THE GEOMETRISATION OF $\mathbb{N}$-MANIFOLDS OF DEGREE 2.

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ABSTRACT. This paper describes an equivalence of the canonical category of $\mathbb{N}$-manifolds of degree 2 with a category of involutive double vector bundles. More precisely, we show how involutive double vector bundles are in duality with double vector bundles endowed with a linear metric. We describe then how special sections of the metric double vector bundle that is dual to a given involutive double vector bundle are the generators of a graded manifold of degree 2 over the double base.

We discuss how split Poisson $\mathbb{N}$-manifolds of degree 2 are equivalent to *self-dual representations up to homotopy* and so, following Gracia-Saz and Mehta, to linear splittings of a certain class of VB-algebroids. In other words, the equivalence of categories above induces an equivalence between so called *Poisson involutive double vector bundles*, which are the dual objects to metric VB-algebroids, and Poisson $\mathbb{N}$-manifolds of degree 2.

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CONTENTS

1. Introduction

Graded manifolds are omnipresent objects in the field of Poisson geometry. In this context, they were first considered in the early 00's in Voronov's study of Lie bialgebroids [25] and in Roytenberg's supergeometric approach to Courant algebroids [23], see also [24]. One of our goals in this series of papers is to make more precise the $\mathbb{N}$-geometric approach to Courant algebroids, VB-algebroids and LA-courant algebroids [13].

An $\mathbb{N}$-graded manifold over a smooth manifold M is a sheaf of $\mathbb{N}$-graded, graded commutative, associative, unital $C^\infty(M)$-algebras over M, that is locally freely generated by finitely many elements of strictly positive degree. $\mathbb{N}$-manifolds of degree 1, and geometric stuctures thereon, are fully understood as the exterior algebras of sections of smooth vector bundles. Such a geometric understanding of $\mathbb{N}$-graded manifold of higher degrees does not exist in the literature. In the degree 2 case, graded manifolds with a homological vector field (Lie 2-algebroids) were linked to VB-Courant algebroids by Li-Bland in his thesis [13], building up on the correspondence of Courant algebroids with symplectic Lie 2-algebroids [23, 24]: the cotangent space of a Lie 2-algebroid becomes a symplectic Lie 2-algebroid with its canonical symplectic structure. The linear properties of the cotangent space constuction translate to an additional linear structure on the obtained Courant algebroid.

While this nice idea is very simple to understand, the obtained correspondence lacks very much of the concreteness of the well-understood correspondence of Lie 1-algebroids with Lie algebroids, and it gives little geometric insight on the meaning of the generators of the graded algebra in terms of the double vector bundle underlying the VB-Courant algebroid. Furthermore, morphisms are not studied in this correspondence.

This paper remedies to this by geometrising $\mathbb{N}$-manifolds of degree 2 via a certain class of double vector bundles – that was already defined and considered by Pradines in his first work on double vector bundles and nonholonomic jets [22] – and explaining the full and rich picture behind Li-Bland's observations. Our main result is an explicit equivalence between the category of degree 2 $\mathbb{N}$-manifolds with a category of double vector bundles with identical sides and an involution exchanging the sides and restricting to minus the identity on the core. Such double vector bundles were called 'symmetric double vector bundles with inverse symmetry" by Pradines [22]. For simplicity, we call them here *involutive double vector bundles*. We prove that the dual objects are double vector bundles endowed with a linear metric.

In particular, split $\mathbb{N}$-manifolds of degree 2 are equivalent in this manner to *involutive* splittings of involutive double vector bundles – "symmetric charts" in [22] – or equivalently to *Lagrangian* splittings of the dual metric double vector bundles. Our approach is a classical one; an extension of the construction of vector bundles over a manifold M from free and locally finitely generated sheaves of $C^\infty(M)$-modules, using the double vector bundle charts in [22].

Let us stress out here that while the equivalence of metric double vector bundles with [2]-manifolds can be seen as a special case of Li-Bland's correspondence [13] of Lie 2-algebroids with VB-Courant algebroids – namely the one of a trivial homological vector field versus a trivial Courant bracket – this example is neither explored in [13] nor completely straightfoward to deduce from the proof of Li-Bland's correspondence. We believe that this example, and in particular our new interpretation of metric double vector bundles as the dual objects to involutive double vector bundles, is in fact of significant importance since it could lead to geometrisations of $\mathbb{N}$-manifolds of higher degrees via relatively simple 'classical' geometric objects.

From our main theorem follow many enlightening results on geometric structures on degree 2 $\mathbb{N}$-manifolds and on their counterparts on metric double vector bundles: in sequels of this paper [11, 12] we recover for instance in a constructive manner Li-Bland's equivalences between the category of Lie 2-algebroids and the category of VB-Courant algebroids, and between the category of Poisson Lie 2-algebroids and the category of LA-Courant algebroids [13], providing along the way a more handy definition of the latter objects than the one already existing [13]. Most importantly, our geometric approach to graded manifolds of degree 2 allows us to describe several new classes of examples of those objects, and later to explain the links between 2-representations and Lie 2-algebroids, and between the different notions of doubles associated to Lie bialgebroids; namely the cotangent double and the bicrossproduct Lie algebroids [11].

In the second part of this paper we focus on Poisson structures of degree -2 on $\mathbb{N}$-manifolds of degree 2 (*Poisson* $\mathbb{N}$*-manifolds of degree* 2). We show that their splittings are the same as *self-dual* 2*-term representations up to homotopy.* We deduce from the equivalence of 2-term representations up to homotopy with linear splittings of VB-algebroids [9] that the equivalence of categories in our main theorem induces an equivalence of Poisson $\mathbb{N}$-manifolds of degree 2 with *metric VB-algebroids*, and we explain the dual picture of Poisson involutive double vector bundles. In particular, we find that symplectic [2]-manifolds are equivalent in this manner to cotangent doubles of metric vector bundles, a particular class of involutive double vector bundles, together with, up to a sign, the Poisson structure induced by the canonical symplectic form.

Note that Grabowski, Grabowska and Bruce propose in [4] an alternative geometric characterisation of $\mathbb{N}$-manifolds via double graded bundles, which simplifies to our description in the degree 2 case. Note also that in his PhD thesis [5], Fernando del Carpio-Marek finds independently, mostly through different methods, results that are similar to some of ours. Since his results will not be published, we summarise where appropriate the main lines of his approach and we bridge some of his results to ours.

Outline, main results and applications. This paper is organised as follows.

Section 2 starts by recalling how vector bundle morphisms are equivalent to morphisms of the sheaves of sections of the dual bundles. We then discuss in detail the necessary background on double vector bundles and their dualisations and splittings. We recall how linear splittings of VB-algebroids induce 2-term representations up to homotopy [9].

Section 3 recalls the definition of $\mathbb{N}$-manifolds and the equivalence of $\mathbb{N}$-manifolds of degree 1 with vector bundles. We define metric double vector bundles and their Lagrangian splittings, the existence of which we prove. We define the dual objects, the involutive double vector bundles, and we describe their morphisms. We construct an equivalence of this category with the category of $\mathbb{N}$-manifolds of degree 2.

Finally we discuss the geometric meaning of the generators of the graded algebras in terms of functions on the corresponding involutive double vector bundles: they can be understood as functions on the involutive double vector bundles that are polynomial in their sides, and on which the pullback under the involution is just multiplication by -1.

Section 4 studies Poisson structures of degree -2 on $\mathbb{N}$-manifolds of degree 2. We show how a Poisson structure of degree -2 on a split $\mathbb{N}$-manifold of degree 2 is equivalent to a 2-term representation up to homotopy that is dual to itself. Then we give the geometrisation of Poisson $\mathbb{N}$-manifolds of degree 2; namely linear Lie algebroids structures on metric double vector bundles, that are compatible with the metric, or equivalently, double linear Poisson structures on involutive double vector bundles, such that the involution is anti-Poisson. We prove that the equivalence of categories established in the previous section induces an equivalence of the category of Poisson $\mathbb{N}$-manifolds of degree 2 with the category of Poisson involutive double vector bundles. Finally, we discuss some examples of Poisson $\mathbb{N}$-manifolds of degree 2, and of the corresponding metric VB-algebroids and Poisson involutive double vector bundles.

We discuss in detail symplectic $\mathbb{N}$-manifolds of degree 2, and show how they correspond to symplectic cotangent doubles of metric vector bundles.

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Notation and conventions. We write $p_M\colon TM \to M$, $q_E\colon E \to M$ for vector bundle projections and $\pi_A\colon D \to A$ and $\pi_B\colon D \to B$ for the two "top" vector bundle projections of a double vector bundle. For a vector bundle $Q \to M$ we often identify without further mentioning the vector bundle $(Q^*)^*$ with Q via the canonical isomorphism. We write $\langle \cdot\,, \cdot\rangle$ or $\langle \cdot\,, \cdot\rangle_M$ for the canonical pairing of a vector bundle with its dual; i.e. $\langle a_m, \alpha_m\rangle = \alpha_m(a_m)$ for $a_m \in A$ and $\alpha_m \in A^*$. We use many different pairings; in general, which pairing is used is clear from its arguments. Given a section ε of E^*, we write $\ell_\varepsilon\colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m\rangle$ for all $e_m \in E$.

We assume all manifolds to be connected. Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the sheaf of sections of E is written $\Gamma(E)$. Let $f\colon M \to N$ be a smooth map between two smooth manifolds M and N. Then two vector fields $X \in \mathfrak{X}_U(M)$ and $Y \in \mathfrak{X}_V(N)$ are said to be ***f*-related** if $Tf \circ X = Y \circ f$ on $U \cap f^{-1}(V)$. We write then $X \sim_f Y$.

We write "$[n]$-manifold" for "$\mathbb{N}$-manifold of degree n". We write "2-representations" for "2-term representations up to homotopy".

2. Background and definitions on double vector bundles and VB-algebroids

We collect in this section background on vector bundles and their morphisms, on double vector bundles and their linear splittings, on VB-algebroids and their encoding by 2-representations. Further references will be given throughout the text.

2.1. Vector bundles and morphisms.

Let $A \to M$ and $B \to N$ be vector bundles and $\omega\colon A \to B$ a morphism of vector bundles over a smooth map $\omega_0\colon M \to N$. First we introduce a few notations. We say that $a \in \Gamma_U(A)$ is ω-related to $b \in \Gamma_V(B)$ if

$$\omega \circ a = b \circ \omega_0$$

on $U \cap \omega_0^{-1}(V)$. We write then $a \sim_\omega b$. We write $\omega_0^* B \to M$ for the pullback of B under ω_0; for $m \in M$, elements of $(\omega_0^* B)(m)$ are pairs $(m, b_{\omega_0(m)})$ with $b_{\omega_0(m)} \in B(\omega_0(m))$.

The dual of a morphism $\omega\colon A \to B$ over $\omega_0\colon M \to N$ is in general not a morphism of vector bundles, but a morphism $\omega^\star$ of modules over $\omega_0^*\colon C^\infty(N) \to C^\infty(M)$:

$$\omega^\star\colon \Gamma(B^*) \to \Gamma(A^*), \qquad \omega^\star(\beta)(m) = \omega_m^* \beta_{\omega_0(m)} \tag{1}$$

for all $\beta \in \Gamma(B^*)$ and $m \in M$. We prove the following lemma in Appendix A.

Lemma 2.1. *The map $\cdot^\star$, that sends a morphism of vector bundles $\omega\colon A \to B$ over $\omega_0\colon M \to N$ to the morphism $\omega^\star\colon \Gamma(B^*) \to \Gamma(A^*)$ of modules over $\omega_0^*\colon C^\infty(N) \to C^\infty(M)$, is a bijection.*

Note finally that for $\beta \in \Gamma_N(B^*)$ and $\ell_\beta \in C^{\infty,\mathrm{lin}}(B)$, we have

$$\ell_{\omega^\star\beta} = \omega^* \ell_\beta \in C^{\infty,\mathrm{lin}}(A). \tag{2}$$

2.2. **Double vector bundles.** We briefly recall the definitions of double vector bundles, of their *linear* and *core* sections, and of their *linear splittings* and *lifts*. We refer to [22, 17, 9] for more details.

A **double vector bundle** $(D; A, B; M)$ is a smooth manifold D with two vector bundle structures over the total spaces A and B of two vector bundles with base M, such that the square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

of vector bundle projections is commutative and the following conditions are satisfied:

(1) π_B is a vector bundle morphism over q_A;
(2) $+_B \colon D \times_B D \to D$ is a vector bundle morphism over $+ \colon A \times_M A \to A$, where $+_B$ is the addition map for the vector bundle $D \to B$, and
(3) the scalar multiplication $\mathbb{R} \times D \to D$ in the bundle $D \to B$ is a vector bundle morphism over the scalar multiplication $\mathbb{R} \times A \to A$.

(Note that the notation $(D; A, B; M)$ is omissive: all structure maps are of course part of the data of a double vector bundle.)

The corresponding statements to (1)–(3) for the operations in the bundle $D \to A$ follow. Note that the condition that each addition in D is a morphism with respect to the other is exactly

$$(d_1 +_A d_2) +_B (d_3 +_A d_4) = (d_1 +_B d_3) +_A (d_2 +_B d_4) \tag{3}$$

for $d_1, d_2, d_3, d_4 \in D$ with $\pi_A(d_1) = \pi_A(d_2)$, $\pi_A(d_3) = \pi_A(d_4)$ and $\pi_B(d_1) = \pi_B(d_3)$, $\pi_B(d_2) = \pi_B(d_4)$.

Given a double vector bundle $(D; A, B; M)$, the vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and of π_B. By (3), adding over A or over B elements of the core yields the same result, and C gets a natural vector bundle structure over M, the projection of which we call $q_C \colon C \to M$. The inclusion $C \hookrightarrow D$ is usually denoted by $C_m \ni c \longmapsto \overline{c} \in \pi_A^{-1}(0_m^A) \cap \pi_B^{-1}(0_m^B)$.

Given a double vector bundle $(D; A, B; M)$, the space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by two distinguished classes of sections (see [18]), the *linear* and the *core sections* which we now describe. For a smooth section $c \colon M \to C$, the corresponding **core section** $c^\dagger \colon B \to D$ is defined as

$$c^\dagger(b_m) = 0^D_{b_m} +_A \overline{c(m)}, \; m \in M, \, b_m \in B_m. \tag{4}$$

We denote the corresponding core section $A \to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B is written $\Gamma_B^c(D)$. A section $\xi \in \Gamma_B(D)$ is called **linear** if $\xi \colon B \to D$ is a bundle morphism from $B \to M$ to $D \to A$ over a section $a \in \Gamma(A)$. The space of linear sections of D over B is denoted by $\Gamma_B^\ell(D)$. A section $\psi \in \Gamma(B^* \otimes C)$ defines a linear section $\widetilde{\psi} \colon B \to D$ over the zero section $0^A \colon M \to A$ by $\widetilde{\psi}(b_m) = 0^D_{b_m} +_A \overline{\psi(b_m)}$ for all $b_m \in B$. We call $\widetilde{\psi}$ a **core-linear section**.

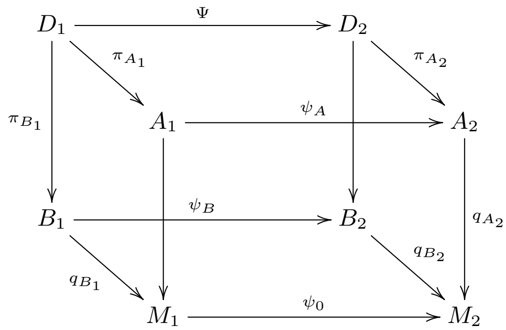
Example 2.2. *Let A, B, C be vector bundles over M and consider $D = A \times_M B \times_M C$. With the vector bundle structures $D = q_A^!(B \oplus C) \to A$ and $D = q_B^!(A \oplus C) \to B$, one finds that $(D; A, B; M)$ is a double vector bundle called the decomposed double vector bundle with sides A and B and core C. The core sections are given by $c^\dagger \colon b_m \mapsto (0_m^A, b_m, c(m))$, where $m \in M$,*

$b_m \in B_m$, $c \in \Gamma(C)$, and similarly for $c^\dagger \colon A \to D$. The space of linear sections $\Gamma^\ell_B(D)$ is naturally identified with $\Gamma(A) \oplus \Gamma(B^* \otimes C)$ via

$$(a, \psi) : b_m \mapsto (a(m), b_m, \psi(b_m)), \text{ where } \psi \in \Gamma(B^* \otimes C),\ a \in \Gamma(A).$$

In particular, the fibered product $A \times_M B$ is a double vector bundle over the sides A and B and its core is the trivial bundle over M.

Definition 2.3. *Let $(D_1; A_1, B_1; M_1)$ and $(D_2; A_2, B_2; M_2)$ be two double vector bundles. A double vector bundle morphism $(\Psi; \psi_A, \psi_B; \psi_0)$ from D_1 to D_2 is a commutative cube*



where all the faces are vector bundle morphisms.

Given a double vector bundle morphism $(\Psi; \psi_A, \psi_B; \psi_0)$, its restriction to the core bundles induces a vector bundle morphism $\psi_c \colon C_1 \to C_2$:

$$\Psi(\overline{\tau}) = \overline{\psi_c(\tau)}$$

for all $\tau \in C_1$. ψ_C is called the **core morphism** of Ψ.

Note that given $c \in \Gamma(C_2)$, we have $\Psi^\star(c^\dagger) = (\psi_c^\star(c))^\dagger$ for $c^\dagger \in \Gamma^c_{A_2}(D_2)$ or $c^\dagger \in \Gamma^c_{B_2}(D_2)$. If $\chi \in \Gamma^l_{A_2}(D_2)$ is linear over $b \in \Gamma(B_2)$, then $\Psi^\star(\chi) \in \Gamma^l_{A_1}(D_1)$ is linear over $\psi^\star_B(b)$. Similarly, if $\chi \in \Gamma^l_{B_2}(D_2)$ is linear over $a \in \Gamma(A_2)$, then $\Psi^\star(\chi) \in \Gamma^l_{B_1}(D_1)$ is linear over $\psi^\star_A(a)$. Furthermore, we have $\Psi^\star(q^*_{A_2} f \cdot \chi) = q^*_{A_1}(\psi_0^* f) \cdot \Psi^\star(\chi)$ for all $f \in C^\infty(M_2)$ and $\chi \in \Gamma^l_{A_2}(D_2)$.

2.2.1. *Linear splittings and lifts.* A **linear splitting** of $(D; A, B; M)$ is an injective morphism of double vector bundles $\Sigma \colon A \times_M B \hookrightarrow D$ over the identity on the sides A and B. That every double vector bundle admits local linear splittings was proved by [7] (see also [5] for a more elementary proof). Local linear splittings are equivalent to double vector bundle charts. Pradines originally defined double vector bundles as topological spaces with an atlas of double vector bundle charts [21] (see Definition 3.22). Using a partition of unity, he proved that (provided the double base is a smooth manifold) this implies the existence of a global double splitting [22]. Hence, any double vector bundle in the sense of our definition admits a (global) linear splitting.

Note that a linear splitting of D is equivalent to a **decomposition** of D, i.e. an isomorphism $\mathbb{I} \colon A \times_M B \times_M C \to D$ of double vector bundles over the identities on the sides and inducing the identity on the core. Given a linear splitting Σ, the corresponding decomposition $\mathbb{I}$ is given by $\mathbb{I}(a_m, b_m, c_m) = \Sigma(a_m, b_m) +_B (\tilde{0}_{b_m} +_A \overline{c_m})$. Given a decomposition $\mathbb{I}$, the corresponding linear splitting Σ is given by $\Sigma(a_m, b_m) = \mathbb{I}(a_m, b_m, 0^C_m)$.

