.$$

Computations reveal that the Courant algebroid structure is just the tangent of the Courant algebroid structure on $A \oplus A^*$, and so that the two notions of double of a Lie bialgebroid are just as above two descriptions of the same object.

Appendix A. Proof of Theorem 4.7

Let $(\mathbb{E}; Q, B; M)$ be a VB-Courant algebroid and choose a Lagrangian splitting $\Sigma\colon Q \times_M B$. We prove here that the obtained split linear Courant algebroid is equivalent to a Dorfman 2-representation. Recall the construction of the objects $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ in §4.3.1, and recall that $\mathcal{S} \subseteq \Gamma_B(\mathbb{E})$ is the subset $\{\tau^\dagger \mid \tau \in \Gamma(Q^*)\} \cup \{\sigma_Q(q) \mid q \in \Gamma(Q)\} \subseteq \Gamma(\mathbb{E})$. Recall also that the tangent double $(TB \to B; TM \to M)$ has a VB-Lie algebroid structure, which is described in Example 2.1.

We start with the proofs of two useful lemmas.

Lemma A.1. *For $\beta \in \Gamma(B^*)$, we have*

$$\mathcal{D}(\ell_\beta) = \sigma_Q(\partial_B^*\beta) + \widetilde{\nabla_\cdot^*\beta},$$

where $\nabla_\cdot^*\beta$ is seen as follows as a section of $\Gamma(\operatorname{Hom}(B, Q^*))$: $(\nabla_\cdot^*\beta)(b) = \langle \nabla_\cdot^*\beta, b\rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$.

Proof. For $\beta \in \Gamma(B^*)$, the section $\mathbf{d}\ell_\beta$ is a linear section of $T^*B \to B$. Since the anchor Θ is linear, the section $\mathcal{D}\ell_\beta = \Theta^*\mathbf{d}\ell_\beta$ is linear. Since for any $\tau \in \Gamma(Q^*)$,

$$\langle \mathcal{D}(\ell_\beta), \tau^\dagger\rangle = \Theta(\tau^\dagger)(\ell_\beta) = q_B^*\langle \partial_B\tau, \beta\rangle,$$

we find that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta) \in \Gamma(\ker \pi_Q)$. Hence, $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta)$ is a core-linear section of $\mathbb{E} \to B$ and there exists a section ϕ of $\operatorname{Hom}(B, Q^*)$ such that $\mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta) = \widetilde{\phi}$. We have

$$\ell_{\langle \phi, q\rangle} = \langle \widetilde{\phi}, \sigma_Q(q)\rangle = \langle \mathcal{D}(\ell_\beta) - \sigma_Q(\partial_B^*\beta), \sigma_Q(q)\rangle = \Theta(\sigma_Q(q))(\ell_\beta) = \ell_{\nabla_q^*\beta}$$

and so $\phi(b) = \langle \nabla^*_\cdot \beta, b\rangle \in \Gamma(Q^*)$ for all $b \in \Gamma(B)$. □

For each $q \in \Gamma(Q)$, ∇_q and Δ_q define a derivation $\lozenge_q$ of $\Gamma(\operatorname{Hom}(B, Q^*))$: for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$ and $b \in \Gamma(B)$, we have

$$(\lozenge_q \phi)(b) = \Delta_q(\phi(b)) - \phi(\nabla_q b).$$

Lemma A.2. *For $q \in \Gamma(Q)$ and $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, we have*

$$\left[\!\left[\sigma_Q(q), \widetilde{\phi}\right]\!\right] = \widetilde{\lozenge_q \phi}.$$

Proof. Write $\phi = \sum_{i=1}^n \beta_i \otimes \tau_i$ with $\beta_1, \ldots, \beta_n \in \Gamma(B^*)$ and $\tau_1, \ldots, \tau_n \in \Gamma(Q^*)$. Then $\widetilde{\phi} = \sum_{i=1}^n \ell_{\beta_i} \cdot \tau_i^\dagger$ and we can compute

$$\left[\!\left[\sigma_Q(q), \widetilde{\phi}\right]\!\right] = \sum_{i=1}^n \left(\ell_{\nabla^*_q \beta_i} \cdot \tau_i^\dagger + \ell_{\beta_i} \cdot (\Delta_q \tau_i)^\dagger\right) = \sum_{i=1}^n \widetilde{\nabla^*_q \beta_i \otimes \tau_i + \beta_i \otimes \Delta_q \tau_i}$$