A linear splitting Σ of D is also equivalent to a splitting σ_A of the short exact sequence of $C^\infty(M)$-modules

$$0 \longrightarrow \Gamma(B^* \otimes C) \hookrightarrow \Gamma^\ell_B(D) \longrightarrow \Gamma(A) \longrightarrow 0, \tag{5}$$

where the first map sends $\phi \in \Gamma(B^*\otimes C)$ to $\tilde{\phi} \in \Gamma_B^\ell(D)$ and the third map is the map that sends a linear section (ξ, a) to its base section $a \in \Gamma(A)$. The splitting σ_A is called a **horizontal lift**. Given Σ, the horizontal lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$ is given by $\sigma_A(a)(b_m) = \Sigma(a(m), b_m)$ for all $a \in \Gamma(A)$ and $b_m \in B$. By the symmetry of a linear splitting, we find that a lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$ is equivalent to a lift $\sigma_B \colon \Gamma(B) \to \Gamma_A^\ell(D)$. Given a lift $\sigma_A \colon \Gamma(A) \to \Gamma_B^\ell(D)$, the corresponding lift $\sigma_B \colon \Gamma(B) \to \Gamma_A^\ell(D)$ is given by $\sigma_B(b)(a(m)) = \sigma_A(a)(b(m))$ for all $a \in \Gamma(A)$, $b \in \Gamma(B)$.

Note that two linear splittings $\Sigma^1, \Sigma^2 \colon A \times_M B \to D$ differ by a section ϕ_{12} of $A^* \otimes B^* \otimes C \simeq \operatorname{Hom}(A, B^* \otimes C) \simeq \operatorname{Hom}(B, A^* \otimes C)$ in the following sense. For each $a \in \Gamma(A)$ the difference $\sigma_A^1(a) -_B \sigma_A^2(a)$ of horizontal lifts is the core-linear section defined by $\phi_{12}(a) \in \Gamma(B^* \otimes C)$. By symmetry, $\sigma_B^1(b) -_A \sigma_B^2(b) = \widetilde{\phi_{12}(b)}$ for each $b \in \Gamma(B)$.

2.2.2. *The tangent double of a vector bundle.* Let $q_E \colon E \to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE \to TM$ are the derivatives of the structure maps of $E \to M$. The space TE is a double vector bundle with core bundle $E \to M$.

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

The map $\bar{\ }\colon E \to p_E^{-1}(0^E) \cap (Tq_E)^{-1}(0^{TM})$ sends $e_m \in E_m$ to $\bar{e}_m = \frac{d}{dt}\big|_{t=0} te_m \in T_{0^E_m}E$. The core vector field corresponding to $e \in \Gamma(E)$ is the vertical lift $e^\uparrow \colon E \to TE$, i.e. the vector field with flow $\phi \colon E \times \mathbb{R} \to E$, $\phi(e'_m, t) = e'_m + te(m)$. An element of $\Gamma_E^\ell(TE) = \mathfrak{X}^\ell(E)$ is called a **linear vector field**. Since its flow is a flow of vector bundle morphisms, a linear vector field sends linear functions to linear functions and pullbacks to pullbacks. It is well-known (see e.g. [17]) that a linear vector field $\xi \in \mathfrak{X}^l(E)$ covering $X \in \mathfrak{X}(M)$ is equivalent to a derivation $\delta_\xi^* \colon \Gamma(E^*) \to \Gamma(E^*)$ over $X \in \mathfrak{X}(M)$, and hence to the dual derivation $\delta_\xi \colon \Gamma(E) \to \Gamma(E)$ over $X \in \mathfrak{X}(M)$. The precise correspondence is given by

$$\xi(\ell_\varepsilon) = \ell_{\delta_\xi^*(\varepsilon)} \quad \text{and} \quad \xi(q_E^* f) = q_E^*(X(f)) \tag{6}$$

for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$. We write $\widehat{\delta}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation δ of $\Gamma(E)$. The choice of a linear splitting Σ for $(TE; TM, E; M)$ is equivalent to the choice of a connection on E: Since a linear splitting gives us a linear vector field $\sigma_{TM}(X) \in \mathfrak{X}^l(E)$ for each $X \in \mathfrak{X}(M)$, we can define $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by $\sigma_{TM}(X) = \widehat{\nabla_X}$ for all $X \in \mathfrak{X}(M)$. Conversely, a connection $\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ defines a lift σ_{TM}^∇ for $(TE; TM, E; M)$ and a linear splitting $\Sigma^\nabla \colon TM \times_M E \to TE$.

We recall as well the relation between the connection and the Lie bracket of vector fields on E. Given ∇, it is easy to see using the equalities in (6) that, writing σ for σ_{TM}^∇:

$$[\sigma(X), \sigma(Y)] = \sigma[X, Y] - \widetilde{R_\nabla(X, Y)}, \qquad \left[\sigma(X), e^\uparrow\right] = (\nabla_X e)^\uparrow, \qquad \left[e^\uparrow, e'^\uparrow\right] = 0, \tag{7}$$

for all $X, Y \in \mathfrak{X}(M)$ and $e, e' \in \Gamma(E)$. That is, the Lie bracket of vector fields on M and the connection encode completely the Lie bracket of vector fields on E.

Now let us have a quick look at the other structure on the double vector bundle TE. The lift $\sigma_E^\nabla \colon \Gamma(E) \to \Gamma_{TM}^\ell(TE)$ is given by

$$\sigma_E^\nabla(e)(v_m) = T_m e(v_m) +_{TM} (T_m 0^E(v_m) -_E \overline{\nabla_{v_m} e}), \ v_m \in TM, \ e \in \Gamma(E).$$

Further, for $e \in \Gamma(E)$, the core section $e^\dagger \in \Gamma_{TM}(TE)$ is given by

$$e^\dagger(v_m) = T_m 0^E(v_m) +_E \left.\frac{d}{dt}\right|_{t=0} te(m).$$

2.2.3. *Dualisation and lifts.* Double vector bundles can be dualised in two distinct ways. We denote by D^*_A the dual of D as a vector bundle over A and likewise for D^*_B. The dual D^*_A is again a double vector bundle, with side bundles A and C^* and core B^* [16, 18].

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array} \qquad \begin{array}{ccc} D^*_A & \xrightarrow{\pi_{C^*}} & C^* \\ \downarrow & & \downarrow{\scriptstyle q_{C^*}} \\ A & \xrightarrow[q_A]{} & M \end{array} \qquad \begin{array}{ccc} D^*_B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle q_B} \\ C^* & \xrightarrow[q_{C^*}]{} & M \end{array}$$

The projection $\pi_{C^*}\colon D^*_A \to C^*$ is defined as follows: if $\Phi \in D^*_A$ projects to $\pi_A(\Phi) = a_m$, then $\pi_{C^*}(\Phi) \in C^*_m$ is defined by $\pi_{C^*}(\Phi)(c_m) = \Phi(0^D_{a_m} +_B \overline{c_m})$ for all $c_m \in C_m$. The addition in the fibers of the vector bundle $D^*_A \to C^*$ is defined as follows: if Φ_1 and $\Phi_2 \in D^*_A$ satisfy $\pi_{C^*}(\Phi_1) = \pi_{C^*}(\Phi_2)$, $\pi_A(\Phi_1) = a^1_m$ and $\pi_A(\Phi_2) = a^2_m$, then $\Phi_1 +_{C^*} \Phi_2$ is defined by

$$(\Phi_1 +_{C^*} \Phi_2)(d_1 +_B d_2) = \Phi_1(d_1) + \Phi_2(d_2)$$

for all $d_1, d_2 \in D$ with $\pi_B(d_1) = \pi_B(d_2)$ and $\pi_A(d_1) = a^1_m$, $\pi_A(d_2) = a^2_m$. The core element $\overline{\beta_m} \in D^*_A$ defined by $\beta_m \in B^*$ is given by $\overline{\beta_m}(d) = \beta_m(\pi_B(d))$ for all $d \in D$ with $\pi_A(d) = 0^A_m$. By playing with the vector bundle structures on D^*_A and (3), one can show that each core element of D^*_A is of this form. We encourage the reader who is not familiar with the dualisations of double vector bundles to check this, and also to find out where the projection to C^* is relevant in the definition of the addition over C^*.

Given a linear splitting $\Sigma\colon A \times_M B \to D$ the "dual" linear spliting $\Sigma^\star\colon A \times_M C^* \to D^*_A$ is defined by

$$\langle \Sigma^\star(a_m, \gamma_m), \Sigma(a_m, b_m)\rangle_A = 0, \qquad \langle \Sigma^\star(a_m, \gamma_m), c^\dagger(a_m)\rangle_A = \langle \gamma_m, c(m)\rangle \tag{8}$$

for all $a_m \in A$, $c \in \Gamma(C)$, $b_m \in B$ and $\gamma_m \in C^*$. (See [8] for a more complicated construction of the dual splitting. We let the reader check that the two constructions yield the same splitting.)

2.2.4. *Canonical (up to sign) pairing of D^*_A with D^*_B.* The vector bundles $D^*_A \to C^*$ and $D^*_B \to C^*$ are, up to a sign, naturally in duality to each other [17]. The pairing

$$\langle\!\langle \cdot\,,\cdot \rangle\!\rangle\colon D^*_A \times_{C^*} D^*_B \to \mathbb{R}$$

is defined as follows: for $\Phi \in D^*_A$ and $\Psi \in D^*_B$ projecting to the same element γ_m in C^*, choose $d \in D$ with $\pi_A(d) = \pi_A(\Phi)$ and $\pi_B(d) = \pi_B(\Psi)$. Then $\langle \Phi, d\rangle_A - \langle \Psi, d\rangle_B =: \langle\!\langle \Phi, \Psi \rangle\!\rangle$ does not depend on the choice of d. This implies in particular that D^*_A is canonically (up to a sign) isomorphic to $(D^*_B)^*_{C^*}$ and D^*_B is isomorphic to $(D^*_A)^*_{C^*}$.

2.3. **VB-algebroids.** Let $(D; A, B; M)$ be a double vector bundle with core C. Then $(D \to B; A \to M)$ is a **VB-algebroid** ([15]; see also [9]) if $D \to B$ has a Lie algebroid structure the anchor of which is a bundle morphism $\Theta_B\colon D \to TB$ over $\rho_A\colon A \to TM$ and such that the Lie bracket is linear:

$$[\Gamma^\ell_B(D), \Gamma^\ell_B(D)] \subset \Gamma^\ell_B(D), \qquad [\Gamma^\ell_B(D), \Gamma^c_B(D)] \subset \Gamma^c_B(D), \qquad [\Gamma^c_B(D), \Gamma^c_B(D)] = 0.$$

The vector bundle $A \to M$ is then also a Lie algebroid, with anchor ρ_A and bracket defined as follows: if $\xi_1, \xi_2 \in \Gamma^\ell_B(D)$ are linear over $a_1, a_2 \in \Gamma(A)$, then the bracket $[\xi_1, \xi_2]$ is linear over $[a_1, a_2]$. We also say that the Lie algebroid structure on $D \to B$ is linear over the Lie algebroid $A \to M$.

Since the anchor Θ_B is linear, it sends a core section $c^\dagger$, $c \in \Gamma(C)$ to a vertical vector field on B. This defines the **core-anchor** $\partial_B \colon C \to B$; for $c \in \Gamma(C)$ we have $\Theta_B(c^\dagger) = (\partial_B c)^\uparrow$ (see [14]).

Example 2.4. *It is easy to see from the considerations in §2.2.2 that the tangent double $(TE; E, TM; M)$ of a vector bundle $E \to M$ has a VB-algebroid structure $(TE \to E, TM \to M)$.*

2.4. **Representations up to homotopy.** Let $A \to M$ be a Lie algebroid and consider an A-connection ∇ on a vector bundle $E \to M$. Then the space $\Omega^\bullet(A, E)$ of E-valued Lie algebroid forms has an induced degree 1 operator $\mathbf{d}_\nabla$ given by:

$$\mathbf{d}_\nabla \omega(a_1, \ldots, a_{k+1}) = \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{k+1}) + \sum_i (-1)^{i+1} \nabla_{a_i}(\omega(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1}))$$

for all $\omega \in \Omega^k(A, E)$ and $a_1, \ldots, a_{k+1} \in \Gamma(A)$. We have $\mathbf{d}_\nabla(\alpha \wedge \omega) = \mathbf{d}_A \alpha \wedge \omega + (-1)^{|\alpha|} \alpha \wedge \mathbf{d}_\nabla \omega$ for $\alpha \in \Gamma(\wedge A^*)$ and $\omega \in \Omega(A, E)$ and $\mathbf{d}_\nabla^2 = 0$ if and only if the connection ∇ is flat; that is, if and only if ∇ defines a representation of A on E. Let $\mathcal{E} = \bigoplus_{k \in \mathbb{Z}} E_k[k]$ be now a graded vector bundle. Consider the space $\Omega(A, \mathcal{E})$ with grading given by $\Omega(A, \mathcal{E})[k] = \bigoplus_{i+j=k} \Omega^i(A, E_j)$.

Definition 2.5. [1][9] *A representation up to homotopy of A on $\mathcal{E}$ is a map $\mathcal{D} \colon \Omega(A, \mathcal{E}) \to \Omega(A, \mathcal{E})$ with total degree 1 and such that $\mathcal{D}^2 = 0$ and*

$$\mathcal{D}(\alpha \wedge \omega) = \mathbf{d}_A \alpha \wedge \omega + (-1)^{|\alpha|} \alpha \wedge \mathcal{D}(\omega),$$

for $\alpha \in \Gamma(\wedge A^*)$, $\omega \in \Omega(A, \mathcal{E})$, where $\mathbf{d}_A \colon \Gamma(\wedge A^*) \to \Gamma(\wedge A^*)$ is the Lie algebroid differential.

Let A be a Lie algebroid. The representations up to homotopy which we consider are always on graded vector bundles $\mathcal{E} = E_0[0] \oplus E_1[1]$ concentrated on degrees 0 and 1, so called *2-term graded vector bundles*. These representations are equivalent to the following data (see [1, 9]):

(1) a vector bundle morphism $\partial \colon E_0 \to E_1$,
(2) two A-connections, ∇^0 and ∇^1 on E_0 and E_1, respectively, such that $\partial \circ \nabla^0 = \nabla^1 \circ \partial$,
(3) an element $R \in \Omega^2(A, \operatorname{Hom}(E_1, E_0))$ such that $R_{\nabla^0} = R \circ \partial$, $R_{\nabla^1} = \partial \circ R$ and $\mathbf{d}_{\nabla^{\mathrm{Hom}}} R = 0$, where ∇^{Hom} is the connection induced on $\operatorname{Hom}(E_1, E_0)$ by ∇^0 and ∇^1.

For brevity we call such a 2-term representation up to homotopy a **2-representation**.

2.5. **2-representations and VB-algebroids.** Let $(D \to B, A \to M)$ be a VB-algebroid and choose a linear splitting $\Sigma \colon A \times_M B \to D$. Since the anchor of a linear section is linear, for each $a \in \Gamma(A)$ the vector field $\Theta_B(\sigma_A(a))$ defines a derivation of $\Gamma(B)$ with symbol $\rho(a)$ (see §2.2.2). This defines a linear connection $\nabla^B \colon \Gamma(A) \times \Gamma(B) \to \Gamma(B)$:

$$\Theta_B(\sigma_A(a)) = \widehat{\nabla_a^B}$$

for all $a \in \Gamma(A)$. Since the bracket of a linear section with a core section is again a core section, we find a linear connection $\nabla^C \colon \Gamma(A) \times \Gamma(C) \to \Gamma(C)$ such that

$$[\sigma_A(a), c^\dagger] = (\nabla_a^C c)^\dagger$$

for all $c \in \Gamma(C)$ and $a \in \Gamma(A)$. The difference $\sigma_A[a_1, a_2] - [\sigma_A(a_1), \sigma_A(a_2)]$ is a core-linear section for all $a_1, a_2 \in \Gamma(A)$. This defines a vector valued Lie algebroid form $R \in \Omega^2(A, \operatorname{Hom}(B, C))$ such that

$$[\sigma_A(a_1), \sigma_A(a_2)] = \sigma_A[a_1, a_2] - \widetilde{R(a_1, a_2)},$$

for all $a_1, a_2 \in \Gamma(A)$. See [9] for more details on these constructions. The following theorem is proved in [9].

Theorem 2.6. *Let $(D \to B; A \to M)$ be a VB-algebroid and choose a linear splitting $\Sigma\colon A \times_M B \to D$. The triple (∇^B, ∇^C, R) defined as above is a 2-representation of A on the complex $\partial_B\colon C \to B$, where ∂_B is the core-anchor.*

Conversely, let $(D; A, B; M)$ be a double vector bundle such that A has a Lie algebroid structure and choose a linear splitting $\Sigma\colon A \times_M B \to D$. Then if (∇^B, ∇^C, R) is a 2-representation of A on a complex $\partial_B\colon C \to B$, then the three equations above and the core-anchor ∂_B define a VB-algebroid structure on $(D \to B; A \to M)$.

In the situation of the previous theorem, we have

$$\left[\sigma_A(a), \widetilde{\phi}\right] = \widetilde{\nabla_a^{\mathrm{Hom}}\phi} \qquad \text{and} \qquad \left[c^\dagger, \widetilde{\phi}\right] = (\phi(\partial_B c))^\dagger$$

for all $a \in \Gamma(A)$, $\phi \in \Gamma(\mathrm{Hom}(B, C))$ and $c \in \Gamma(C)$, see for instance [8].

Remark 2.7. *If $\Sigma_1, \Sigma_2\colon A \times_M B \to D$ are two linear splittings of a VB-algebroid $(D \to B, A \to M)$ and $\phi_{12} \in \Gamma(A^* \otimes B^* \otimes C)$ is the change of splitting, then the two corresponding 2-representations are related by the following identities* [9].

$$\nabla_a^{B,2} = \nabla_a^{B,1} + \partial_B \circ \phi_{12}(a), \quad \nabla_a^{C,2} = \nabla_a^{C,1} + \phi_{12}(a) \circ \partial_B \tag{9}$$

and

$$\begin{aligned} R^2(a_1, a_2) =& R^1(a_1, a_2) + (\mathbf{d}_{\nabla^{\mathrm{Hom}}}\phi_{12})(a_1, a_2) \\ &+ \phi_{12}(a_1)\partial_B\phi_{12}(a_2) - \phi_{12}(a_2)\partial_B\phi_{12}(a_1) \end{aligned} \tag{10}$$

for all $a, a_1, a_2 \in \Gamma(A)$.

Given a 2-representation $\mathcal{D}$ of A on $E_0[0] \oplus E_1[1]$ and a tensor $\phi \in \Gamma(A^* \otimes E_1^* \otimes E_0)$, we say that the new 2-representation defined by (9) *and* (10) *is the **ϕ-twist of** $\mathcal{D}$.*

Example 2.8. *Choose a linear connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ and consider the corresponding linear splitting Σ^∇ of TE as in §2.2.2. The description of the Lie bracket of vector fields in* (7) *shows that the 2-representation induced by Σ^∇ is the 2-representation of TM on $\mathrm{Id}_E\colon E \to E$ given by $(\nabla, \nabla, R_\nabla)$.*

Example 2.9 (The tangent of a Lie algebroid)**.** *Let $(A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid. Then the tangent $TA \to TM$ has a Lie algebroid structure with bracket defined by $[Ta_1, Ta_2] = T[a_1, a_2]$, $[Ta_1, a_2^\dagger] = [a_1, a_2]^\dagger$ and $[a_1^\dagger, a_2^\dagger] = 0$ for all $a_1, a_2 \in \Gamma(A)$. The anchor of Ta is $\widehat{[\rho(a), \cdot]} \in \mathfrak{X}(TM)$ and the anchor of $a^\dagger$ is $\rho(a)^\uparrow$ for all $a \in \Gamma(A)$. This defines a VB-algebroid structure $(TA \to TM; A \to M)$ on $(TA; TM, A; M)$.*

*Given a TM-connection on A, and so a linear splitting Σ^∇ of TA as in §2.2.2, the 2-representation of A on $\rho\colon A \to TM$ encoding this VB-algebroid is the **adjoint 2-representation** $(\nabla^{\mathrm{bas}}, \nabla^{\mathrm{bas}}, R_\nabla^{\mathrm{bas}})$* [9]*, where the connections are defined by*

$$\nabla^{\mathrm{bas}}\colon \Gamma(A) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad \nabla_a^{\mathrm{bas}} X = [\rho(a), X] + \rho(\nabla_X a)$$

and

$$\nabla^{\mathrm{bas}}\colon \Gamma(A) \times \Gamma(A) \to \Gamma(A), \qquad \nabla_{a_1}^{\mathrm{bas}} a_2 = [a_1, a_2] + \nabla_{\rho(a_2)} a_1,$$

and $R_\nabla^{\mathrm{bas}} \in \Omega^2(A, \mathrm{Hom}(TM, A))$ is given by

$$R_\nabla^{\mathrm{bas}}(a_1, a_2)X = -\nabla_X[a_1, a_2] + [\nabla_X a_1, a_2] + [a_1, \nabla_X a_2] + \nabla_{\nabla_{a_2}^{\mathrm{bas}} X} a_1 - \nabla_{\nabla_{a_1}^{\mathrm{bas}} X} a_2$$

for all $X \in \mathfrak{X}(M)$, $a, a_1, a_2 \in \Gamma(A)$.