on the one hand, and on the other hand

$$(\lozenge_q \phi)(b) = \Delta_q\left(\sum_{i=1}^n \langle \beta_i, b\rangle \tau_i\right) - \sum_{i=1}^n \langle \beta_i, \nabla_q b\rangle \tau_i = \left(\sum_{i=1}^n \beta_i \otimes \Delta_q \tau_i + \nabla^*_q \beta_i \otimes \tau_i\right)(b)$$

for any $b \in \Gamma(B)$. □

Now we can express all the conditions of Lemma 3.3 in terms of the objects $\partial_B, \Delta, \nabla, [\![\cdot\,,\cdot]\!]_\sigma, R$ found in §4.3.1.

Proposition A.3. *The anchor satisfies $\Theta \circ \Theta^* = 0$ if and only if*

1. *(1)* $\rho_Q \circ \partial_B^* = 0$ *and*
2. *(2)* $\nabla^*_{\partial_B^* \beta_1} \beta_2 + \nabla^*_{\partial_B^* \beta_2} \beta_1 = 0$ *for all* $\beta_1, \beta_2 \in \Gamma(B^*)$.

Proof. The composition $\Theta \circ \Theta^*$ vanishes if and only if $(\Theta \circ \Theta^*)\mathbf{d}F = 0$ for all linear and pullback functions $F \in C^\infty(B)$. For $f \in C^\infty(M)$, $\Theta(\Theta^* \mathbf{d}(q_B^* f)) = ((\partial_B \circ \rho_Q^*)\mathbf{d}f)^\uparrow$. For $\beta \in \Gamma(B^*)$, we find using Lemma A.1 $\Theta(\Theta^* \mathbf{d}\ell_\beta) = \Theta(\mathcal{D}\ell_\beta) = \Theta(\sigma_Q(\partial_B^* \beta) + \widetilde{\nabla^*_\cdot \beta}) = \widehat{\nabla_{\partial_B^* \beta}} + \widehat{\partial_B \circ \langle \nabla^*_\cdot \beta, \cdot\rangle}$. Here, $\partial_B \circ \langle \nabla^*_\cdot \beta, \cdot\rangle$ is as follows a morphism $B \to B$; $b \mapsto \partial_B(\langle \nabla^*_\cdot \beta, b\rangle)$. On a linear function $\ell_{\beta'}$, $\beta' \in \Gamma(B^*)$, $\Theta(\Theta^* \mathbf{d}\ell_\beta)(\ell_{\beta'}) = \ell_{\nabla^*_{\partial_B^* \beta} \beta'} + \ell_{\nabla^*_{\partial_B^* \beta'} \beta}$. On a pullback $q_B^* f$, $f \in C^\infty(M)$, this is $q_B^*(\pounds_{(\rho_Q \circ \partial_B^*)(\beta)} f)$. □

Proposition A.4. *The compatibility of Θ with the Courant algebroid bracket $[\![\cdot\,,\cdot]\!]$ implies*

1. *(1)* $\partial_B \circ R(q_1, q_2) = R_\nabla(q_1, q_2)$,
2. *(2)* $\rho_Q \circ [\![\cdot\,,\cdot]\!]_\sigma = [\cdot\,,\cdot] \circ (\rho_Q, \rho_Q)$, *or* $\Delta_q(\rho_Q^* \mathbf{d}f) = \rho_Q^* \mathbf{d}(\rho_Q(q)(f))$ *for all* $q \in \Gamma(Q)$ *and* $f \in C^\infty(M)$, *and*
3. *(3)* $\partial_B \circ \Delta = \nabla \circ \partial_B$.