2.5.1. *Dualisation and* 2*-representations.* Let $(D \to B, A \to M)$ be a VB-algebroid. Then $(D^*_A \to C^*, A \to M)$ has in induced VB-algebroid structure [18]. While this can be defined in an abstract and natural manner (i.e. without the use of splittings), we characterise for simplicity the linear Lie algebroid structure on $D^*_A \to C^*$ using Theorem 2.6.

Let $\Sigma\colon A \times_M B \to D$ be a linear splitting of D and denote by (∇^B, ∇^C, R) the induced 2-representation of the Lie algebroid A on $\partial_B\colon C \to B$. We have seen above that the linear splitting Σ induces a linear splitting $\Sigma^\star\colon A \times_M C^* \to D^*_A$. The induced VB-algebroid structure on $(D^*_A \to C^*, A \to M)$ is given in this splitting by the 2-representation $(\nabla^{C^*}, \nabla^{B^*}, -R^*)$ of A on the complex $\partial_B^*\colon B^* \to C^*$. This is proved in the appendix of [6]. (Note that the construction of the "dual" linear splitting of D^*_A, given a linear splitting of D, is done in [6] by dualising the corresponding decomposition and taking its inverse. The resulting linear splitting of D^*_A is the same as ours.)

3. [2]-manifolds and metric double vector bundles

In this section we recall the definitions of ℕ-manifolds of degree 2 and of their morphisms. Then we introduce linear metrics on double vector bundles, and the dual objects, involutive double vector bundles. We define morphisms of involutive double vector bundles and we prove our main result: an equivalence between the category of ℕ-manifolds of degree 2 and the obtained category of involutive double vector bundles.

We illustrate the theory with two standard classes of metric double vector bundles: the tangent double $TE \to TM$ of a metric vector bundle E, and the Pontryagin bundle $TE \oplus T^*E \to E$ of a vector bundle E. We describe the dual involutive double vector bundles.

3.1. ℕ-manifolds.

Here we give the definitions of ℕ-manifolds. We are particularly interested in ℕ-manifolds of degree 2. We refer to [20, 3] for more details.

Definition 3.1. *An* ℕ***-manifold*** $\mathcal{M}$ *of degree* n *and dimension* $(m; r_1, \ldots, r_n)$ *is a sheaf of* ℕ*-graded, graded commutative, associative, unital* $C^\infty(M)$*-algebras over a smooth* m*-dimensional manifold* M*, that is locally freely generated by* $r_1 + \ldots + r_n$ *elements* $\xi_1^1, \ldots, \xi_1^{r_1}$, $\xi_2^1, \ldots, \xi_2^{r_2}, \ldots,$ $\xi_n^1, \ldots, \xi_n^{r_n}$ *with* ξ_i^j *of degree* i *for* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, r_i\}$.

A morphism of ℕ*-manifolds* $\mu\colon \mathcal{N} \to \mathcal{M}$ *over a smooth map* $\mu_0\colon N \to M$ *of the underlying smooth manifolds is a morphism* $\mu^\star\colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{N})$ *of sheaves of graded algebras over* $\mu_0^*\colon C^\infty(M) \to C^\infty(N)$.

Note that the degree 0 elements of $C^\infty(\mathcal{M})$ are precisely the smooth functions on M. We call $[n]$**-manifold** an ℕ-manifold of degree $n \in \mathbb{N}$. We write $|\xi|$ for the degree of a homogeneous element $\xi \in C^\infty(\mathcal{M})$, i.e. an element which can be written as a sum of functions of the same degree and we write $C^\infty(\mathcal{M})^i$ for the elements of degree i in $C^\infty(\mathcal{M})$. Note that a [1]**-manifold** over a smooth manifold M is equivalent to a locally free and finitely generated sheaf of $C^\infty(M)$-modules.

Our goal in this section is to prove that [2]-manifolds are equivalent to double vector bundles endowed with a linear metric (Theorem 3.23). We begin with a few observations on the equivalence of smooth vector bundles with locally free and finitely generated sheaves of C^∞-modules, i.e. [1]-manifolds. Theorem 3.23 will generalise this result to the degree 2 case.

3.1.1. *Vector bundles and* [1]*-manifolds.* Here we recall the equivalence of categories between degree [1]-manifolds (or locally free and finitely generated sheaves of C^∞-modules) and smooth vector bundles (see for instance [26, Theorem II.1.13]). This subsection can be seen as introductory to the methods in §3.4.

Let VB be the category of smooth vector bundles. Let $E \to M$ and $F \to N$ be vector bundles. Recall from Lemma 2.1 that a morphism $\phi\colon F \to E$ of vector bundles over $\phi_0\colon N \to M$ is equivalent to a map $\phi^\star\colon \Gamma(E^*) \to \Gamma(F^*)$ defined as in (1) and satisfying

$$\phi^\star(f \cdot \varepsilon) = \phi_0^* f \cdot \phi^\star(\varepsilon)$$

for all $f \in C^\infty(M)$ and $\varepsilon \in \Gamma(E^*)$.

Let [1]-Man be the category of [1]-manifolds. We now establish an equivalence between VB and [1]-Man. The functor $\Gamma(\cdot)\colon$ VB $\to$ [1]-Man sends a vector bundle $E \to M$ to the set of sections $\Gamma(E^*)$, a locally free and finitely generated sheaf of $C^\infty(M)$-modules. We call this [1]-manifold $E[-1]$. The functor $\Gamma(\cdot)$ sends a morphism $\Phi = (\phi, \phi_0)\colon F \to E$ as above to the morphism $\phi^\star\colon \Gamma(E^*) \to \Gamma(F^*)$ over $\phi_0^*\colon C^\infty(M) \to C^\infty(N)$, defining a morphism $\Gamma(\Phi)\colon F[-1] \to E[-1]$ of [1]-manifolds.

Next choose a [1]-manifold $\mathcal{M}$ over a smooth manifold M. There exists a maximal open covering $\{U_\alpha\}$ of M such that $C^\infty_{U_\alpha}(\mathcal{M})$ is finitely generated by generators $\xi_1^\alpha, \ldots, \xi_m^\alpha$. For two indices α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, we can write each generator in an unique manner as $\xi_j^\beta = \sum_{i=1}^m \psi_{\alpha\beta}^{ij} \xi_i^\alpha$ with smooth functions $\psi_{\alpha\beta}^{ij} \in C^\infty(U_\alpha \cap U_\beta)$. We define $A_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathrm{Gl}(\mathbb{R}^m))$ by $A_{\alpha\beta} = (\psi_{\alpha\beta}^{ij})_{i,j=1,\ldots,m}$. We have then immediately

$$A_{\gamma\alpha} \cdot A_{\alpha\beta} = A_{\gamma\beta}, \tag{11}$$

where $\cdot$ is the pointwise multiplication of matrices. Next we consider the disjoint union $\tilde{E} = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^m$ and identify for $x \in U_\alpha \cap U_\beta \neq \emptyset$

$$(x, v) \in U_\beta \times \mathbb{R}^m \quad \text{with} \quad (x, A_{\alpha\beta}(x)v) \in U_\alpha \times \mathbb{R}^m.$$

By (11), this defines an equivalence relation on $\tilde{E}$ and the quotient E has a smooth vector bundle structure with vector bundle charts given by the inclusions $U_\alpha \times \mathbb{R}^m \hookrightarrow E$, and changes of charts the cocycles $A_{\alpha\beta}$. We set $E(\mathcal{M}) := E^*$. Note that the maps $e_i^\alpha\colon U_\alpha \to U_\alpha \times \mathbb{R}^m$, $x \mapsto (x, e_i)$ define smooth local sections of E and $e_i^\beta = \sum_{j=1}^n \psi_{\alpha\beta}^{ji} e_j^\alpha$ for α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Hence, we can identify ξ_i^α with the section e_i^α and we see that a morphism $\mu\colon \mathcal{N} \to \mathcal{M}$ over $\mu_0\colon N \to M$ defines a morphism $E(\mu)^\star\colon \Gamma(E(\mathcal{M})^*) \to \Gamma(E(\mathcal{N})^*)$ of modules over $\mu_0^*\colon C^\infty(M) \to C^\infty(N)$, and so by Lemma 2.1 a vector bundle morphism $E(\mathcal{N}) \to E(\mathcal{M})$ over $\mu_0\colon N \to M$. Hence we have constructed a functor $E(\cdot)\colon$ [1]-Man $\to$ VB.

Next we show that the two functors are part of an equivalence of categories. The functor $E(\cdot) \circ \Gamma(\cdot)\colon$ VB $\to$ VB sends a vector bundle to the abstract vector bundle defined by its trivialisations and cocycles. There is an obvious natural isomorphism between this functor and the identity functor VB $\to$ VB. The functor $\Gamma(\cdot) \circ E(\cdot)\colon$ [1]-Man $\to$ [1]-Man sends a [1]-manifold $\mathcal{M}$ over M with local generators $\xi_i^\alpha \in C^\infty_{U_\alpha}(\mathcal{M})^1$ and cocycles $A_{\alpha\beta}$ to the sheaf of sections of $E(\mathcal{M})^*$, with local basis sections $\varepsilon_i^\alpha \in \Gamma_{U_\alpha}(E(\mathcal{M})^*)$ and cocycles $A_{\alpha\beta}$. There is an obvious natural isomorphism between this functor and the identity functor [1]-Man $\to$ [1]-Man.

3.1.2. *Split* $\mathbb{N}$*-manifolds.* Next we quickly discuss split $\mathbb{N}$-manifolds and we recall how each $\mathbb{N}$-manifold is noncanonically isomorphic to a split $\mathbb{N}$-manifold of the same degree and of the same dimension.

Let E be a smooth vector bundle of rank r over a smooth manifold M of dimension p and assign the degree n to the fiber coordinates of E. This defines $E[-n]$, an $[n]$-manifold of dimension $(p; r_1 = 0, \ldots, r_{n-1} = 0, r_n = r)$ with $C^\infty(E[-n])^n = \Gamma(E^*)$.

Now let $E_1, E_2, \ldots, E_n$ be smooth vector bundles of finite ranks $r_1, \ldots, r_n$ over M and assign the degree i to the fiber coordinates of E_i, for each $i = 1, \ldots, n$. The direct sum $E = E_1 \oplus \ldots \oplus E_n$ is a graded vector bundle with grading concentrated in degrees $1, \ldots, n$. The

$[n]$-manifold $E_1[-1]\oplus\ldots\oplus E_n[-n]$ has local basis sections of E_i^* as local generators of degree i, for $i=1,\ldots,n$, and so dimension $(p;r_1,\ldots,r_n)$. The $[n]$-manifold $\mathcal{M}=E_1[-1]\oplus\ldots\oplus E_n[-n]$ is called a **split $[n]$-manifold**.

In this paper, we are exclusively interested in the cases $n=2$ and $n=1$. Choose two vector bundles E_1 and E_2 of ranks r_1 and r_2 over a smooth manifold M. Consider $\mathcal{M}=E_1[-1]\oplus E_2[-2]$. We find $C^\infty(\mathcal{M})^0=C^\infty(M)$, $C^\infty(\mathcal{M})^1=\Gamma(E_1^*)$ and $C^\infty(\mathcal{M})^2=\Gamma(E_2^*\oplus\wedge^2E_1^*)$.

A morphism $\mu\colon F_1[-1]\oplus F_2[-2]\to E_1[-1]\oplus E_2[-2]$ of split [2]-manifolds over the bases N and M, respectively, consists of a smooth map $\mu_0\colon N\to M$, three vector bundle morphisms $\mu_1\colon F_1\to E_1$, $\mu_2\colon F_2\to E_2$ and $\mu_{12}\colon \wedge^2 F_1\to E_2$ over μ_0. The map $\mu^\star$ sends a degree 1 function $\xi\in\Gamma(E_1^*)$ to $\mu_1^\star\xi\in\Gamma(F_1^*)$ and a degree 2-function $\xi\in\Gamma(E_2^*)$ to

$$\mu_2^\star\xi+\mu_{12}^\star\xi\in\Gamma(F_2^*\oplus\wedge^2F_1^*).$$

Any ℕ-manifold is non-canonically diffeomorphic to a split ℕ-manifold. Further, the categories of split ℕ-manifolds and of ℕ-manifolds are equivalent. This is proved for instance in [3], following the proof of the $\mathbb{Z}/2\mathbb{Z}$-graded version of this theorem, which is known as *Batchelor's theorem* [2].

Theorem 3.2 ([2, 3])**.** *Any $[n]$-manifold is non-canonically diffeomorphic to a split $[n]$-manifold.*

We give here the proof by [3] in the case $n=2$. We are especially interested in the morphism of split [2]-manifolds induced by a change of splitting of a [2]-manifold and we emphasize this in the proof.

Sketch of Proof, [3]. Consider a [2]-manifold $\mathcal{M}$ over a smooth base manifold M. Since $C^\infty(\mathcal{M})^0=C^\infty(M)$ and $C^\infty(\mathcal{M})^0\cdot C^\infty(\mathcal{M})^1\subset C^\infty(\mathcal{M})^1$, the sheaf $C^\infty(\mathcal{M})^1$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules and there exists a vector bundle $E_1\to M$ such that $C^\infty(\mathcal{M})^1\simeq\Gamma(E_1^*)$. Now let $\mathcal{A}_1$ be the subalgebra of $C^\infty(\mathcal{M})$ generated by $C^\infty(\mathcal{M})^0\oplus C^\infty(\mathcal{M})^1$. We find easily that $\mathcal{A}_1\simeq\Gamma(\wedge^\bullet E_1^*)$ and $\mathcal{A}_1\cap C^\infty(\mathcal{M})^2=\wedge^2C^\infty(\mathcal{M})^1$ is a proper $C^\infty(M)$-submodule of $C^\infty(\mathcal{M})^2$. Since the quotient $C^\infty(\mathcal{M})^2/\wedge^2C^\infty(\mathcal{M})^1$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules, we have $C^\infty(\mathcal{M})^2/\wedge^2C^\infty(\mathcal{M})^1\simeq\Gamma(E_2^*)$, for a vector bundle E_2 over M. The short exact sequence

$$0\to\wedge^2C^\infty(\mathcal{M})^1\hookrightarrow C^\infty(\mathcal{M})^2\to\Gamma(E_2^*)\to 0$$

of $C^\infty(M)^0$-modules is non canonically split. Let us choose a splitting and identify $\Gamma(E_2^*)$ with a submodule of $C^\infty(\mathcal{M})^2$:

$$C^\infty(\mathcal{M})^2\simeq\wedge^2C^\infty(\mathcal{M})^1\oplus\Gamma(E_2^*)=\Gamma(\wedge^2E_1^*\oplus E_2^*)\ .$$

Hence, the considered [2]-manifold is isomorphic, modulo the chosen splitting, to the split [2]-manifold $E_1[-1]\oplus E_2[-2]$.

Note finally that a change of splitting is equivalent to a section ϕ of $\operatorname{Hom}(\wedge^2E_1,E_2)$ and induces an isomorphism of split [2]-manifolds over the identity on M: $\mu^\star(\xi)=\xi+\phi^\star\xi\in\Gamma(E_2^*\oplus\wedge^2E_1^*)$ for all $\xi\in\Gamma(E_2^*)$ and $\mu^\star(\xi)=\xi$ for all $\xi\in\Gamma(E_1^*)$. □

Note that [1]-manifolds are automatically split. As we have seen in §3.1.1, [1]-manifolds are just vector bundles with a degree shifting in the fibers, i.e. $\mathcal{M}=E[-1]$ for some vector bundle $E\to M$ and $C^\infty(\mathcal{M})=\Gamma(\wedge^\bullet E^*)$, the exterior algebra of E.

3.1.3. *Vector fields on $[n]$-manifolds.* Let us quickly introduce the notion of *vector field* on an $\mathbb{N}$-manifold. Let $\mathcal{M}$ be an $[n]$-manifold. A **vector field of degree** j on $\mathcal{M}$ is a degree j derivation ϕ of $C^\infty(\mathcal{M})$: $|\phi(\xi)| = j + |\xi|$ for a homogeneous element $\xi \in C^\infty(\mathcal{M})$. We write $\operatorname{Der}(C^\infty(\mathcal{M}))$ for the sheaf of graded derivations of $C^\infty(\mathcal{M})$.

The vector fields on $\mathcal{M}$ and their Lie bracket defined by $[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$ satisfy the following conditions:

1. $\phi(\xi\eta) = \phi(\xi)\eta + (-1)^{|\phi||\xi|}\xi\phi(\eta)$,
2. $[\phi, \psi] = (-1)^{1+|\phi||\psi|}[\psi, \phi]$,
3. $[\phi, \xi\psi] = \phi(\xi)\psi + (-1)^{|\phi||\xi|}\xi[\phi, \psi]$,
4. $(-1)^{|\phi||\gamma|}[\phi, [\psi, \gamma]] + (-1)^{|\psi||\phi|}[\psi, [\gamma, \phi]] + (-1)^{|\gamma||\psi|}[\gamma, [\phi, \psi]] = 0$

for ϕ, ψ, γ homogeneous elements of $\operatorname{Der}(C^\infty(\mathcal{M}))$ and ξ, η homogeneous elements of $C^\infty(\mathcal{M})$. For instance, given an open set U of M where $C^\infty(\mathcal{M})$ is freely generated by ξ^i_j , the derivation $\partial_{\xi^i_j}$ of $C^\infty_U(\mathcal{M})$ sends ξ^i_j to 1 and the other local generators to 0. It is hence a derivation of degree $-j$. $\operatorname{Der}(C^\infty_U(\mathcal{M}))$ is freeely generated as a $C^\infty_U(\mathcal{M})$-module by ∂_{x_k}, $k = 1, \ldots, p$ and $\partial_{\xi^i_j}$, $j = 1, \ldots, n$, $i = 1, \ldots, r_j$.