Proof. We have

$$\Theta\, [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!] = [\Theta(\sigma_Q(q_1)), \Theta(\sigma_Q(q_2))] = \left[\widehat{\nabla_{q_1}}, \widehat{\nabla_{q_2}}\right]$$

and

$$\Theta\left(\sigma_Q([\![q_1, q_2]\!]_\sigma) - \widetilde{R(q_1, q_2)}\right) = \widehat{\nabla_{[\![q_1, q_2]\!]_\sigma}} - \widehat{\partial_B \circ R(q_1, q_2)}.$$

Applying both derivations to a pullback function q_B^*f for $f\in C^\infty(M)$ yields

$$\left[\widehat{\nabla_{q_1}},\widehat{\nabla_{q_2}}\right](q_B^*f)=q_B^*([\rho_Q(q_1),\rho_Q(q_2)]f).$$

and

$$\left(\widehat{\nabla_{[\![q_1,q_2]\!]_\sigma}}-\partial_B\circ\widetilde{R(q_1,q_2)}\right)(q_B^*f)=q_B^*(\rho_Q[\![q_1,q_2]\!]_\sigma(f))$$

Applying both vector fields to a linear function $\ell_\beta\in C^\infty(B)$, $\beta\in\Gamma(B^*)$, we get

$$\left[\widehat{\nabla_{q_1}},\widehat{\nabla_{q_2}}\right](\ell_\beta)=\ell_{\nabla^*_{q_1}\nabla^*_{q_2}\beta-\nabla^*_{q_2}\nabla^*_{q_1}\beta}$$

and

$$\left(\widehat{\nabla_{[\![q_1,q_2]\!]_\sigma}}-\partial_B\circ\widetilde{R(q_1,q_2)}\right)(\ell_\beta)=\ell_{\nabla^*_{[\![q_1,q_2]\!]_\sigma}\beta-R(q_1,q_2)^*\partial_B^*\beta}.$$

Since $R_{\nabla^*}(q_1,q_2)=-(R_\nabla(q_1,q_2))^*$, we find that

$$\Theta\,[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]=[\Theta(\sigma_Q(q_1)),\Theta(\sigma_Q(q_2))]$$

for all $q_1,q_2\in\Gamma(Q)$ if and only if (1) and (2) are satisfied.

In the same manner we compute for $q\in\Gamma(Q)$ and $\tau\in\Gamma(Q^*)$:

$$\Theta\left([\![\sigma_Q(q),\tau^\dagger]\!]\right)=(\partial_B\Delta_q\tau)^\uparrow$$

and

$$\left[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)\right]=\left[\widehat{\nabla_q},(\partial_B\tau)^\uparrow\right]=(\nabla_q(\partial_B\tau))^\uparrow.$$

Hence, $\Theta\left([\![\sigma_Q(q),\tau^\dagger]\!]\right)=\left[\Theta(\sigma_Q(q)),\Theta(\tau^\dagger)\right]$ if and only if $\partial_B(\Delta_q\tau)=\nabla_q(\partial_B\tau)$. □

Proposition A.5. *The condition (3) of Lemma 3.3 is equivalent to*

1. $R(q_1,q_2)=-R(q_2,q_1)$ *and*
2. $[\![q_1,q_2]\!]_\sigma+[\![q_2,q_1]\!]_\sigma=0$

for $q_1,q_2\in\Gamma(Q)$.

Proof. Choose q_1,q_2 in $\Gamma(Q)$. Then we have

$$[\![\sigma_Q(q_1),\sigma_Q(q_2)]\!]+[\![\sigma_Q(q_2),\sigma_Q(q_1)]\!]=\sigma_Q([\![q_1,q_2]\!]_\sigma+[\![q_2,q_1]\!]_\sigma)-\widetilde{R(q_1,q_2)}-\widetilde{R(q_2,q_1)}.$$

By the choice of the splitting, we have $\mathcal{D}\langle\sigma_Q(q_1),\sigma_Q(q_2)\rangle=\mathcal{D}(0)=0$. Hence, (3) of Lemma 3.3 is true on horizontal lifts of sections of Q if and only if $R(q_1,q_2)=-R(q_2,q_1)$ and $[\![q_1,q_2]\!]_\sigma+[\![q_2,q_1]\!]_\sigma=0$ for all $q_1,q_2\in\Gamma(Q)$. Further, we have $[\![\sigma_Q(q),\tau^\dagger]\!]=(\Delta_q\tau)^\dagger$ and $[\![\tau^\dagger,\sigma_Q(q)]\!]=(-\Delta_q\tau+\rho_Q^*\mathbf{d}\langle\tau,q\rangle)^\dagger$ by definition. On core sections (3) is trivially satisfied since both the pairing and the bracket of two core sections vanish. □

Proposition A.6. *The derivation formula (2) in Lemma 3.3 is equivalent to*

1. Δ *is dual to* $[\![\cdot\,,\cdot]\!]_\sigma$*, that is* $[\![\cdot\,,\cdot]\!]_\sigma=[\![\cdot\,,\cdot]\!]_\Delta$*,*
2. $[\![q_1,q_2]\!]_\sigma+[\![q_2,q_1]\!]_\sigma=0$ *for all* $q_1,q_2\in\Gamma(Q)$ *and*
3. $R(q_1,q_2)^*q_3=-R(q_1,q_3)^*q_2$ *for all* $q_1,q_2,q_3\in\Gamma(Q)$.