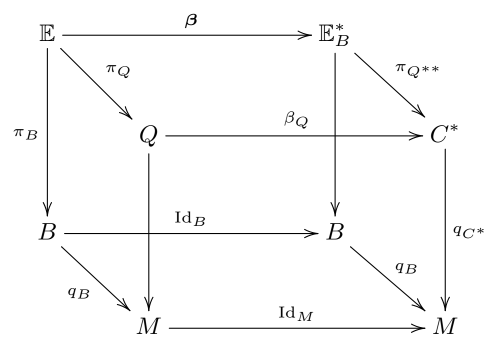
Finally note that if an $[n]$-manifold $\mathcal{M}$ splits as $E_1[-1] \oplus E_2[-2] \oplus \ldots \oplus E_n[-n]$, then each section e of E_j defines a derivation $\hat{e}$ of degree $-j$ on $\mathcal{M}$: $\hat{e}(f) = 0$, $\hat{e}(\varepsilon^i_j) = \langle e, \varepsilon^j_j\rangle$, and $\hat{e}(\varepsilon^i_k) = 0$ for $k \neq j$. We find $\hat{e^i_j} = \partial_{\varepsilon^i_j}$ if $\{e^1_j, \ldots, e^{r_j}_j\}$ is a local basis of E_j and $\{\varepsilon^1_j, \ldots, \varepsilon^{r_j}_j\}$ is the dual basis of E^*_j. Further, a vector field ϕ of degree 0 on $\mathcal{M}$ can be written as a sum $X + \delta_1 + \delta_2 + \ldots + \delta_n$, with $X \in \mathfrak{X}(M)$ and each δ_i a derivation of E_i with symbol $X \in \mathfrak{X}(M)$. The derivation $X + \delta_1 + \ldots + \delta_n$ sends $\varepsilon_i \in \Gamma(E^*_i)$ to $\delta^*_i(\varepsilon_i)$. In particular, if for each j the map $\delta^j : \mathfrak{X}(M) \to \operatorname{Der}(E_j)$ is a morphism of $C^\infty(M)$-modules that sends a vector field X to a derivation $\delta^j(X)$ over X, then

$$\{X + \delta^1(X) + \ldots + \delta^n(X) \mid X \in \mathfrak{X}(M)\} \cup \{\hat{\varepsilon} \mid \varepsilon \in \Gamma(E_j) \text{ for some } j\} \tag{12}$$

span $\operatorname{Der}(C^\infty(\mathcal{M}))$ as a $C^\infty(\mathcal{M})$-module.

3.2. **Metric double vector bundles.** Next we introduce linear metrics on double vector bundles.

Definition 3.3. *A **metric double vector bundle** is a double vector bundle $(\mathbb{E}, Q; B, M)$ equipped with a **symmetric non-degenerate pairing** $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$ that is also **linear over** Q, i.e. such that the map $\boldsymbol{\beta} \colon \mathbb{E} \to \mathbb{E}^*_B$*



*defined by the pairing $\langle\cdot,\cdot\rangle$ is an **isomorphism of double vector bundles**. In particular, the core $C \to M$ of $\mathbb{E}$ is isomorphic to $Q^* \to M$.*

Note that, equivalently, a pairing $\langle\cdot\,,\cdot\rangle$ on $\mathbb{E} \to B$ is linear if and only if

$$\langle e_1 +_Q e_2, f_1 +_Q f_2\rangle_{\mathbb{E}} = \langle e_1, f_1\rangle_{\mathbb{E}} + \langle e_2, f_2\rangle_{\mathbb{E}} \tag{13}$$

for $e_1, e_2, f_1, f_2 \in \mathbb{E}$ with $\pi_B(e_i) = \pi_B(f_i)$, $i = 1, 2$. In terms of sections, a bilinear pairing $\langle \cdot\,, \cdot \rangle_{\mathbb{E}} \colon \mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$ is symmetric, nondegenerate and linear over Q if and only if the core of $\mathbb{E}$ is isomorphic to Q^* and, via this isomorphism,

(1) $\langle \tau_1^\dagger, \tau_2^\dagger \rangle = 0$ for $\tau_1, \tau_2 \in \Gamma(Q^*)$,
(2) $\langle \chi, \tau^\dagger \rangle = q_B^* \langle q, \tau \rangle$ for $\chi \in \Gamma_B^l(\mathbb{E})$ linear over $q \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$ and
(3) $\langle \chi_1, \chi_2 \rangle$ is a linear function on B for $\chi_1, \chi_2 \in \Gamma_B^l(\mathbb{E})$.

In the following, we always identify with Q^* the core of a metric double vector bundle $(\mathbb{E}, Q; B, M)$.

3.2.1. *Lagrangian decompositions of a metric double vector bundle.*

Definition 3.4. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. A linear splitting $\Sigma \colon Q \times_M B \to \mathbb{E}$ is said to be* ***Lagrangian*** *if its image is (maximal) isotropic in $\mathbb{E} \to B$. The corresponding horizontal lifts and the corresponding decomposition of $\mathbb{E}$ are then also said to be* ***Lagrangian****.*

Note that by definition, a horizontal lift $\sigma_Q \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ is Lagrangian if and only if $\langle \sigma_Q(q_1), \sigma_Q(q_2) \rangle = 0$ for all $q_1, q_2 \in \Gamma(Q)$.

Let $\sigma_Q \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ be an arbitrary horizontal lift. We have seen that by the definition of a linear metric on $\mathbb{E} \to B$, the pairing of two linear sections is a linear function on B. This implies with

$$\sigma_Q(fq) = q_B^* f \cdot \sigma_Q(q) \text{ and } \ell_{f\beta} = q_B^* f \cdot \ell_\beta \text{ for all } f \in C^\infty(M), q \in \Gamma(Q) \text{ and } \beta \in \Gamma(B^*)$$

the existence of a symmetric tensor $\Lambda \in \Gamma(S^2(Q, B^*))$ such that

$$\langle \sigma_Q(q_1), \sigma_Q(q_2) \rangle_{\mathbb{E}} = \ell_{\Lambda(q_1, q_2)}. \tag{14}$$

In particular, $\Lambda(q, \cdot) : Q \to B^*$ is a morphism of vector bundles for each $q \in \Gamma(Q)$. Define a new horizontal lift $\sigma_Q' \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ by $\sigma_Q'(q) = \sigma_Q(q) - \frac{1}{2}\widetilde{\Lambda(q, \cdot)^*}$ for all $q \in \Gamma(Q)$. Since for $\phi \in \Gamma(\operatorname{Hom}(B, Q^*))$, $\langle \widetilde{\phi}, \chi \rangle = \ell_{\phi^*(q)}$ if $\chi \in \Gamma_B^l(\mathbb{E})$ is linear over $q \in \Gamma(Q)$, we find then

$$\langle \sigma_Q'(q_1), \sigma_Q'(q_2) \rangle_{\mathbb{E}} = \langle \sigma_Q(q_1), \sigma_Q(q_2) \rangle_{\mathbb{E}} - \frac{1}{2}\ell_{\Lambda(q_1, q_2)} - \frac{1}{2}\ell_{\Lambda(q_2, q_1)} = 0$$

for all $q_1, q_2 \in \Gamma(Q)$. This proves the following result.

Proposition 3.5. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Then there exists a Lagrangian splitting of $\mathbb{E}$.*

Next we show that a change of Lagrangian splitting corresponds to a skew-symmetric element of $\Gamma(Q^* \otimes B^* \otimes Q^*)$.

Proposition 3.6. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle and choose a Lagrangian horizontal lift $\sigma_Q^1 \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$. Then a second horizontal lift $\sigma_Q^2 \colon \Gamma(Q) \to \Gamma_B^l(\mathbb{E})$ is Lagrangian if and only if the change of lift $\phi_{12} \in \Gamma(Q^* \otimes B^* \otimes Q^*)$ satisfies the following equality:*

$$\langle \phi_{12}(q), q' \rangle = -\langle \phi_{12}(q'), q \rangle \in \Gamma(B^*)$$

for all $q, q' \in \Gamma(Q)$, i.e. if and only if $\phi_{12} \in \Gamma(Q^* \wedge Q^* \otimes B^*)$.

Proof. For $q \in \Gamma(Q)$ we have $\langle \widetilde{\phi_{12}(q)}, \chi \rangle = \ell_{\langle \phi_{12}(q), q' \rangle}$ for any linear section $\chi \in \Gamma_B^l(\mathbb{E})$ over $q' \in \Gamma(Q)$. Hence we find

$$\begin{aligned} &\langle \sigma_Q^1(q), \sigma_Q^1(q') \rangle_{\mathbb{E}} - \langle \sigma_Q^2(q), \sigma_Q^2(q') \rangle_{\mathbb{E}} \\ &= \langle \sigma_Q^1(q) - \sigma_Q^2(q), \sigma_Q^1(q') \rangle_{\mathbb{E}} + \langle \sigma_Q^2(q), \sigma_Q^1(q') - \sigma_Q^2(q') \rangle_{\mathbb{E}} = \ell_{\langle \phi_{12}(q), q' \rangle} + \ell_{\langle q, \phi_{12}(q') \rangle}. \end{aligned}$$

□

The last proposition shows that not any linear section of $\mathbb{E}$ over B can be obtained as the Lagrangian horizontal lift of a section of Q. This is easy to understand in Example 3.11.

Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. Choose a Lagrangian splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$ and set

$$\mathcal{C}(\mathbb{E}) := \sigma_B(\Gamma(B)) + \{\tilde\omega \mid \omega\in\Omega^2(Q)\}. \tag{15}$$

Note that $\mathcal{C}(\mathbb{E})$ together with $\Gamma_Q^c(\mathbb{E})\simeq\Gamma(Q^*)$ span $\mathbb{E}$ as a vector bundle over Q. Note also that $\mathcal{C}(\mathbb{E})$ is a sheaf of $C^\infty(M)$-modules: for $\chi\in\mathcal{C}(\mathbb{E})$ and $f\in C^\infty(M)$, the product $q_Q^*f\cdot\chi$ is again an element of $\mathcal{C}(\mathbb{E})$. In particular, for a Lagrangian splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$, $q_Q^*f\cdot(\sigma_B(b)+\tilde\omega)=\sigma_B(fb)+\widetilde{f\omega}$ for all $b\in\Gamma(B)$ and all $\omega\in\Omega^2(Q)$.

We begin by giving an intrinsic geometric description of $\mathcal{C}(\mathbb{E})$.

Proposition 3.7. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. The space $\mathcal{C}(\mathbb{E})\subseteq\Gamma_Q^l(\mathbb{E})$ is the set of linear sections $Q\to\mathbb{E}$ with isotropic image relative to $\langle\cdot\,,\cdot\rangle$.*

Proof. We have already seen that for any section $\chi\in\mathcal{C}(\mathbb{E})$, the pairing $\langle\chi(q),\chi(q')\rangle$ vanishes for all $(q,q')\in Q\times_M Q$. Conversely, consider a linear section $\chi\in\Gamma_Q^l(\mathbb{E})$ with isotropic image. Let $b\in\Gamma(B)$ be the basis section of χ and choose a Lagrangian splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$. Then $\chi=\sigma_B(b)+\widetilde{\phi}$ with $\phi\in\Gamma(Q^*\otimes Q^*)$ and we get for all $q,q'\in Q$ with $q_Q(q)=q_Q(q')=m\in M$:

$$\begin{aligned} 0\ &= \langle\chi(q),\chi(q')\rangle = \left\langle \sigma_B(b)(q)+_Q\big(0^{\mathbb{E}}_q +_B \overline{\phi(q)}\big), \sigma_B(b)(q')+_Q\big(0^{\mathbb{E}}_{q'} +_B \overline{\phi(q')}\big)\right\rangle\\ &\overset{(3)}{=} \left\langle \sigma_B(b)(q)+_B\big(0^{\mathbb{E}}_{b(m)} +_Q \overline{\phi(q)}\big), \sigma_B(b)(q')+_B\big(0^{\mathbb{E}}_{b(m)} +_Q \overline{\phi(q')}\big)\right\rangle \overset{(13)}{=} \phi(q)(q')+\phi(q')(q).\end{aligned}$$

Therefore, $\phi\in\Omega^2(Q)$. □

Proposition 3.8. *Let $(\mathbb{E}, B; Q, M)$ be a metric double vector bundle. The space $\mathcal{C}(\mathbb{E})\subseteq\Gamma_Q^l(\mathbb{E})$ is a locally free and finitely generated sheaf of $C^\infty(M)$-modules, that fits in the following short exact sequence of sheaves of $C^\infty(M)$-modules:*

$$0\longrightarrow\Omega^2(Q)\hookrightarrow\mathcal{C}(\mathbb{E})\longrightarrow\Gamma(B)\longrightarrow 0. \tag{16}$$

The maps are the restrictions of the maps in (5). *Lagrangian splittings of $\mathbb{E}$ are equivalent to splittings of* (16).

Proof. Choose a Lagrangian splitting $\Sigma\colon B\times_M Q\to\mathbb{E}$ of $\mathbb{E}$. Let r_1 be the rank of Q and r_2 the rank of B. Choose $p\in M$. Then there exists an open neighborhood U of p in M that trivialises both Q and B. Choose local basis sections $b_1,\ldots,b_{r_2}\in\Gamma_U(B)$ of B over U and local basis sections $\tau_1,\ldots,\tau_{r_1}\in\Gamma_U(Q^*)$ of Q^* over U. Then by (15) and the considerations below it, $\mathcal{C}_U(\mathbb{E})$ is freely generated as a $C_U^\infty(M)$-module by $\{\sigma_B(b_1),\ldots,\sigma_B(b_{r_2})\}\cup\{\widetilde{\tau_i\wedge\tau_j}\mid 1\le i<j\le r_1\}$.

It is easy to check as in the proof of Proposition 3.7 that isotropic core-linear sections of $\mathbb{E}\to Q$ are exactly the sections $\widetilde{\phi}$ for $\phi\in\Omega^2(Q)$. Since the inclusion $\Gamma(Q^*\otimes Q^*)\hookrightarrow\Gamma_Q^l(\mathbb{E})$ is injective (see (5)), its restriction to $\Omega^2(Q)\hookrightarrow\mathcal{C}(\mathbb{E})$ is also injective. The rest follows from the construction of $\mathcal{C}(\mathbb{E})$ in (15), or more precisely from the existence of Lagrangian splittings. □

Corollary 3.9. *There exists a vector bundle $\widehat{B}$ over M which set of sections is isomorphic to $\mathcal{C}(\mathbb{E})$. The short exact sequence* (16) *induces a short exact sequence of vector bundles over M:*

$$0\longrightarrow Q^*\wedge Q^*\hookrightarrow\widehat{B}\longrightarrow B\longrightarrow 0.$$

We end this section with a characterisation of Lagrangian splittings that will be useful in §4. Recall from Section 2.2.3 that given a linear splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$, one can construct the dual linear splitting $\Sigma^\star\colon Q^{**}\times_M B\to\mathbb{E}^*_B$.

Lemma 3.10. *Let $(\mathbb{E};Q,B;M)$ be a metric double vector bundle and choose a linear splitting Σ of $\mathbb{E}$. Then Σ is Lagrangian if and only if the linear map $\boldsymbol{\beta}\colon\mathbb{E}\to\mathbb{E}^*_B$ sends $\sigma_B(b)$ to $\sigma^\star_B(b)$ for all $b\in\Gamma(B)$.*

Proof. Recall from (8) that given a horizontal lift $\sigma_B\colon\Gamma(B)\to\Gamma^l_Q(\mathbb{E})$, the dual horizontal lift $\sigma^\star_B\colon\Gamma(B)\to\Gamma^l_{Q^{**}}(\mathbb{E}^*_B)$ can be defined by

$$\langle\sigma^\star_B(b)(p_m),\sigma_B(b)(q_m)\rangle_B=0,\qquad\langle\sigma^\star_B(b)(p_m),\tau^\dagger(b(m))\rangle_B=\langle p_m,\tau(m)\rangle$$

for all $b\in\Gamma(B)$, $\tau\in\Gamma(Q^*)$, $q_m\in Q$ and $p_m\in Q^{**}\simeq Q$.

On the other hand, if $\Sigma\colon B\times_M Q\to\mathbb{E}$ is a Lagrangian splitting, we have

$$\begin{aligned}\langle\boldsymbol{\beta}(\sigma_B(b)(p(m))),\sigma_B(b)(q(m))\rangle_B&=\langle\sigma_B(b)(p(m)),\sigma_B(b)(q(m))\rangle_{\mathbb{E}}\\&=\langle\sigma_Q(p),\sigma_Q(q)\rangle_{\mathbb{E}}(b(m))=0\end{aligned}$$

for all $q,p\in\Gamma(Q)$ and $b\in\Gamma(B)$, and

$$\begin{aligned}\langle\boldsymbol{\beta}(\sigma_B(b)(p(m))),\tau^\dagger(b(m))\rangle_B&=\langle\sigma_B(b)(p(m)),\tau^\dagger(b(m))\rangle_{\mathbb{E}}\\&=\langle\sigma_Q(p)(b(m)),\tau^\dagger(b(m))\rangle_{\mathbb{E}}=\langle p,\tau\rangle(m)\end{aligned}$$

for all $\tau\in\Gamma(Q^*)$. This proves that $\boldsymbol{\beta}$ sends the linear section $\sigma_B(b)\in\Gamma^l_Q(\mathbb{E})$ to $\sigma^\star_B(b)$ in $\Gamma^l_{Q^{**}}(\mathbb{E}^*_B)$. It is easy to see from the four equalities above that this condition is necessary for Σ to be Lagrangian. □

3.2.2. *Examples of metric double vector bundles.* Next we describe a couple of examples of metric double vector bundles.

Example 3.11. *Let $E\to M$ be a metric vector bundle, i.e. a vector bundle endowed with a symmetric non-degenerate pairing $\langle\cdot\,,\cdot\rangle\colon E\times_M E\to\mathbb{R}$. Then $E\simeq E^*$ and the tangent double is a metric double vector bundle $(TE,E;TM,M)$ with pairing $TE\times_{TM}TE\to\mathbb{R}$ the tangent of the pairing $E\times_M E\to\mathbb{R}$. In particular, we have*

$$\langle Te_1,Te_2\rangle_{TE}=\ell_{\mathbf{d}\langle e_1,e_2\rangle},\quad\langle Te_1,e_2^\dagger\rangle_{TE}=p_M^*\langle e_1,e_2\rangle\quad\textit{and }\langle e_1^\dagger,e_2^\dagger\rangle_{TE}=0$$

for $e_1,e_2\in\Gamma(E)$.

Recall from §2.2.2 that linear splittings of TE are equivalent to linear connections $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$. We have then for all $e_1,e_2\in\Gamma(E)$:

$$\left\langle\sigma_E^\nabla(e_1),e_2^\dagger\right\rangle=\left\langle Te_1-\widetilde{\nabla_\cdot e_1},e_2^\dagger\right\rangle=p_M^*\langle e_1,e_2\rangle$$

and

$$\langle\sigma_E^\nabla(e_1),\sigma_E^\nabla(e_2)\rangle=\left\langle Te_1-\widetilde{\nabla_\cdot e_1},Te_2-\widetilde{\nabla_\cdot e_2}\right\rangle=\ell_{\mathbf{d}\langle e_1,e_2\rangle-\langle e_2,\nabla_\cdot e_1\rangle-\langle e_1,\nabla_\cdot e_2\rangle}.$$

*The Lagrangian splittings of TE are hence exactly the linear splittings that correspond to **metric** connections, i.e. linear connections $\nabla\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ that preserve the metric: $\langle\nabla_\cdot e_1,e_2\rangle+\langle e_1,\nabla_\cdot e_2\rangle=\mathbf{d}\langle e_1,e_2\rangle$ for $e_1,e_2\in\Gamma(E)$.*

More generally, the isotropic linear vector fields on E are the linear vector fields corresponding to derivations of E that preserve the pairing.