Proof. We compute (CA2) for linear and core sections. First of all, the equations

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\tau_3^\dagger\rangle=\langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\tau_3^\dagger\rangle+\langle\tau_2^\dagger,[\![\tau_1^\dagger,\tau_3^\dagger]\!]\rangle,$$

$$\Theta(\tau_1^\dagger)\langle\tau_2^\dagger,\sigma_Q(q)\rangle=\langle[\![\tau_1^\dagger,\tau_2^\dagger]\!],\sigma_Q(q)\rangle+\langle\tau_2^\dagger,[\![\tau_1^\dagger,\sigma_Q(q)]\!]\rangle$$

and

$$\Theta(\sigma_Q(q))\langle\tau_1^\dagger,\tau_2^\dagger\rangle=\langle[\![\sigma_Q(q),\tau_1^\dagger]\!],\tau_2^\dagger\rangle+\langle\tau_1^\dagger,[\![\sigma_Q(q),\tau_2^\dagger]\!]\rangle$$

are trivially satisfied for all $\tau_1, \tau_2, \tau_3 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$. Next we have for $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$:

$$
\begin{aligned}
&\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \tau^\dagger\rangle - \langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \tau^\dagger\rangle - \langle \sigma_Q(q_2), [\![\sigma_Q(q_1), \tau^\dagger]\!]\rangle \\
&= \widehat{\nabla_{q_1}}(q_B^*\langle q_2, \tau\rangle) - q_B^*\langle [\![q_1, q_2]\!]_\sigma, \tau\rangle - q_B^*\langle q_2, \Delta_{q_1}\tau\rangle \\
&= q_B^*\Big(\rho_Q(q_1)\langle q_2, \tau\rangle - \langle [\![q_1, q_2]\!]_\sigma, \tau\rangle - \langle q_2, \Delta_{q_1}\tau\rangle\Big)
\end{aligned}
$$

Hence $\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \tau^\dagger\rangle = \langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \tau^\dagger\rangle + \langle \sigma_Q(q_2), [\![\sigma_Q(q_1), \tau^\dagger]\!]\rangle$ for all $q_1, q_2 \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ if and only if Δ and $[\![\cdot\,,\cdot]\!]_\sigma$ are dual to each other. Using this, we compute

$$
\begin{aligned}
&\Theta(\tau^\dagger)\langle \sigma_Q(q_1), \sigma_Q(q_2)\rangle - \langle [\![\tau^\dagger, \sigma_Q(q_1)]\!], \sigma_Q(q_2)\rangle - \langle \sigma_Q(q_1), [\![\tau^\dagger, \sigma_Q(q_2)]\!]\rangle \\
&= 0 - \langle -(\Delta_{q_1}\tau)^\dagger + (\rho_Q^*\mathbf{d}\langle q_1, \tau\rangle)^\dagger, \sigma_Q(q_2)\rangle - \langle \sigma_Q(q_1), -(\Delta_{q_2}\tau)^\dagger + (\rho_Q^*\mathbf{d}\langle q_2, \tau\rangle)^\dagger\rangle \\
&= \ -q_B^*\langle [\![q_1, q_2]\!]_\sigma + [\![q_2, q_1]\!]_\sigma, \tau\rangle.
\end{aligned}
$$

Finally we have $\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \sigma_Q(q_3)\rangle = 0$ for all $q_1, q_2, q_3 \in \Gamma(Q)$, and $\langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)\rangle = \ell_{-R(q_1,q_2)^*q_3}$. This shows that

$$
\Theta(\sigma_Q(q_1))\langle \sigma_Q(q_2), \sigma_Q(q_3)\rangle = \langle [\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)\rangle + \langle \sigma_Q(q_2), [\![\sigma_Q(q_1), \sigma_Q(q_3)]\!]\rangle
$$

if and only if $0 = -R(q_1, q_2)^*q_3 - R(q_1, q_3)^*q_2$. □

Proposition A.7. *Assume that Δ and $[\![\cdot\,,\cdot]\!]_\sigma$ are dual to each other. The Jacobi identity in Leibniz form for sections in $\mathcal{S}$ is equivalent to*