Example 3.12. *Let $q_E\colon E\to M$ be a vector bundle and consider the double vector bundle*

$$\begin{array}{ccc} TE\oplus T^*E & \xrightarrow{\Phi_E:=(q_{E*},r_E)} & TM\oplus E^* \\ \downarrow{\scriptstyle \pi_E} & & \downarrow \\ E & \xrightarrow[q_E]{} & M \end{array}$$

with sides E and $TM\oplus E^*\to M$, and with core $E\oplus T^*M\to M$. The projection $r_E\colon T^*E\to E^*$ is defined by

$$r_E(\theta_{e_m})\in E^*_m,\qquad \langle r_E(\theta_{e_m}),e'_m\rangle=\left\langle \theta_{e_m},\,\left.\frac{d}{dt}\right|_{t=0} e_m+te'_m\right\rangle,$$

and is a fibration of vector bundles over the projection $q_E\colon E\to M$. The core elements are identified in the following manner with elements of $E\oplus T^*M\to M$. For $m\in M$ and $(e_m,\theta_m)\in E_m\times T^*_mM$, the pair

$$\left(\left.\frac{d}{dt}\right|_{t=0} te_m,(T_{0^E_m}q_E)^*\theta_m\right)$$

projects to $(0^{TM}_m,0^{E^*}_m)$ under Φ_E and to 0^E_m under π_E. Conversely, any element of $TE\oplus T^*E$ in the double kernel can be written in this manner. Next recall that $TE\oplus T^*E\to E$ has a natural symmetric nondegenerate pairing given by

$$\langle(v^1_{e_m},\theta^1_{e_m}),(v^2_{e_m},\theta^2_{e_m})\rangle=\theta^1_{e_m}(v^2_{e_m})+\theta^2_{e_m}(v^1_{e_m}),\tag{17}$$

the natural pairing underlying the standard Courant algebroid structure on $TE\oplus T^*E\to E$.

Dull algebroids and Dorfman connections were introduced in [10]*. A* ***Dorfman** $TM\oplus E^*$**-connection on its dual*** $E\oplus T^*M$ *is an $\mathbb{R}$-bilinear map*

$$\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$$

satisfying

1. $\Delta_q(f\cdot\tau)=f\cdot\Delta_q\tau+\pounds_{\mathrm{pr}_{TM}(q)}(f)\cdot\tau$,
2. $\Delta_{f\cdot q}\tau=f\cdot\Delta_q\tau+\langle q,\tau\rangle\cdot(0,\mathbf{d}f)$, *and*
3. $\Delta_q(0,\mathbf{d}f)=(0,\mathbf{d}(\pounds_{\mathrm{pr}_{TM}q}f))$

*for all $q\in\Gamma(TM\oplus E^*)$, $\tau\in\Gamma(E\oplus T^*M)$ and $f\in C^\infty(M)$. The first axiom says that Δ defines a map $\Delta\colon q\mapsto\Delta_q\in\mathrm{Der}(E\oplus T^*M)$. The dual of this map in the sense of derivations defines a **dull bracket on sections of** $TM\oplus E^*$, i.e. an $\mathbb{R}$-bilinear map*

$$[\![\cdot\,,\cdot]\!]_\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(TM\oplus E^*)\to\Gamma(TM\oplus E^*)$$

satisfying

1. $\mathrm{pr}_{TM}[\![q_1,q_2]\!]_\Delta=[\mathrm{pr}_{TM}q_1,\mathrm{pr}_{TM}q_2]$,
2. $[\![f_1q_1,f_2q_2]\!]=f_1f_2[\![q_1,q_2]\!]_\Delta+f_1\pounds_{\mathrm{pr}_{TM}q_1}(f_2)q_2-f_2\pounds_{\mathrm{pr}_{TM}q_2}(f_1)q_1$

for all $q_1,q_2\in\Gamma(TM\oplus E^*)$ and $f_1,f_2\in C^\infty(M)$.

We prove in [10] *that linear splittings of $TE\oplus T^*E$ are in bijection with dull brackets on sections of $TM\oplus E^*$, or equivalently with Dorfman connections $\Delta\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$. Choose such a Dorfman connection. For any pair $(X,\epsilon)\in\Gamma(TM\oplus E^*)$, the horizontal lift $\sigma:=\sigma^\Delta_{TM\oplus E^*}\colon\Gamma(TM\oplus E^*)\to\Gamma_E(TE\oplus T^*E)=\mathfrak{X}(E)\times\Omega^1(E)$ is given by*

$$\sigma(X,\epsilon)(e_m)=(T_meX(m),\mathbf{d}\ell_\epsilon(e_m))-\Delta_{(X,\epsilon)}(e,0)^\dagger(e_m)$$

for all $e_m\in E$, where for $(e,\theta)\in\Gamma(e\oplus T^*M)$, the pair $(e,\theta)^\dagger\in\Gamma_E(TE\oplus T^*E)$ is defined by $(e,\theta)^\dagger=(e^\uparrow,q_E^*\theta)$.

Since the vector bundle $TM \oplus E^*$ is anchored by the morphism $\mathrm{pr}_{TM} \colon TM \oplus E^* \to TM$, the TM-part of $[\![q_1, q_2]\!]_\Delta + [\![q_2, q_1]\!]_\Delta$ is trivial and this sum can be seen as an element of $\Gamma(E^*)$. We proved the following result in [10].

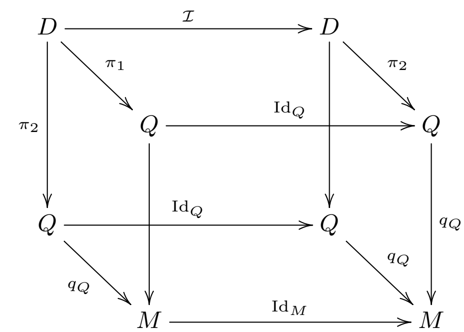
Theorem 3.13. *Choose $q, q_1, q_2 \in \Gamma(TM \oplus E^*)$ and $\tau, \tau_1, \tau_2 \in \Gamma(E \oplus T^*M)$. The natural pairing on fibres of $TE \oplus T^*E \to E$ is given by*

(1) $\langle \sigma(q_1), \sigma(q_2) \rangle = \ell_{[\![q_1,q_2]\!]_\Delta + [\![q_2,q_1]\!]_\Delta}$,
(2) $\left\langle \sigma(q), \tau^\dagger \right\rangle = q_E^* \langle q, \tau \rangle$.
(3) $\left\langle \tau_1^\dagger, \tau_2^\dagger \right\rangle = 0$.

As a consequence, the natural pairing on fibres of $TE \oplus T^*E \to E$ is a linear metric on $(TE \oplus T^*E; TM \oplus E^*, E; M)$ and the Lagrangian splittings are equivalent to skew-symmetric dull brackets on sections of the anchored vector bundle $(TM \oplus E^*, \mathrm{pr}_{TM})$.

3.3. **Involutive double vector bundles.** We begin here the study of involutive double vector bundles, which we find to be in duality with metric double vector bundles. Note that in the original work (see [22]) where double vector bundles were introduced, Pradines already introduced involutive double vector bundles, called *fibrés vectoriels doubles à symétrie inverse*[1], for his study of nonholonomic jets.

Definition 3.14. *An **involutive double vector bundle** is a double vector bundle (D, Q, Q, M) with core B^* equipped with a morphism*



satisfying $\mathcal{I}^2 = \mathrm{Id}_D$ and $\pi_1 \circ \mathcal{I} = \pi_2$, $\pi_2 \circ \mathcal{I} = \pi_1$ and with core morphism $-\mathrm{Id}_{B^*} \colon B^* \to B^*$.

We begin by proving that metric double vector bundles are dual to involutive double vector bundles.

Proposition 3.15. (1) *Let (D, Q, Q, M) be an involutive double vector bundle with involution $\mathcal{I}$ and core B^*. Then the dual $(\mathbb{E} := D^*_{\pi_1}, Q, B, M)$ inherits a linear metric $\mathbb{E} \times_B \mathbb{E} \to \mathbb{R}$ defined by*

$$\langle e_1, e_2 \rangle_{\mathbb{E}} = \langle e_1, d \rangle_Q + \langle e_2, \mathcal{I}(d) \rangle_Q$$

for $(e_1, e_2) \in \mathbb{E} \times_B \mathbb{E}$ and any $d \in D$ with $\pi_Q(e_1) = \pi_1(d)$ and $\pi_Q(e_2) = \pi_2(d)$.

(2) *Conversely, consider a metric double vector bundle $(\mathbb{E}, Q, B, M)$ with core Q^*. Then the dual $D = \mathbb{E}^*_Q$ with sides $\pi_1 \colon \mathbb{E}^*_Q \to Q$, $\pi_2 \colon \mathbb{E}^*_Q \to Q^{**} \simeq Q$ and with core B^* is an involutive double vector bundle with $\mathcal{I} \colon D \to D$ defined by*

$$\langle \mathcal{I}(d), e \rangle_Q = \langle\!\langle e, d \rangle\!\rangle$$

[1]*Double vector bundles with inverse symmetry* in [22]. A symmetry of a double vector bundle is an involution as in Definition 3.14, but without the condition on the core morphism. It is *direct* if the induced morphism on the core is the identity, and *inverse* if it is minus the identity.

for $d \in D$ and $e \in \mathbb{E} \simeq \mathbb{E}^*_B$ with $\pi_2(d) = \pi_Q(e)$.

(3) *The constructions in (1) and (2) are inverse to each other.*

Proof. (1) We begin by proving that the pairing is well-defined. Choose $e_1, e_2 \in \mathbb{E} \times_B \mathbb{E}$ and any $d \in D$ with $\pi_Q(e_1) = \pi_1(d) = q_1 \in Q_m$ and $\pi_Q(e_2) = \pi_2(d) = q_2 \in Q_m$. Then, for any $\beta \in B^*_m$, we have by(3) $d' := d +_1 (0^D_{q_1} +_2 \overline{\beta}) = d +_2 (0^D_{q_2} +_1 \overline{\beta})$ and thefore $\pi_Q(e_1) = q_1 = \pi_1(d') \in Q_m$ and $\pi_Q(e_2) = q_2 = \pi_2(d') \in Q_m$. Conversely, any $d' \in D$ with $\pi_Q(e_1) = \pi_1(d') \in Q_m$ and $\pi_Q(e_2) = \pi_2(d') \in Q_m$ can be obtained in this manner. We compute

$$\begin{aligned}
\langle e_1, d' \rangle_Q + \langle e_2, \mathcal{I}(d') \rangle_Q &= \langle e_1, d +_1 (0^1_{q_1} +_2 \overline{\beta}) \rangle_Q + \langle e_2, \mathcal{I}(d +_1 (0^1_{q_1} +_2 \overline{\beta})) \rangle_Q \\
&= \langle e_1, d +_1 (0^1_{q_1} +_2 \overline{\beta}) \rangle_Q + \langle e_2, \mathcal{I}(d) +_2 (0^2_{q_1} +_1 \overline{-\beta})) \rangle_Q \\
&= \langle e_1, d +_1 (0^1_{q_1} +_2 \overline{\beta}) \rangle_Q + \langle e_2, \mathcal{I}(d) +_1 (0^1_{q_2} +_2 \overline{-\beta})) \rangle_Q \\
&= \langle e_1, d \rangle_Q + \langle e_2, \mathcal{I}(d) \rangle_Q + \langle \pi_B(e_1), \beta \rangle - \langle \pi_B(e_2), \beta \rangle \\
&= \langle e_1, d \rangle_Q + \langle e_2, \mathcal{I}(d) \rangle_Q.
\end{aligned}$$

To check the symmetry of the pairing, recall that if d is as above, then $\mathcal{I}(d)$ satisfies $\pi_1(\mathcal{I}(d)) = q_2 = \pi_Q(e_2)$ and $\pi_2(\mathcal{I}(d)) = q_1 = \pi_Q(e_1)$. Hence, we have by definition:

$$\langle e_2, e_1 \rangle_{\mathbb{E}} = \langle e_2, \mathcal{I}(d) \rangle_Q + \langle e_1, \mathcal{I}^2(d) \rangle_Q = \langle e_2, \mathcal{I}(d) \rangle_Q + \langle e_1, d \rangle_Q = \langle e_1, e_2 \rangle_{\mathbb{E}}.$$

Finally, consider $e \in \mathbb{E}$ with $\langle e, e' \rangle_{\mathbb{E}} = 0$ for all $e' \in \mathbb{E}$ with $\pi_B(e) = \pi_B(e') = b_m$. In particular, we find for all $\tau \in Q^*_m$:

$$0 = \langle e, 0^{\mathbb{E}}_{b_m} +_Q \overline{\tau} \rangle = \langle e, 0^1_q \rangle_Q + \langle 0^{\mathbb{E}}_{b_m} +_Q \overline{\tau}, \mathcal{I}(0^1_q) \rangle_Q = \langle e, 0^1_q \rangle_Q + \langle \tau^\dagger(b_m), 0^2_q \rangle_Q = \langle \tau, q \rangle.$$

This shows that $q = \pi_1(e) \in Q_m$ must vanish, and so that $e = 0^{\mathbb{E}}_{b_m} +_Q \overline{\eta}$ for some $\eta \in Q^*_m$. In the same manner as above, we find then that $\langle \eta, q' \rangle = 0$ for all $q' \in Q_m$, and so that $\eta = 0$. This shows that $e = 0^{\mathbb{E}}_{b_m}$. The linearity of $\langle \cdot\,, \cdot \rangle_{\mathbb{E}}$ is immediate and its proof is left to the reader.

(2) A straightforward computation shows that both

$$\begin{array}{ccc}
D & \xrightarrow{\mathcal{I}} & D \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
Q & \xrightarrow[\mathrm{Id}_Q]{} & Q
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
D & \xrightarrow{\mathcal{I}} & D \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
Q & \xrightarrow[\mathrm{Id}_Q]{} & Q
\end{array}$$

are morphisms of vector bundles. To find the core morphism of $\mathcal{I}$, consider $\beta \in B^*_m$ and $0^{\mathbb{E}}_b +_Q \overline{\tau}$ for some $b \in B_m$ and $\tau \in Q^*_m$. Then

$$\begin{aligned}
\langle \mathcal{I}(\overline{\beta}), 0^{\mathbb{E}}_b +_Q \overline{\tau} \rangle_Q = \boldsymbol{\langle} 0^{\mathbb{E}}_b +_Q \overline{\tau}, \overline{\beta} \boldsymbol{\rangle} &= \langle 0^{\mathbb{E}}_b +_Q \overline{\tau}, 0^{\mathbb{E}}_b \rangle_{\mathbb{E}} - \langle \overline{\beta}, 0^{\mathbb{E}}_b \rangle_Q \\
&= -\langle \beta, b \rangle = \left\langle \overline{-\beta}, 0^{\mathbb{E}}_b +_Q \overline{\tau} \right\rangle_Q .
\end{aligned}$$

Since any element of $\pi_Q^{-1}(0^Q_m) \subseteq \mathbb{E}$ can be written $0^{\mathbb{E}}_b +_Q \overline{\tau}$ for some $b \in B_m$ and $\tau \in Q^*_m$, we have proved that $\mathcal{I}(\overline{\beta}) = \overline{-\beta}$ for all $\beta \in B^*$. We prove that $\mathcal{I}^2 = \mathrm{Id}_D$. Choose $d \in D$ and $e \in \mathbb{E}$ with $\pi_Q(e) = \pi_1(d) = \pi_2(\mathcal{I}(d))$. Then, by definition, with $e' \in \mathbb{E}$ such that $\pi_B(e') = \pi_B(e)$ and $\pi_Q(e') = \pi_2(d) = \pi_1(\mathcal{I}(d))$:

$$\begin{aligned}
\langle \mathcal{I}^2(d), e \rangle_Q = \boldsymbol{\langle} e, \mathcal{I}(d) \boldsymbol{\rangle} &= \langle e, e' \rangle_{\mathbb{E}} - \langle \mathcal{I}(d), e' \rangle_Q \\
&= \langle e, e' \rangle_{\mathbb{E}} - \boldsymbol{\langle} e', d \boldsymbol{\rangle} = \langle e, e' \rangle_{\mathbb{E}} - \langle e', e \rangle_{\mathbb{E}} + \langle d, e \rangle_Q
\end{aligned}$$

(3) We start from an involutive double vector bundle (D, Q, Q, M) with core B^* and involution $\mathcal{I}$. We build the dual metric double vector bundle $(\mathbb{E} = D^*_{\pi_1}, Q, Q, M)$ as in (i), and

construct the dual involutive double vector bundle as in (ii). Let $\mathcal{I}'$ be the new involution obtained in this manner on D. By definition, we have for $d \in D$ and $e \in \mathbb{E}$ with $\pi_Q(e) = \pi_2(d)$:

$$\langle \mathcal{I}'(d), e\rangle_Q = \langle\!\langle e, d\rangle\!\rangle = \langle e, e'\rangle_{\mathbb{E}} - \langle d, e'\rangle_Q = \langle \mathcal{I}(d), e\rangle_Q + \langle \mathcal{I}^2(d), e'\rangle_Q - \langle d, e'\rangle_Q = \langle \mathcal{I}(d), e\rangle_Q$$

for any $e' \in \mathbb{E}$ with $\pi_B(e') = \pi_B(e)$ and $\pi_Q(e') = \pi_1(d)$. This shows $\mathcal{I} = \mathcal{I}'$.

Conversely, if we start with a metric double vector bundle $(\mathbb{E}, Q, B, M)$ and take the dual involutive double vector bundle $(D = \mathbb{E}^*_Q, Q, Q, M)$ with core B^* and involution $\mathcal{I}$, the involution defines a new metric $\langle\cdot\,,\cdot\rangle'$ on $\mathbb{E}$. We have for all $(e_1, e_2) \in \mathbb{E} \times_B \mathbb{E}$:

$$\langle e_1, e_2\rangle' = \langle d, e_1\rangle_Q + \langle \mathcal{I}(d), e_2\rangle_Q = \langle d, e_1\rangle_Q + \langle\!\langle e_2, d\rangle\!\rangle = \langle e_1, e_2\rangle_{\mathbb{E}}$$

with d any element of D satisfying $\pi_1(d) = \pi_Q(e_1)$ and $\pi_2(d) = \pi_Q(e_2)$. □

Let (D, Q, Q, M) be an involutive double vector bundle with involution $\mathcal{I}$ and core B^*. Take a Lagrangian linear splitting $\Sigma\colon Q \times_M B \to \mathbb{E}$ of the metric double vector bundle $\mathbb{E} = D^*_{\pi_1}$ and the dual splitting $\Sigma^\star\colon Q \times_M Q \to D$ of D. Consider $(q_1, q_2) \in Q \times_M Q$. Each element $e \in \mathbb{E}$ with $\pi_Q(e) = q_2$ and $\pi_B(e) =: b$ can be written $e = \Sigma(q_2, b) +_Q (0^{\mathbb{E}}_{q_2} +_B \overline{\tau})$ with a core element $\tau \in Q^*$ with $q_{Q^*}(\tau) = q_Q(q_2) = q_B(b)$:

$$\begin{aligned}\langle \mathcal{I}(\Sigma^\star(q_1, q_2)), e\rangle_Q &= \langle\!\langle e, \Sigma^\star(q_1, q_2)\rangle\!\rangle = \langle\!\langle \Sigma(q_2, b) +_Q (0^{\mathbb{E}}_{q_2} +_B \overline{\tau}), \Sigma^\star(q_1, q_2)\rangle\!\rangle\\ &= \langle \Sigma(q_2, b) +_Q (0^{\mathbb{E}}_{q_2} +_B \overline{\tau}), \Sigma(q_1, b)\rangle_{\mathbb{E}} - \langle \Sigma^\star(q_1, q_2), \Sigma(q_1, b)\rangle_Q\\ &= 0 + \langle \tau, q_1\rangle - 0 = \langle \tau, q_1\rangle.\end{aligned}$$

Since also $\langle \Sigma^\star(q_2, q_1), e\rangle_Q = \langle \Sigma^\star(q_2, q_1), \Sigma(q_2, b) +_Q (0^{\mathbb{E}}_{q_2} +_B \overline{\tau})\rangle_Q = \langle \tau, q_1\rangle$, we find that $\mathcal{I}(\Sigma^\star(q_1, q_2)) = \Sigma^\star(q_2, q_1)$. We call such a splitting an **involutive splitting** of D.