(1) $R(q_1, q_2) \circ \partial_B = R_\Delta(q_1, q_2)$ *and*
(2)

$$
\begin{aligned}
&R(q_1, [\![q_2, q_3]\!]_\Delta) - R(q_2, [\![q_1, q_3]\!]_\Delta) - R([\![q_1, q_2]\!]_\Delta), q_3) \\
&+ \lozenge_{q_1}(R(q_2, q_3)) - \lozenge_{q_2}(R(q_1, q_3)) + \lozenge_{q_3}(R(q_1, q_2)) = \nabla^*_\cdot\,(R(q_1, q_2)^*q_3)
\end{aligned}
$$

for all $q_1, q_2, q_3 \in \Gamma(Q)$.

If R is skew-symmetric as in (1) of Proposition A.5, then the second equation is the same as (D4) in Definition 3.9.

Proof. The Jacobi identity is trivially satisfied on core sections since the bracket of two core sections is 0. Similarly, for $\tau_1, \tau_2 \in \Gamma(Q^*)$ and $q \in \Gamma(Q)$, we find $[\![\sigma_Q(q), [\![\tau_1^\dagger, \tau_2^\dagger]\!]]\!] = 0$ and $[\![[\![\sigma_Q(q), \tau_1^\dagger]\!], \tau_2^\dagger]\!] + [\![\tau_1^\dagger, [\![\sigma_Q(q), \tau_2^\dagger]\!]]\!] = 0$. We have

$$
\begin{aligned}
&\Big[\!\Big[\sigma_Q(q_1), \Big[\!\Big[\sigma_Q(q_2), \tau^\dagger\Big]\!\Big]\Big]\!\Big] - \Big[\!\Big[\sigma_Q(q_2), \Big[\!\Big[\sigma_Q(q_1), \tau^\dagger\Big]\!\Big]\Big]\!\Big] \\
&= \Big[\!\Big[\sigma_Q(q_1), (\Delta_{q_2}\tau)^\dagger\Big]\!\Big] - \Big[\!\Big[\sigma_Q(q_2), (\Delta_{q_1}\tau)^\dagger\Big]\!\Big] \\
&= (\Delta_{q_1}\Delta_{q_2}\tau)^\dagger - (\Delta_{q_2}\Delta_{q_1}\tau)^\dagger,
\end{aligned}
$$

and

$$
\begin{aligned}
\Big[\!\Big[[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!]\,, \tau^\dagger\Big]\!\Big] &= \Big[\!\Big[\sigma_Q([\![q_1, q_2]\!]_\Delta) - \widetilde{R(q_1, q_2)}, \tau^\dagger\Big]\!\Big] \\
&= (\Delta_{[\![q_1,q_2]\!]_\Delta}\tau)^\dagger + (R(q_1, q_2)(\partial_B\tau))^\dagger
\end{aligned}
$$

by Lemma 4.5. We now choose $q_1, q_2, q_3 \in \Gamma(Q)$ and compute

$$
\begin{aligned}
&[\![[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)]\!]\\
&= \left[\!\!\left[\sigma_Q([\![q_1, q_2]\!]_\Delta) - \widetilde{R(q_1, q_2)}, \sigma_Q(q_3)\right]\!\!\right]\\
&= \sigma_Q([\![[\![q_1, q_2]\!]_\Delta, q_3]\!]_\Delta) - \widetilde{R([\![q_1, q_2]\!]_\Delta, q_3)} - \mathcal{D}\ell_{\langle R(q_1,q_2)\cdot, q_3\rangle} + \lozenge_{q_3}\widetilde{R(q_1, q_2)}\\
&= \sigma_Q([\![[\![q_1, q_2]\!]_\Delta, q_3]\!]_\Delta) - \widetilde{R([\![q_1, q_2]\!]_\Delta, q_3)}\\
&\qquad - \sigma_Q(\partial_B^*\langle R(q_1, q_2)\cdot, q_3\rangle) - \nabla^*_\cdot\widetilde{\langle R(q_1, q_2)\cdot, q_3\rangle} + \lozenge_{q_3}\widetilde{R(q_1, q_2)}
\end{aligned}
$$