Note that the existence of involutive splittings can also be proved directly. Take an arbitrary splitting $\Sigma\colon Q \times_M Q \to D$ of D and set $\Sigma'\colon Q \times_M Q \to D$,

$$\Sigma'(q_1, q_2) := \frac{1}{2} \cdot_1 (\Sigma(q_1, q_2) +_1 \mathcal{I}(\Sigma(q_2, q_1))) = \frac{1}{2} \cdot_2 (\Sigma(q_1, q_2) +_2 \mathcal{I}(\Sigma(q_2, q_1))).$$

It is easy to check that Σ' is a linear splitting of D. The involutivity of Σ' is immediate.

We leave to the reader the proof of the following lemma.

Lemma 3.16. *Let $(D; Q, Q; M)$ be an involutive double vector bundle and consider the dual metric double vector bundle $(\mathbb{E} = D^*_{\pi_1}, B, Q, M)$. Then a section $\chi \in \Gamma^l_Q(\mathbb{E})$ lies in $\mathcal{C}(\mathbb{E})$ if and only if $\mathcal{I}^*(\ell_\chi) = -\ell_\chi$.*

Further, given $\tau \in \Gamma(Q^*)$, the morphism $\mathcal{I}^*\colon C^\infty(M) \to C^\infty(M)$ sends $\ell_{\tau^\dagger} = \pi_2^*\ell\tau$ to $\pi_1^*\ell_\tau$ and consequently $\pi_1^*\ell_\tau$ to $\ell_{\tau^\dagger}$. For $f \in C^\infty(M)$, $\mathcal{I}^*(\pi_1^* q_Q^* f) = \pi_1^* q_Q^* f$.

We end this section with a few examples.

Example 3.17. *Consider the metric double vector bundle TE in Example 3.11. The dual over E is (T^*E, E, E^*, M). with core T^*M. Since $E \simeq E^*$ via the metric, we find (T^*E, E, E, M) with the involution $\mathcal{I}$ sending $\mathbf{d}_{e_1(m)}\ell_{e_2} + (T_{e_1(m)}q_E)^*(\theta_m)$ to $\mathbf{d}_{e_2(m)}\ell_{e_1} - (T_{e_2(m)}q_E)^*(\theta_m + \mathbf{d}\langle e_1, e_2\rangle$ for $m \in M$, $e_1, e_2 \in \Gamma(E)$ and $\theta_m \in T^*_mM$. Up to the identification of E with E^*, the isomorphism $\mathcal{I}$ is the* reversal isomorphism *$T^*E \simeq T^*E^*$ in* [19].

Example 3.18. *Consider the metric double vector bundle $TE \oplus T^*E$ in Example 3.12. The dual $(TE)^*_{TM}$ is isomorphic to TE^*. The dual $(T^*E)^*_{E^*}$ is TE^*, modulo the reversal isomorphism*

$$-R\colon T^*E \to T^*E^*,\ -R(\mathbf{d}_{e(m)}\ell_\varepsilon +_E (T_{e(m)}q_E)^*\theta_m) = -\mathbf{d}_{\varepsilon(m)}\ell_e +_E (T_{\varepsilon(m)}q_{E^*})^*(\theta_m + \mathbf{d}\langle e, \varepsilon\rangle)$$

[19]. *Hence, the dual over $TM \oplus E^*$ is $(TE^* \oplus TE^*; TM \oplus E^*, E^* \oplus TM, M)$ with core E^*. More precisely, consider an element $(T_m\epsilon_1 v_m + \eta_1^\uparrow(\epsilon_1(m)), T_m\epsilon_2 w_m + \eta_2^\uparrow(\epsilon_2(m)))$ of $TE^* \oplus TE^*$ with*

$\epsilon_1, \epsilon_2, \eta_1, \eta_2 \in \Gamma(E^*)$, $v_m, w_m \in T_mM$. Its projection π_1 is $(v_m, \epsilon_2(m))$ and its projection π_2 is $(\epsilon_1(m), w_m)$. Its pairing over $(v_m, \epsilon_2(m))$ with an element $(T_m e v_m + (e')^\uparrow(e(m)), \mathbf{d}_{e(m)}\ell_{\epsilon_2} + (T_{e(m)}q_E)^*\theta_m)$ of $TE \oplus T^*E$ over $e(m) \in E$ and $(v_m, \epsilon_2(m)) \in TM \oplus E^*$ is

$$\begin{aligned} &\langle T_m\epsilon_1 v_m + \eta_1^\uparrow(\epsilon_1(m)), T_m e v_m + (e')^\uparrow(e(m))\rangle \\ &+ \langle T_m\epsilon_2 w_m + \eta_2^\uparrow(\epsilon_2(m)), -\mathbf{d}_{\epsilon_2(m)}\ell_e + (T_{\epsilon_2(m)}q_{E^*})^*(\theta_m + \mathbf{d}\langle\epsilon_2, e\rangle), \end{aligned}$$

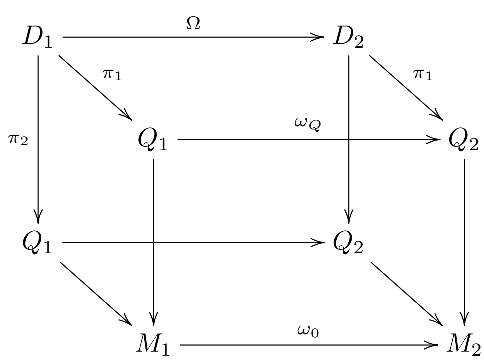
which is easily computed to be $v_m\langle\epsilon_1, e\rangle + \langle e, \eta_1 - \eta_2\rangle(m) + \langle\theta_m, w_m\rangle + \langle e', \epsilon_1\rangle$. In particular, the core of $(TE \oplus T^*E)^*_{TM\oplus E^*}$ is identified as follows with E^*:

$$\eta_m \in E^* \mapsto \left(\left.\frac{d}{dt}\right|_{t=0} 0_m^{E^*} + t\frac{\eta_m}{2}, \left.\frac{d}{dt}\right|_{t=0} 0_m^{E^*} - t\frac{\eta_m}{2} \right).$$

A computation yields the equality of $\mathcal{I}\left(T_m\epsilon_1(v_m) + \eta_1^\uparrow(\epsilon_1(m)), T_m\epsilon_2(v_m) + \eta_2^\uparrow(\epsilon_2(m))\right)$ with $\left(T_m\epsilon_2(v_m) + \eta_2^\uparrow(\epsilon_2(m)), T_m\epsilon_1(v_m) + \eta_1^\uparrow(\epsilon_1(m))\right)$. The morphism $\mathcal{I}$ is minus the identity on the core, and it exchanges π_1 and π_2.

3.3.1. *The category of involutive double vector bundles.*

Definition 3.19. *A* ***morphism*** *$\Omega\colon D_1 \to D_2$ of* ***involutive double vector bundles*** *is a morphism*



of double vector bundles such that

$$\Omega \circ \mathcal{I}_1 = \mathcal{I}_2 \circ \Omega.$$

We write IDVB *for the obtained category of involutive double vector bundles.*

We call ω_{B^*} the induced morphism $B_1^* \to B_2^*$ on the cores. We let the reader check that the morphism on the second side of the diagram must coincide with $\omega_Q\colon Q_1 \to Q_2$.

Theorem 3.20. *Let $(D_1; Q_1, Q_1; M_1)$ and $(D_2; Q_2, Q_2; M_2)$ be two involutive double vector bundles and consider the dual metric double vector bundles $(\mathbb{E}_1 = D_1{}^*_{\pi_1}, B_1, Q_1, M_1)$ and $(\mathbb{E}_2 = D_2{}^*_{\pi_1}, B_2, Q_2, M_2)$. A morphism $\Omega\colon D_1 \to D_2$ of involutive double vector bundles is equivalent to a pair of morphisms of modules*

$$\omega^\star\colon \mathcal{C}(\mathbb{E}_2) \to \mathcal{C}(\mathbb{E}_1), \quad \omega_Q^\star\colon \Gamma(Q_2^*) \to \Gamma(Q_1^*)$$

over a smooth map $\omega_0\colon M_1 \to M_2$, such that $\omega^\star\left(\widetilde{\tau_1 \wedge \tau_2}\right) = \widetilde{\omega_Q^\star\tau_1 \wedge \omega_Q^\star\tau_2}$ for all $\tau_1, \tau_2 \in \Gamma(Q_2^*)$.

Recall that this means in particular that $\omega^\star(q_Q^* f \cdot \chi) = q_{Q_1}^*(\omega_0^* f) \cdot \omega^\star(\chi)$ and $\omega_Q^\star(f \cdot \tau) = \omega_0^* f \cdot \omega_Q^\star\tau$ for all $\tau \in \Gamma(Q_2^*)$, $f \in C^\infty(M_2)$ and $\chi \in \mathcal{C}(\mathbb{E}_2)$.

Proof. Recall that the restriction of $\Omega^\star$ to core sections $\tau^\dagger$, $\tau\in\Gamma(Q_2^*)$ is given by $\Omega^\star(\tau^\dagger)=(\omega_Q^\star(\tau))^\dagger$ (see §2.1).

Recall further that $\Omega^\star$ restricts to a morphism $\Omega^\star\colon\Gamma^l_{Q_2}(\mathbb{E}_2)\to\Gamma^l_{Q_1}(\mathbb{E}_1)$ of modules over $\omega_0^*\colon C^\infty(M_2)\to C^\infty(M_1)$. Choose $\chi\in\mathcal{C}(\mathbb{E}_2)$. Since by (2)

$$\mathcal{I}_1^*(\ell_{\Omega^\star\chi})=\ell_{\mathcal{I}_1^\star\Omega^\star\chi}=\ell_{\Omega^\star\mathcal{I}_2^\star\chi}=\Omega^*\mathcal{I}_2^*\ell_\chi=\Omega^*(-\ell_\chi)=-\ell_{\Omega^\star\chi},$$

we find that $\Omega^\star\chi\in\mathcal{C}(\mathbb{E}_1)$ by Lemma 3.16. Therefore, $\Omega^\star$ restricts to a morphism $\omega^\star\colon\mathcal{C}(\mathbb{E}_2)\to\mathcal{C}(\mathbb{E}_1)$ of modules over $\omega_0^*\colon C^\infty(M_2)\to C^\infty(M_1)$. Next we choose $\tau_1,\tau_2\in\Gamma(Q_2^*)$. Then

$$\frac{1}{2}\Omega^\star\left(\pi_1^*\ell_{\tau_1}\tau_2^\dagger-\pi_1^*\ell_{\tau_2}\tau_1^\dagger\right)=\frac{1}{2}\left(\pi_1^*\ell_{\omega_Q^\star\tau_1}(\omega_Q^\star\tau_2)^\dagger-\pi_1^*\ell_{\omega_Q^\star\tau_2}(\omega_Q^\star\tau_1)^\dagger\right)$$

shows that $\Omega^\star\left(\widetilde{\tau_1\wedge\tau_2}\right)=\omega_Q^\star\widetilde{\tau_1\wedge\omega_Q^\star}\tau_2$.

Since $\mathcal{C}(\mathbb{E})$ and $\Gamma_Q^c(\mathbb{E})$ span pointwise $\mathbb{E}=D^*_{\pi_1}$, we find that the morphism Ω is completely encoded by the two maps $\omega^\star$ and $\omega_Q^\star$. □

Remark 3.21. *Recall that the morphism*

$$\omega_{B^*}^\star\colon\Gamma(B_2)\to\Gamma(B_1)$$

of modules over $\omega_0^*\colon C^\infty(M)\to C^\infty(N)$, i.e. the vector bundle morphism $\omega_{B^*}\colon B_1^*\to B_2^*$ is induced as follows by the two maps in the theorem. If $\chi\in\Gamma^l_{Q_2}(\mathbb{E}_2)$ is linear over $b\in\Gamma(B_2)$, then $\omega^\star(\chi)$ is linear over $\omega^\star_{B^*}(b)$.

Further, a morphism $\Omega\colon Q_1\times_{M_1}Q_1\times_{M_1}B_1^*\to Q_2\times_{M_2}Q_2\times_{M_2}B_2^*$ of decomposed metric double vector bundles is described by $\omega_Q\colon Q_1\to Q_2$, $\omega_{B^*}\colon B_1^*\to B_2^*$ and $\omega_{12}\colon Q_1\wedge Q_1\to B_2^*$, all morphisms of vector bundles over a smooth map $\omega_0\colon M_1\to M_2$:

$$\Omega(q,q',\beta)=(\omega_Q(q),\omega_Q(q'),\omega_{B^*}(\beta)+\omega_{12}(q,q')).$$

For $b\in\Gamma(B_1)$ the isotropic section $b^l\in\Gamma^l_{Q_2}(B_2\times_{M_2}Q_2\times_{M_2}Q_2^*)$, $b^l(q_m)=(b(m),q_m,0_m^{Q_2^*})$, is sent by $\omega^\star$ to the isotropic section $(\omega_B^\star(b))^l+\widetilde{\omega_{12}^\star(b)}\in\Gamma^l_{Q_1}(B_1\times_{M_1}Q_1\times_{M_1}Q_1^*)$.

3.4. **Equivalence of [2]-manifolds and involutive double vector bundles.** In this section we describe the equivalence of the category of involutive double vector bundles with the category of [2]-manifolds.

3.4.1. *The functor $\mathcal{M}(\cdot)\colon$ IDVB $\to$ [2]-Man.* Let (D,Q,Q,M) be an involutive double vector bundle with core B^* and consider the dual metric double vector bundle $(\mathbb{E}=D_Q^*,Q,B,M)$. We construct a [2]-manifold by assigning the degree 0 to elements of $C^\infty(M)$, the degree 1 to elements of $\Gamma(Q^*)$ and the degree 2 to elements of $\mathcal{C}(\mathbb{E})$.

Recall from Corollary 3.9 the existence of the vector bundle $\widehat{B}$ over M with $\Gamma(\widehat{B})=\mathcal{C}(\mathbb{E})$. We construct as follows the sheaf $C^\infty(\mathcal{M}(D))^\bullet$ of ℕ-graded, graded commutative, associative, unital $C^\infty(M)$-algebras. For an arbitrary open set $U\subseteq M$ we set $C^\infty_U(\mathcal{M}(D))^0:=C^\infty_U(M)$. For $k\geq 0$ we set $C^\infty_U(\mathcal{M}(D))^{2k}=\Gamma_U(S^k\widehat{B})$, that is, the space of symmetric elements of $\Gamma(\otimes_k\widehat{B})$, and we set $C^\infty_U(\mathcal{M}(D))^{2k+1}=\Gamma_U(Q^*\otimes S^k\widehat{B})$. In particular, we have defined $C^\infty_U(\mathcal{M}(D))^2$ to be $\Gamma(\widehat{B})\simeq\mathcal{C}(\mathbb{E})$ and $C^\infty_U(\mathcal{M}(D))^1$ to be $\Gamma(Q^*)$. For each $i\in\mathbb{N}$, the sheaf $C^\infty_U(\mathcal{M}(D))^i$ is a sheaf of $C^\infty(M)$-modules, so the multiplication of $f\in C^\infty_U(M)$ with $\xi\in C^\infty_U(\mathcal{M}(D))^i$ is already given. Note that elements $\frac{1}{k!}\sum_{\sigma\in S_k}\chi_{\sigma(1)}\otimes\ldots\otimes\chi_{\sigma(k)}=:\chi_1\cdot\ldots\cdot\chi_k$ with $\chi_1,\ldots,\chi_k\in\widehat{B}_p$ generate $S^k\widehat{B}_p$ over a point $p\in M$. The symmetric product $(\cdot)\colon S^k\widehat{B}\otimes S^l\widehat{B}\to S^{k+l}\widehat{B}$ sends generators $\xi\otimes\eta=(\chi_1\cdot\ldots\cdot\chi_k)\otimes(\chi_{k+1}\cdot\ldots\cdot\chi_{k+l})$ to $\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\chi_{\sigma(1)}\otimes\ldots\otimes\chi_{\sigma(k+l)}$, and induces a product $(\cdot)\colon\Gamma(S^k\widehat{B})\otimes\Gamma(S^l\widehat{B})\to\Gamma(S^{k+l}\widehat{B})$, wich gives us the product

$(\cdot)\colon C^\infty_U(\mathcal{M}(D))^{2k}\otimes C^\infty_U(\mathcal{M}(D))^{2l}\to C^\infty_U(\mathcal{M}(D))^{2(k+l)}$ for $k,l\geq 0$. We further define the products $(\cdot)\colon (Q^*\otimes S^k\widehat{B})\otimes S^l\widehat{B}\to Q^*\otimes S^{k+l}\widehat{B}$ and $(\cdot)\colon S^k\widehat{B}\otimes(Q^*\otimes S^l\widehat{B})\to Q^*\otimes S^{k+l}\widehat{B}$ by

$$(\tau\otimes\chi_1\cdot\ldots\cdot\chi_k)\cdot(\chi_1\cdot\ldots\cdot\chi_{k+l})=\tau\otimes(\chi_1\cdot\ldots\cdot\chi_{k+l})=(\chi_1\cdot\ldots\cdot\chi_k)\cdot(\tau\otimes\chi_1\cdot\ldots\cdot\chi_{k+l}),$$

and the skew-symmetric product

$$(\cdot)\colon (Q^*\otimes S^k\widehat{B})\otimes(Q^*\otimes S^l\widehat{B})\to S^{k+l+1}\widehat{B},$$
$$(\tau_1\otimes\chi_1\cdot\ldots\cdot\chi_k)\cdot(\tau_2\otimes\chi_{k+1}\cdot\ldots\cdot\chi_{k+l})=\widetilde{\tau_1\wedge\tau_2}\cdot\chi_1\cdot\ldots\cdot\chi_{k+l}.$$

These pointwise multiplications induce an associative, graded-commutative multiplication

$$(\cdot)\colon C^\infty(\mathcal{M}(D))^\bullet\times C^\infty(\mathcal{M}(D))^\blacktriangle\to C^\infty(\mathcal{M}(D))^{\bullet+\blacktriangle}.$$

Choose now $p\in M$ and a coordinate neighborhood $U\ni p$ such that Q^* and B are trivialised on U by the basis frames $(\tau_1,\ldots,\tau_{r_1})$ and $(b_1,\ldots,b_{r_2})$. Recall also that after the choice of a Lagrangian splitting $\Sigma\colon Q\times_M B\to\mathbb{E}$, each element $\chi\in\mathcal{C}(\mathbb{E})$ over $b\in\Gamma(B)$ can be written as $\sigma_B(b)+\widetilde{\phi}$ with $\phi\in\Omega^2(Q)$. Then $C^\infty_U(\mathcal{M}(D))$ is generated on U as a $C^\infty_U(M)$-algebra by $\{\tau_1,\ldots,\tau_k,\sigma_B(b_1),\ldots,\sigma_B(b_l)\}$ (see also the proof of Proposition 3.8), where $\sigma_B(b_1),\ldots,\sigma_B(b_{r_2})$ are considered as sections of $\widehat{B}$. We obtain so the [2]-manifold $\mathcal{M}(D)$ of dimension $(m;r_1,r_2)$, where m is the dimension of m, r_1 is the rank of Q and r_2 is the rank of B.

We have constructed a map $\mathcal{M}(\cdot)$ sending involutive double vector bundles to [2]-manifolds. By Theorem 3.20 a morphism $\Omega\colon D_1\to D_2$ of metric double vector bundles is the same as a triple of maps

$$\omega_0\colon M_1\to M_2\quad\Leftrightarrow\quad\omega_0^*\colon C^\infty(M_2)\to C^\infty(M_1),$$
$$\omega^\star\colon\mathcal{C}(\mathbb{E}_2)\to\mathcal{C}(\mathbb{E}_1)\quad\text{and}\quad\omega_Q^\star\colon\Gamma(Q_2^*)\to\Gamma(Q_1^*)$$

with

$$\omega^\star\left(\widetilde{\tau_1\wedge\tau_2}\right)=\widetilde{\omega_Q^\star\tau_1\wedge\omega_Q^\star\tau_2},\qquad q_{Q_1}^*\omega_0^*f\cdot\omega^\star(\chi)=\omega^\star(q_{Q_2}^*f\cdot\chi)$$

and

$$\omega_0^*f\cdot\omega_Q^\star(\tau)=\omega^\star(f\cdot\tau)$$

for $f\in C^\infty(M_2)$, $\tau\in\Gamma(Q_2^*)$ and $\chi\in\mathcal{C}(\mathbb{E}_2)$. Hence we find that the triple $(\omega^\star,\omega_Q^\star,\omega_0^*)$ defines in this manner a morphism $\mathcal{M}(\Omega)\colon\mathcal{M}(D_1)\to\mathcal{M}(D_2)$ of the [2]-manifolds constructed above.

We have so defined a covariant functor $\mathcal{M}(\cdot)\colon$ IDVB $\to$ [2]-Man from the category of involutive double vector bundles to the category of [2]-manifolds.

3.4.2. *The functor $\mathcal{G}$: [2]-Man $\to$ IDVB.* (The letter $\mathcal{G}$ stands for *geometrisation.*) Conversely, we construct explicitly a metric double vector bundle associated to a given [2]-manifold $\mathcal{M}$. The idea is to adapt the construction of the equivalence of locally free and finitely generated sheaves of $C^\infty(M)$-modules with vector bundles over M (see §3.1.1).

First we give Pradines' original definition of a double vector bundle [22] (in the smooth and finite-dimensional case).

Definition 3.22. [22, C. §1] *Let M be a smooth manifold and $\mathbb{E}$ a set with a map $\Pi\colon\mathbb{E}\to M$. A **double vector bundle chart** is a quintuple $c=(U,\Theta,V_1,V_2,V_0)$, where U is an open set in M, V_1,V_2,V_3 are three vector spaces and $\Theta\colon\Pi^{-1}(U)\to U\times V_1\times V_2\times V_0$ is a bijection such that $\Pi=\mathrm{pr}_1\circ\Theta$.*

*Two double vector bundle charts c and c' are **compatible** if the "change of chart" $\Theta'\circ\Theta^{-1}$ over $U\cap U'$ has the following form:*

$$(x,v_1,v_2,v_0)\mapsto(x,A_1(x)v_1,A_2(x)v_2,A_0(x)v_0+\omega(x)(v_1,v_2))$$

with $x \in U \cap U'$, $v_i \in V_i$, $A_i \in C^\infty(M, \mathrm{Gl}(V_i))$ for $i = 0, 1, 2$ and $\omega \in C^\infty(M, \mathrm{Hom}(V_1 \otimes V_2, V_0))$.

*A **double vector bundle atlas** $\mathfrak{A}$ on $\mathbb{E}$ is a set of double vector bundle charts of $\mathbb{E}$ that are pairwise compatible and such that the set of underlying open sets in M is a covering of M. As usual, two double vector bundle atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are **equivalent** if their union is an atlas. A double vector bundle structure on $\mathbb{E}$ is an equivalence class of double vector bundle atlases on $\mathbb{E}$.*

Given a [2]-manifold $\mathcal{M}$, we interpret its local generators as sections of a double vector bundle atlas, and we show that the obtained double vector bundle has a natural metric structure.

Let M be the smooth manifold underlying $\mathcal{M}$ and assume that $\mathcal{M}$ has dimension $(l; m, n)$. Choose a maximal open covering $\{U_\alpha\}$ of M such that $C^\infty_{U_\alpha}(\mathcal{M})$ is freely generated by $\xi_1^\alpha, \ldots, \xi_m^\alpha$ (in degree 1) and $\eta_1^\alpha, \ldots, \eta_n^\alpha$ (degree 2 generators). Choose now α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Then each generator ξ_i^β can be written in a unique manner as $\sum_{j=1}^m \omega_{\alpha\beta}^{ji} \xi_j^\alpha$ with $\omega^{ji} \in C^\infty(U_\alpha \cap U_\beta)$. Each generator η_i^β can be written as

$$\eta_i^\beta = \sum_{j=1}^n \psi_{\alpha\beta}^{ji} \cdot \left(\eta_j^\alpha + \sum_{1 \le k < l \le m} \rho_{\alpha\beta}^{jkl} \cdot \xi_k^\alpha \wedge \xi_l^\alpha \right)$$

with $\psi_{\alpha\beta}^{ij}, \rho_{\alpha\beta}^{ikl} \in C^\infty(U_\alpha \cap U_\beta)$. Set $A_1^{\alpha\beta} = (\omega_{\alpha\beta}^{ij})_{i,j} \in C^\infty(M, \mathrm{Gl}(\mathbb{R}^{m*}))$, $A_2^{\alpha\beta} = (\psi_{\alpha\beta}^{ij})_{i,j} \in C^\infty(M, \mathrm{Gl}(\mathbb{R}^n))$. Define $\nu^{\alpha\beta} \in C^\infty(M, \mathrm{Hom}(\mathbb{R}^m \otimes \mathbb{R}^n, \mathbb{R}^{m*}))$ by $\nu^{\alpha\beta}(e_i, e_j)(e_l) = \rho_{\alpha\beta}^{jil}$ for $1 \le i < l \le m$ and $j = 1, \ldots, n$. Then by construction

$$A_1^{\gamma\alpha} \cdot A_1^{\alpha\beta} = A_1^{\gamma\beta}, \qquad A_2^{\gamma\alpha} \cdot A_2^{\alpha\beta} = A_2^{\gamma\beta} \qquad \text{and}$$

$$\begin{aligned} \nu^{\gamma\beta}({A_1^{\beta\gamma}}^*(e_i), A_2^{\gamma\beta}(e_j))(e_l) &= \nu^{\gamma\alpha}({A_1^{\beta\gamma}}^*(e_i), A_2^{\gamma\beta}(e_j), e_l) \\ &\quad + \nu^{\alpha\beta}({A_1^{\beta\alpha}}^*(e_i), A_2^{\alpha\beta}(e_j))({A_1^{\gamma\alpha}}^* e_l). \end{aligned} \tag{18}$$

Set $\tilde{\mathbb{E}} = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$ (the disjoint union) and identify

$$(x, v_1, v_2, l_0) \in U_\beta \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$$

with

$$\left(x, (A_1^{\beta\alpha}(x))^*(v_1), A_2^{\alpha\beta}(x)(v_2), A_1^{\alpha\beta}(x)(l_0) + \nu^{\alpha\beta}(x)((A_1^{\beta\alpha}(x))^*(v_1), A_2^{\alpha\beta}(x)(v_2)) \right)$$

in $U_\alpha \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m*}$ for $x \in U_\alpha \cap U_\beta$. The cocycle equations (18) imply that this defines an equivalence relation on $\tilde{\mathbb{E}}$. The quotient space is $\mathbb{E}$, a double vector bundle: The map $\Pi\colon \mathbb{E} \to M$, $(x, v_1, v_2, l_0) \mapsto x$ is well-defined and, by construction, the charts $c = (U_\alpha, \Theta_\alpha = \mathrm{Id}, \mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^{m*})$ define a double vector bundle atlas on $\mathbb{E}$. Since the covering was chosen to be maximal, the obtained double vector bundle $\mathbb{E}$ does not depend on any choices.

Recall from the proof of Theorem 3.2 that there are two vector bundles E_1 and E_2 associated canonically to a [2]-manifold $\mathcal{M}$ (only the inclusion of $\Gamma(E_2)$ in $C^\infty(\mathcal{M})^2$ is non-canonical). E_1 and E_2^* are the sides of $\mathbb{E}$ and E_1^* is the core of $\mathbb{E}$. The vector bundle E_1^* can be defined by the transition functions $A_1^{\alpha\beta}$ and $\tilde{E}_1 = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^n$, $(x, v) \sim (x, A_1^{\alpha\beta}(x)(v))$ for $x \in U_\alpha \cap U_\beta$. The vector bundle E_2^* can be defined in the same manner using $A_2^{\alpha\beta}$ and the model space $\mathbb{R}^m$.

We use this to define a linear metric on $\mathbb{E}$. Over a chart domain U_α we set

$$\langle (x, v_1, v_2, l_0), (x, v_1', v_2, l_0') \rangle = l_0(v_1') + l_0'(v_1).$$

By construction, this does not depend on the choice of α with $x \in U_\alpha$.

By dualising $\mathbb{E}$ over E_1, we get an involutive double vector bundle, which we call $\mathcal{G}(\mathcal{M})$, with sides E_1 and core E_2. Again by definition of the morphisms in the category of [2]-manifolds (Definition 3.1) and in the category of involutive double vector bundles (Theorem 3.20), this defines a functor $\mathcal{G}\colon$ [2]-Man $\to$ IDVB between the two categories.

3.4.3. *Equivalence of categories.* Finally we need to prove that the two obtained functors define an equivalence of categories. The functor $\mathcal{G}\circ\mathcal{M}(\cdot)$ is the functor that sends an involutive double vector bundle to its maximal double vector bundle atlas, hence it is naturally isomorphic to the identity functor.

The functor $\mathcal{M}(\cdot)\circ\mathcal{G}\colon$ [2]-Man $\to$ [2]-Man sends a [2]-manifold $\mathcal{M}$ over M with degree 1 local generators ξ^i_α and cocycles $A^1_{\alpha\beta}$ and degree 2 generators η^i_α and cocycles $A^2_{\alpha\beta}$ and $\nu_{\alpha\beta}$ on U_α to the sheaf of core and coisotropic linear sections of the dual of $\mathcal{G}(\mathcal{M})$ with degree 1 local generators ξ^i_α and cocycles $A^1_{\alpha\beta}$ and degree 2 generators η^i_α and cocycles $A^2_{\alpha\beta}$ and $\nu_{\alpha\beta}$ on U_α. There is an obvious natural isomorphism between this functor and the identity functor [2]-Man $\to$ [2]-Man.

Hence we have the following result.

Theorem 3.23. *The functors $\mathcal{G}\colon$ [2]-Man $\to$ IDVB and $\mathcal{M}\colon$ IDVB $\to$ [2]-Man and the two natural isomorphisms above are an equivalence between the category of involutive double vector bundles and the category of [2]-manifolds.*

Remark 3.24. *Note that del Carpio-Marek's defines* self-conjugate double vector bundles *in his thesis* [5]*: those are double vector bundles $(D;Q,Q;M)$ with identical sides and a morphism $\mathcal{H}\colon D\to D$ of double vector bundles satisfying $\mathcal{H}^4=\mathrm{Id}_D$, $\pi_1\circ\mathcal{H}=-\pi_2$, $\pi_2\circ\mathcal{H}=\pi_1$, and restricting to the identity on the core. Morphisms of self-conjugate double vector bundles are defined like our morphisms of involutive double vector bundles.* [5] *shows that self-conjugate double vector bundles are dual to metric double vector bundles, and establishes an equivalence between their category and the category of* [2]*-manifolds.*

In (2) of Proposition 3.15, we work with the nondegenerate pairing $\langle\!\langle\,,\rangle\!\rangle\colon \mathbb{E}^*_B\times_Q\mathbb{E}^*_Q$ to define the involution on the dual E^*_Q of a metric double vector bundle $(\mathbb{E};B,Q;M)$. The difference between our definition and del Carpio-Marek's is due to his choice of the nondegenerate pairing $\mathbb{E}^*_Q\times_Q\mathbb{E}^*_B$, which equals $\langle\!\langle\,,\rangle\!\rangle$ up to a sign. Accordingly, in (1) of Proposition 3.15, the metric on the dual $\mathbb{E}=\mathbb{D}^*_{\pi_1}$ of an involutive double vector bundle in the sense of [5] *would be given by $\langle e_1,e_2\rangle_{\mathbb{E}}=\langle e_1,d\rangle_Q-\langle e_2,\mathcal{H}(d)\rangle_Q$ for $(e_1,e_2)\in\mathbb{E}\times_B\mathbb{E}$ and any $d\in D$ with $\pi_Q(e_1)=\pi_1(d)$ and $\pi_Q(e_2)=\pi_2(d)$.*

Del Carpio-Marek's analogue of Proposition 3.15 is phrased as follows: he shows that the defining map $\mathcal{H}$ of a self-conjugate double vector bundle (D,Q,Q,M) with core B^* is equivalent to an isomorphism $D^*_{\pi_1}\simeq(D^*_{\pi_1})^*_B$, and so to a linear metric on $D^*_{\pi_1}$. This result is (up to a sign) equivalent to our Proposition 3.15, and the two approaches are therefore very similar in nature (although developed independently). Del Carpio-Marek's equivalence of self-conjugate double vector bundles with [2]*-manifolds is then based on an equivalence of self-conjugate double vector bundles with short exact sequences as in Lemma 3.9, the duals of which belong to a family of short exact sequences that was reportedly proved by Bursztyn, Cattaneo, Mehta and Zambon*[2] *to be equivalent to* [2]*-manifolds. Up to the sign convention in the construction of the dual to a given metric double vector bundle, our Theorem 3.23 and del Carpio-Marek's equivalence of categories work the same: the functions of a given* [2]*-manifold are interpreted in both methods as the special sections of two metric double vector bundles in the same isomorphy class.*

[2]In an unpublished work in preparation.

3.4.4. *Correspondence of splittings.* Via the functors above, a decomposed involutive double vector bundles $Q \times_M Q \times_M B^*$ is sent to a split [2]-manifold $Q[-1] \oplus B^*[-2]$ and vice versa.

Choose an involutive double vector bundle $(D; Q, Q; M)$ with core B^* and the corresponding [2]-manifold $\mathcal{M}$. Each choice of an involutive decomposition $\mathbb{I}$ of D is equivalent to a choice of splitting $\mathcal{S}$ of the corresponding [2]-manifold, such that the following diagram commutes

$$\begin{array}{ccc} D & \xrightarrow{\ \mathbb{I}\ } & Q \times_M Q \times_M B^* \\ {\scriptstyle \mathcal{M}(\cdot)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}(\cdot)} \\ \mathcal{M}(D) & \xrightarrow[\ \mathcal{S}\ ]{} & Q[-1] \oplus B^*[-2]. \end{array}$$

Note also that the category of split [2]-manifolds is equivalent to the category of [2]-manifolds, and the category of decomposed metric double vector bundles is equivalent to the category of metric double vector bundles. We will use this in the following section.

3.4.5. *Geometric interpretation of the local generators $C^\infty(\mathcal{M}(D))$.* The equivalence of a vector bundle E over a smooth manifold M with the [1]-manifold $E[-1]$ (see §3.1.1) is often described as follows: *the generators of $C^\infty(E[-1])$ are the linear functions on E.*

In degree 2, the [2]-manifold that corresponds to an involutive double vector bundle (D, Q, Q, M) with core B^* and involution $\mathcal{I}$ is generated by $\Gamma_Q^c(\mathbb{E}) \simeq \Gamma(Q^*)$ in degree 1 and by $\mathcal{C}(\mathbb{E})$ in degree 2, where $\mathbb{E} = D^*_{\pi_1}$. Recall that $\mathbb{E}$ is dual to $\pi_1 \colon D \to Q$ by construction, but also, since $\mathbb{E} \simeq \mathbb{E}^*_B$, dual to $\pi_2 \colon D \to Q$ via $\langle\cdot,\cdot\rangle \colon \mathbb{E} \times_{Q,\pi_2} D \to \mathbb{R}$.

Since both $C^\infty(M)$-modules are contained in $\Gamma_Q(\mathbb{E})$, their elements can be understood in two manners as linear functions on $\mathbb{E}$. The duality of $\mathbb{E}$ with D over π_1 sends $\tau^\dagger \in \Gamma_Q^c(\mathbb{E})$ to $\ell_{\tau^\dagger} = \pi_2^*\ell_\tau \in C^\infty(D)$ and $\chi \in \mathcal{C}(\mathbb{E})$ to $\ell_\chi \in C^\infty(D)$. The duality of $\mathbb{E}$ with D over π_2 sends $\tau^\dagger \in \Gamma_Q^c(\mathbb{E})$ to $\pi_1^*\ell_\tau = \mathcal{I}^*(\ell_{\tau^\dagger}) \in C^\infty(D)$ and $\chi \in \mathcal{C}(\mathbb{E})$ to $\mathcal{I}^*(\ell_\chi) = -\ell_\chi \in C^\infty(D)$. Define $\mathcal{P}(D) \subseteq C^\infty(D)$ to be $C^\infty(M)$-module of functions that are affine linear in the fibers of π_1 and of π_2 and on which $\mathcal{I}^* \colon C^\infty(D) \to C^\infty(D)$ is just multiplication with -1.

A careful study of $\mathcal{P}(\mathbb{E})$ shows that the morphism of $C^\infty(M)$-modules $\psi \colon \Gamma_Q^c(\mathbb{E}) \oplus \mathcal{C}(\mathbb{E}) \to \mathcal{P}(D)$ sending $\tau^\dagger \in \Gamma_Q^c(\mathbb{E})$ to $\frac{1}{2}(\pi_2^*\ell_\tau - \pi_1^*\ell_\tau) = \frac{1}{2}(\ell_{\tau^\dagger} - \mathcal{I}^*\ell_{\tau^\dagger})$ and $\chi \in \mathcal{C}(\mathbb{E})$ to $\ell_\chi = \frac{1}{2}(\ell_\chi - \mathcal{I}^*\ell_\chi)$ is an isomorphism. Given a splitting $\Sigma \colon B \times_M Q \to \mathbb{E}$, $\psi(\tau^\dagger)$ is the function that sends $d = \Sigma^\star(q_1, q_2) +_1 (0^1_{q_1} +_2 \bar{\beta})$ to $\frac{1}{2}\langle \tau, q_2 - q_1\rangle$, and $\psi(\sigma_B(b) + \tilde{\omega})$ is the function that sends d to $\langle b, \beta\rangle + \omega(q_1, q_2)$. This shows that the elements of $\mathcal{P}(D)$ are polynomial in the sides of D.

In this picture, the degrees assigned in a rather artificial manner to the elements of $\Gamma_Q^c(\mathbb{E}) \oplus \mathcal{C}(\mathbb{E})$ become more natural: the elements of $\Gamma_Q^c(\mathbb{E})$ correspond via ψ to functions on D that are polynomial of degree 1 in the sides of D, and the elements of $\mathcal{C}(\mathbb{E})$ correspond via ψ to functions on D that are polynomial of degree 2 in the sides of D.

Finally, for $\tau_1, \tau_2 \in \Gamma(Q^*)$, the function $\psi(\widetilde{\tau_1 \wedge \tau_2}) = \ell_{\widetilde{\tau_1 \wedge \tau_2}} = \frac{1}{2}(\pi_1^*\ell_{\tau_1}\pi_2^*\ell_{\tau_2} - \pi_1^*\ell_{\tau_2}\pi_2^*\ell_{\tau_1})$ equals $\frac{1}{2}\left((\pi_1^*\ell_{\tau_1} + \pi_2^*\ell_{\tau_1})\psi(\tau_2^\dagger) - (\pi_1^*\ell_{\tau_2} + \pi_2^*\ell_{\tau_2})\psi(\tau_1^\dagger)\right)$.

4. Poisson [2]-manifolds, metric VB-algebroids and Poisson involutive double vector bundles.

In this section we study [2]-manifolds endowed with a Poisson structure of degree -2. We show how split Poisson [2]-manifolds are equivalent to a special family of 2-representations. Then we prove that Poisson [2]-manifolds are equivalent to metric double vector bundles endowed with a linear Lie algebroid structure that is compatible with the metric, or equivalenty to involutive double vector bundles with a linear Poisson structure that is $\mathcal{I}$-invariant.