and

$$
\begin{aligned}
[\![\sigma_Q(q_2), [\![\sigma_Q(q_1), \sigma_Q(q_3)]\!]]\!] &= \left[\!\!\left[\sigma_Q(q_2), \sigma_Q([\![q_1, q_3]\!]_\Delta) - \widetilde{R(q_1, q_3)}\right]\!\!\right]\\
&= \sigma_Q([\![q_2, [\![q_1, q_3]\!]_\Delta]\!]_\Delta) - \widetilde{R(q_2, [\![q_1, q_3]\!]_\Delta)} - \lozenge_{q_2}\widetilde{R(q_1, q_3)}.
\end{aligned}
$$

We hence find that

$$[\![[\![\sigma_Q(q_1), \sigma_Q(q_2)]\!], \sigma_Q(q_3)]\!] + [\![\sigma_Q(q_2), [\![\sigma_Q(q_1), \sigma_Q(q_3)]\!]]\!] = [\![\sigma_Q(q_1), [\![\sigma_Q(q_2), \sigma_Q(q_3)]\!]]\!]$$

if and only if

$$[\![[\![q_1, q_2]\!]_\Delta, q_3]\!]_\Delta + [\![q_2, [\![q_1, q_3]\!]_\Delta]\!]_\Delta = [\![q_1, [\![q_2, q_3]\!]_\Delta]\!]_\Delta + \partial_B^*\langle R(q_1, q_2)\cdot, q_3\rangle$$

and

$$
\begin{aligned}
&R([\![q_1, q_2]\!]_\Delta, q_3) + \nabla^*_\cdot\langle R(q_1, q_2)\cdot, q_3\rangle - \lozenge_{q_3}R(q_1, q_2) + R(q_2, [\![q_1, q_3]\!]_\Delta) + \lozenge_{q_2}R(q_1, q_3)\\
&= R(q_1, [\![q_2, q_3]\!]_\Delta) + \lozenge_{q_1}R(q_2, q_3).
\end{aligned}
$$

We conclude using (10). □

A combination of Propositions A.3, A.4, A.5, A.6, A.7 and Lemma 3.3 proves Theorem 4.7.

Appendix B. Dualisation of double vector bundles and linear splittings

Double vector bundles can be dualised in two distinct ways. We denote by $D \mathbin{\overset{*}{|}} A$ the dual of D as a vector bundle over A and likewise for $D \mathbin{\overset{*}{|}} B$. The dual $D \mathbin{\overset{*}{|}} A$ is again a double vector bundle[13], with side bundles A and C^* and core B^* [19, 21].

$$
\begin{array}{ccc}
D & \xrightarrow{\pi_B} & B\\
{\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B}\\
A & \xrightarrow[q_A]{} & M
\end{array}
\qquad
\begin{array}{ccc}
D \mathbin{\overset{*}{|}} A & \xrightarrow{\pi_{C^*}} & C^*\\
{\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_{C^*}}\\
A & \xrightarrow[q_A]{} & M
\end{array}
\qquad
\begin{array}{ccc}
D \mathbin{\overset{*}{|}} B & \xrightarrow{\pi_B} & B\\
{\scriptstyle \pi_{C^*}}\downarrow & & \downarrow{\scriptstyle q_B}\\
C^* & \xrightarrow[q_{C^*}]{} & M
\end{array}
$$

[13]The projection $\pi_{C^*}\colon D \mathbin{\overset{*}{|}} A \to C^*$ is defined as follows: if $\Phi \in D \mathbin{\overset{*}{|}} A$ projects to $\pi_A(\Phi) = a_m$, then $\pi_{C^*}(\Phi) \in C^*_m$ is defined by $\pi_{C^*}(\Phi)(c_m) = \Phi(0^D_{a_m} +_B \overline{c_m})$ for all $c_m \in C_m$. If Φ_1 and $\Phi_2 \in D \mathbin{\overset{*}{|}} A$ satisfy $\pi_{C^*}(\Phi_1) = \pi_{C^*}(\Phi_2)$, $\pi_A(\Phi_1) = a^1_m$ and $\pi_A(\Phi_2) = a^2_m$, then $\Phi_1 +_{C^*} \Phi_2$ is defined by $(\Phi_1 +_{C^*} \Phi_2)(d_1 +_B d_2) = \Phi_1(d_1) + \Phi_2(d_2)$ for all $d_1, d_2 \in D$ with $\pi_B(d_1) = \pi_B(d_2)$ and $\pi_A(d_1) = a^1_m$, $\pi_A(d_2) = a^2_m$. The core element $\overline{\beta_m} \in D \mathbin{\overset{*}{|}} A$ defined by $\beta_m \in B^*$ is $\overline{\beta_m}(d) = \beta_m(\pi_B(d))$ for all $d \in D$ with $\pi_A(d) = 0^A_m$. By playing with the vector bundle structures on $D \mathbin{\overset{*}{|}} A$ and (1), one can show that each core element of $D \mathbin{\overset{*}{|}} A$ is of this form. See [21].

By dualising again $D \ast A$ over C^*, we get

$$\begin{array}{ccc} D \ast A \ast C^* & \xrightarrow{\pi_{C^*}} & C^* \\ {\scriptstyle \pi_B}\downarrow & & \downarrow{\scriptstyle q_{C^*}} \\ B & \xrightarrow[q_B]{} & M, \end{array}$$

with core A^*. In the same manner, we get a double vector bundle $D \ast B \ast C^*$ with sides A and C^* and core B^*.

The vector bundles $D \ast B \to C^*$ and $D \ast A \to C^*$ are, up to a sign, naturally in duality to each other [20]. The pairing

$$\langle\!\langle \cdot\,,\cdot \rangle\!\rangle \colon (D \ast A) \times_{C^*} (D \ast B) \to \mathbb{R}$$

is defined as follows: for $\Phi \in D \ast A$ and $\Psi \in D \ast B$ projecting to the same element γ_m in C^*, choose $d \in D$ with $\pi_A(d) = \pi_A(\Phi)$ and $\pi_B(d) = \pi_B(\Psi)$. Then $\langle \Phi, d\rangle_A - \langle \Psi, d\rangle_B$ does not depend on the choice of d and we set $\langle\!\langle \Phi, \Psi \rangle\!\rangle = \langle \Phi, d\rangle_A - \langle \Psi, d\rangle_B$. This implies in particular that $D \ast A$ is canonically (up to a sign) isomorphic to $D \ast B \ast C^*$ (we identify $D \ast A$ with $D \ast B \ast C^*$ using $-\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$) and $D \ast B$ is isomorphic to $D \ast A \ast C^*$ (we identify $D \ast B$ with $D \ast A \ast C^*$ using $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$).

Given a horizontal lift $\sigma_A \colon \Gamma(A) \to \Gamma^l_B(D)$, the "dual" horizontal lift $\sigma_A^\star \colon \Gamma(A) \to \Gamma^l_{C^*}(D \ast A)$ is defined by

$$\langle \sigma_A^\star(a)(\gamma_m), \sigma_A(a)(b_m)\rangle_A = 0, \qquad \langle \sigma_A^\star(a)(\gamma_m), c^\dagger(a(m))\rangle_A = \langle \gamma_m, c(m)\rangle$$

for all $a \in \Gamma(A)$, $c \in \Gamma(C)$, $b_m \in B$ and $\gamma_m \in C^*$. In the same manner, given a horizontal lift $\sigma_B \colon \Gamma(B) \to \Gamma^l_A(D)$, we define the dual horizontal lift $\sigma_B^\star \colon \Gamma(B) \to \Gamma^l_{C^*}(D \ast B)$. We have the following equations:

$$\langle\!\langle \sigma_A^\star(a), \sigma_B^\star(b) \rangle\!\rangle = 0, \quad \langle\!\langle \sigma_A^\star(a), \alpha^\dagger \rangle\!\rangle = -q_{C^*}^*\langle \alpha, a\rangle, \quad \langle\!\langle \beta^\dagger, \sigma_B^\star(b) \rangle\!\rangle = q_{C^*}^*\langle \beta, b\rangle, \tag{30}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(B)$, $\alpha \in \Gamma(A^*)$ and $\beta \in \Gamma(B^*)$.

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