Definition 4.1. *A Poisson* $[2]$*-manifold is a* $[2]$*-manifold endowed with a Poisson structure of degree* -2*. A morphism of Poisson* $[2]$*-manifolds is a morphism of* $[2]$*-manifolds that preserves the Poisson structure.*

Note that a Poisson bracket of degree -2 on a $[2]$-manifold $\mathcal{M}$ is an $\mathbb{R}$-bilinear map $\{\cdot\,,\cdot\}\colon C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ of the graded sheaves of functions, such that $|\{\xi,\eta\}| = |\xi| + |\eta| - 2$ for homogeneous elements $\xi,\eta \in C^\infty_{\mathcal{M}}(U)$. The bracket is graded skew-symmetric; $\{\xi,\eta\} = -(-1)^{|\xi|\,|\eta|}\{\eta,\xi\}$ and satisfies the graded Leibniz and Jacobi identities

$$\{\xi_1,\xi_2\cdot\xi_3\} = \{\xi_1,\xi_2\}\cdot\xi_3 + (-1)^{|\xi_1|\,|\xi_2|}\xi_2\cdot\{\xi_1,\xi_3\} \tag{19}$$

and

$$\{\xi_1,\{\xi_2,\xi_3\}\} = \{\{\xi_1,\xi_2\},\xi_3\} + (-1)^{|\xi_1|\,|\xi_2|}\{\xi_2,\{\xi_1,\xi_3\}\} \tag{20}$$

for homogeneous $\xi_1,\xi_2,\xi_3 \in C^\infty_{\mathcal{M}}(U)$. A morphism $\mu\colon \mathcal{N} \to \mathcal{M}$ of Poisson $[2]$-manifolds satisfies $\mu^\star\{\xi_1,\xi_2\} = \{\mu^\star\xi_1,\mu^\star\xi_2\}$ for all $\xi_1,\xi_2 \in C^\infty_{\mathcal{M}}(U)$, U open in M.

4.1. **Split Poisson** $[2]$**-manifolds and self-dual** 2**-representations.** We begin by defining self-dual 2-representations. Recall from §2.5.1 the dual of a 2-representation.

Definition 4.2. *Let* $(A,\rho,[\cdot\,,\cdot])$ *be a Lie algebroid. A 2-representation* $(\nabla^Q,\nabla^{Q^*},R)$ *of* A *on a complex* $\partial_Q\colon Q^* \to Q$ *is said to be* ***self-dual*** *if it equals its dual, i.e.* $\partial_Q = \partial_Q^*$*, the connections* ∇^Q *and* ∇^{Q^*} *are dual to each other, and* $R^* = -R \in \Omega^2(A,\operatorname{Hom}(Q,Q^*))$.

We prove the following result.

Theorem 4.3. *There is a bijection between split Poisson* $[2]$*-manifolds and self-dual 2-representations.*

Proof. First let us consider a split 2-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. (For simplicity, we adopt the notation $Q = E_1$ and $B^* = E_2$ for the bases of the metric double vector bundle geometrising $\mathcal{M}$.) That is, Q and B are vector bundles over M and the functions of degree 0 on $\mathcal{M}$ are the elements of $C^\infty(M)$, the functions of degree 1 are sections of Q^* and the functions of degree 2 are sections of $B \oplus Q^* \wedge Q^*$. Let us now take a Poisson bracket $\{\cdot\,,\cdot\}$ of degree -2 on $C^\infty(\mathcal{M})$. In the following, we consider arbitrary $f, f_1, f_2 \in C^\infty(M)$, $\tau,\tau_1,\tau_2 \in \Gamma(Q^*)$, and $b, b_1, b_2 \in \Gamma(B)$.

The brackets $\{f_1,f_2\}$, $\{f,\tau\}$ have degree -2 and -1, respectively, and must hence vanish. The bracket $\{\tau_1,\tau_2\}$ is a function on M because it has degree 0. Since $\{f,\tau\} = 0$ for all $f \in C^\infty(M)$ and $\tau \in \Gamma(Q^*)$, this defines a vector bundle morphism $\partial_Q\colon Q^* \to Q$ by (19): $\langle \tau_2, \partial_Q(\tau_1)\rangle = \{\tau_1,\tau_2\}$. Since $\{\tau_1,\tau_2\} = -(-1)^{|\tau_2|}\{\tau_2,\tau_1\} = \{\tau_2,\tau_1\}$, we find that $\partial_Q^* = \partial_Q$. The Poisson bracket $\{b,f\}$ has degree 0 and is hence an element of $C^\infty(M)$. Again by (19), this defines a derivation $\{b,\cdot\}|_{C^\infty(M)}$ of $C^\infty(M)$, hence a vector field $\rho_B(b) \in \mathfrak{X}(M)$; $\{b,f\} = \rho_B(b)(f)$. By the Leibnitz identity (19) for the Poisson bracket and the equality $\{f_1,f_2\} = 0$ for all $f_1,f_2 \in C^\infty(M)$, we get in this manner a vector bundle morphism (an anchor) $\rho_B\colon B \to TM$. The bracket $\{b,\tau\}$ has degree 1 and is hence a section of Q^*. Since $\{b,f\tau\} = f\{b,\tau\} + \{b,f\}\tau = f\{b,\tau\} + \rho_B(b)(f)\tau$ and $\{fb,\tau\} = f\{b,\tau\} + \{f,\tau\}b = f\{b,\tau\}$, we find a linear B-connection ∇ on Q^* by setting $\nabla_b\tau = \{b,\tau\}$. Let us finally look at the bracket $\{b_1,b_2\}$. This function has degree 2 and is hence the sum of a section of B and an element of $\Omega^2(Q)$. We write $\{b_1,b_2\} = [b_1,b_2] - R(b_1,b_2)$ with $[b_1,b_2] \in \Gamma(B)$ and $R(b_1,b_2) \in \Omega^2(Q)$. By a similar reasoning as before, we find that this defines a skew-symmetric bracket $[\cdot\,,\cdot]$ on $\Gamma(B)$ that satisfies a Leibniz equality with respect to ρ_B, and an element $R \in \Omega^2(B,\operatorname{Hom}(Q,Q^*))$

such that $R^* = -R$. Note also here that the bracket $\{b, \phi\}$ for $\phi \in \Omega^2(Q) \subseteq \Gamma(\mathrm{Hom}(Q, Q^*))$ is just $\nabla_b^{\mathrm{Hom}}\phi$, where ∇^{Hom} is the B-connection induced on $\mathrm{Hom}(Q, Q^*)$ by ∇ and ∇^*.

Now we explain how the dull algebroid structure on B is in reality a Lie algebroid structure, and that (∇, ∇^*, R) is a self-dual 2-representation of B on $\partial_Q\colon Q \to Q^*$. In order to do this, we only need to recall that the Poisson structure $\{\cdot\,,\cdot\}$ satisfies the Jacobi identity. The Jacobi identity for the three functions b_1, b_2, f yields the compatibility of the anchor on B with the bracket on $\Gamma(B)$. The Jacobi identity for b, τ_1, τ_2 yields $\partial_Q \circ \nabla = \nabla^* \circ \partial_Q$, and the Jacobi identity for b_1, b_2, τ yields $R_\nabla = R \circ \partial_Q$. The equality $R_{\nabla^*} = \partial_Q \circ R$ follows using $\partial_Q = \partial_Q^*$, $R^* = -R$ and $R_\nabla^* = -R_{\nabla^*}$. The Jacobi identity for $b_1, b_2, b_3 \in \Gamma(B)$ yields in a straightforward manner the Jacobi identity for $[\cdot\,,\cdot]$ on sections of $\Gamma(B)$ and the equation $\mathbf{d}_{\nabla^{\mathrm{Hom}}} R = 0$.

Take conversely a self dual 2-representation of a Lie algebroid B on a 2-term complex $\partial_Q\colon Q^* \to Q$ and consider the [2]-manifold $\mathcal{M} = Q[-1] \oplus B^*[-2]$. Then the self-dual 2-representation defines as described above a Poisson bracket of degree -2 on $C^\infty(\mathcal{M})$. □

4.2. **(Split) symplectic [2]-manifolds.** Note that an ordinary Poisson manifold $(M, \{\cdot\,,\cdot\})$ is symplectic if and only if the vector bundle morphism $\sharp\colon T^*M \to TM$ defined by $\mathbf{d}f \mapsto X_f$ is surjective, where $X_f \in \mathfrak{X}(M)$ is the derivation $\{f, \cdot\}$. Alternatively, we can say that the Poisson manifold is symplectic if the image of the map $\sharp\colon C^\infty(M) \to \mathfrak{X}(M)$, $f \mapsto \{f, \cdot\}$ generates $\mathfrak{X}(M)$ as a $C^\infty(M)$-module.

In the same manner, if $(\mathcal{M}, \{\cdot\,,\cdot\})$ is a Poisson $[n]$-manifold, the map

$$\sharp\colon C^\infty(\mathcal{M}) \to \mathrm{Der}(C^\infty(\mathcal{M}))$$

sends ξ to $\{\cdot, \xi\}$. Then $(\mathcal{M}, \{\cdot\,,\cdot\})$ is a **symplectic $[n]$-manifold** if the image of this map generates $\mathrm{Der}(C^\infty(\mathcal{M}))$ as a $C^\infty(\mathcal{M})$-module.

Let $(q_E\colon E \to M, \langle\cdot\,,\cdot\rangle)$ be a metric vector bundle, i.e. a vector bundle endowed with a nondegenerate fiberwise pairing $\langle\cdot\,,\cdot\rangle\colon E \times_M E \to \mathbb{R}$. Choose a metric connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then, identifying E with E^* via $\boldsymbol{\beta}\colon E \to E^*$ given by the metric, we find that the 2-representation $(\mathrm{Id}_E\colon E \to E, \nabla, \nabla, R_\nabla)$ is self-dual (an easy calculation shows that if ∇ is metric, then $\langle R_\nabla(X_1, X_2)e_1, e_2\rangle = -\langle R_\nabla(X_1, X_2)e_2, e_1\rangle$ for all $e_1, e_2 \in \Gamma(E)$ and $X_1, X_2 \in \mathfrak{X}(M)$). Consider the split Poisson [2]-manifold $E[-1] \oplus T^*M[-2]$, with the Poisson bracket given by the self-dual 2-representation. That is, the Poisson bracket is given by

$$\{f_1, f_2\} = 0, \quad \{f, e\} = 0, \quad \{e_1, e_2\} = \langle e_1, e_2\rangle,$$

$$\{X, e\} = \nabla_X e, \quad \{X, f\} = X(f)$$

and

$$\{X_1, X_2\} = [X_1, X_2] - R_\nabla(X_1, X_2).$$

Recall from (12) the special derivations that we found on split $[n]$-manifolds. The function $\sharp\colon C^\infty(E[-1] \oplus T^*M[-2]) \to \mathrm{Der}(C^\infty(E[-1] \oplus T^*M[-2]))$ sends a function f of degree 0 to $\widehat{\mathbf{d}f}$, a derivation of degree -2. $\sharp$ sends e to $\hat{e} + \mathbf{d}_\nabla e$, which is a derivation of degree -1. Note that locally, $\mathbf{d}_\nabla e \in \Omega^1(M, E)$ can be written as a sum $\sum_i \nabla_{\partial_{x_i}} e \cdot \widehat{\mathbf{d}x_i}$. Finally $\sharp$ sends X to $X + \nabla_X + [X, \cdot] - R(X, \cdot)$, which is a derivation of degree 0. Note that $R(X, \cdot)$ can be written as $\sum f_{ijk} e_i e_j \widehat{\mathbf{d}x_k}$ for some basis sections $e_1, \ldots, e_n \in \Gamma(E)$ and some functions f_{ijk} in $C^\infty(M)$. Hence, since the derivations $\widehat{\mathbf{d}f}$, $\hat{e}$ and $X + \boldsymbol{\beta} \circ \nabla_X \circ \boldsymbol{\beta}^{-1} + [X, \cdot]$ for $f \in C^\infty(M)$, $e \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, span $\mathrm{Der}(C^\infty(E[-1] \oplus T^*M[-2]))$ as a $C^\infty(E[-1] \oplus T^*M[-2])$-module, we find as a consequence that $E[-1] \oplus T^*M[-2]$ is a symplectic [2]-manifold.

Conversely, take a split Poisson [2]-manifold $Q[-1]\oplus B^*[-2]$, hence a self-dual 2-representation $(\partial_Q\colon Q^*\to Q,\nabla,\nabla^*,R)$ of a Lie algebroid B. Then $\sharp f=\rho_B^*\mathbf{d}f$ for all $f\in C^\infty(M)$, $\sharp\tau=\hat{\partial_Q}\tau-\mathbf{d}_{\nabla^*}\tau$ and $\sharp b=\rho_B(b)+\nabla_b^*+[b,\cdot]-R(b,\cdot)$. A discussion as the one above shows that the Poisson structure is symplectic if and only if $\rho_B\colon B\to TM$ is injective and surjective, hence an isomorphism and $\partial_Q\colon Q^*\to Q$ is surjective, hence an isomorphism. The isomorphism ∂_Q identifies then Q with its dual and Q becomes so a metric vector bundle with the pairing $\langle q_1,q_2\rangle_Q=\langle\partial_Q^{-1}(q_1),q_2\rangle=\{\partial_Q^{-1}q_1,\partial_Q^{-1}q_2\}$. Via the identification $\boldsymbol{\beta}^{-1}=\partial_Q\colon Q^*\xrightarrow{\sim}Q$, the linear connection ∇ is then automatically a metric connection and the self-dual 2-representation is $(\mathrm{Id}_Q\colon Q\to Q,\nabla,\nabla,R_\nabla)$.

We have hence found that split symplectic [2]-manifolds are equivalent to self-dual 2-representations $(\mathrm{Id}_E\colon E\to E,\nabla,\nabla,R_\nabla)$ defined by a metric vector bundle E together with a metric connection ∇, see also [23].

4.3. **Metric VB-algebroids and Poisson involutive double vector bundles.** Next we introduce the notions of metric VB-algebroids, involutive Poisson double vector bundles, and their morphisms.

Definition 4.4. (1) *Let $(\mathbb{E};Q,B;M)$ be a metric double vector bundle (with core Q^*) and assume that $(\mathbb{E}\to Q,B\to M)$ is a VB-algebroid. Then $(\mathbb{E}\to Q,B\to M)$ is a* ***metric VB-algebroid*** *if the isomorphism $\boldsymbol{\beta}\colon\mathbb{E}\to\mathbb{E}_B^*$ is an isomorphism of VB-algebroids.*

(2) *Let (D,Q,Q,M) be an involutive double vector bundle with core B^*, and let $\{\cdot\,,\cdot\}\colon C^\infty(D)\times C^\infty(D)\to C^\infty(D)$ be a Poisson structure on D. Then $(D,\{\cdot\,,\cdot\})$ is an* ***involutive Poisson double vector bundle*** *if the Poisson structure is linear over both sides of D and if $\mathcal{I}$ is an anti-Poisson morphism: $\{\mathcal{I}^*F,\mathcal{I}^*F'\}=-\mathcal{I}^*\{F,F'\}$ for all $F,F'\in C^\infty(D)$.*

(3) *A morphism $\Omega\colon D_1\to D_2$ of Poisson involutive double vector bundles is a morphism of the underlying involutive double vector bundles that is a Poisson map: $\{\Omega^*F,\Omega^*F'\}=\Omega^*\{F,F'\}$ for all $F,F'\in C^\infty(D)$.*

Remark 4.5. *Note that in his thesis* [5]*, del Carpio-Marek defines a Poisson self-conjugate double vector bundle as a self-conjugate double vector bundle (see Remark 3.24) endowed with a Poisson structure such that the self-conjugation $\mathcal{H}$ is a Poisson morphism. This is, again up to a sign, the same setting as the one of Poisson involutive double vector bundles. Del Carpio-Marek recovers independently our results Corollary 4.8 and Theorem 4.10 below.*

Consider a linear VB-algebroid structure $(\mathbb{E}\to Q,[\cdot\,,\cdot],\Theta\colon\mathbb{E}\to TQ)$ on a metric double vector bundle $(\mathbb{E};Q,B;M)$. The linear Lie algebroid structure defines a linear Poisson structure on $D=\mathbb{E}_Q^*$:

$$\{\ell_{\chi_1},\ell_{\chi_2}\}=\ell_{[\chi_1,\chi_2]},\quad\{\ell_\chi,\pi_1^*F\}=\Theta(\chi)(F),\quad\{F_1,F_2\}=0$$

for $\chi,\chi_1,\chi_2\in\Gamma_Q(\mathbb{E})$ and $F,F_1,F_2\in C^\infty(Q)$. This Poisson structure is automatically also linear over the other side $\pi_2\colon D\to Q$ [18]. We will prove below that it is involutive if and only if the corresponding Lie algebroid structure was metric.

Recall from Theorem 2.6 that linear splittings of VB-algebroids define 2-representations. First we prove that Lagrangian splittings of metric VB-algebroids correspond to self-dual 2-representations.

Proposition 4.6. *Let $(\mathbb{E}\to Q,B\to M)$ be a VB-algebroid with core Q^* and assume that $\mathbb{E}$ is endowed with a linear metric. Choose a Lagrangian splitting of $\mathbb{E}$ and consider the* *corresponding 2-representation of B on $\partial_Q\colon Q^*\to Q$. This 2-representation is self-dual if and only if $(\mathbb{E}\to Q, B\to M)$ is a metric VB-algebroid.*

Proof. It is easy to see that $\boldsymbol{\beta}\colon \mathbb{E}\to\mathbb{E}^*_B$ sends core sections $\tau^\dagger\in\Gamma^c_Q(\mathbb{E})$ to core sections $\tau^\dagger\in\Gamma^c_Q(\mathbb{E}^*_B)$. (As always, we identify Q^{**} with Q via the canonical isomorphism.) Let $\Sigma\colon B\times_M Q\to\mathbb{E}$ be a Lagrangian splitting of $\mathbb{E}$. We have seen in Section 2.2.3 that the map $\sigma_B\colon\Gamma(B)\to\Gamma^l_Q(\mathbb{E})$ induces a horizontal lift $\sigma^\star_B\colon\Gamma(B)\to\Gamma^l_Q(\mathbb{E}^*_B)$. Recall from Lemma 3.10 that $\boldsymbol{\beta}$ sends then also the linear sections $\sigma_B(b)$ to $\sigma^\star_B(b)$, for all $b\in\Gamma(B)$.

The double vector bundle $\mathbb{E}^*_B$ has a VB-algebroid structure $(\mathbb{E}^*_B\to Q^{**}, B\to M)$ (see §2.3). Given the splitting $\Sigma^\star\colon B\times_M Q^{**}\to\mathbb{E}^*_B$ defined by a Lagrangian splitting $\Sigma\colon B\times_M Q\to\mathbb{E}$, the VB-algebroid structure is given by the dual of the 2-representation $(\partial_Q\colon Q^*\to Q,\nabla^Q,\nabla^{Q^*},R\in\Omega^2(B,\operatorname{Hom}(Q,Q^*)))$, i.e.

$$\rho_{\mathbb{E}^*_B}(\tau^\dagger)=(\partial_Q^*\tau)^\uparrow\in\mathfrak{X}^c(Q^{**}),\qquad \rho_{\mathbb{E}^*_B}(\sigma^\star_B(b))=\widehat{\nabla^{Q^{**}}}\in\mathfrak{X}^l(Q^{**}),$$
$$\left[\sigma^\star_B(b),\tau^\dagger\right]=(\nabla^{Q^*}{}_b\tau)^\dagger,\quad\text{and}\quad[\sigma^\star_B(b_1),\sigma^\star_B(b_2)]=\sigma^\star_B[b_1,b_2]+R\widetilde{(b_1,b_2)}^*$$

(see §2.5.1). This shows immediately that $\boldsymbol{\beta}$ is an isomorphism of VB-algebroids over the canonical isomorphism $Q\to Q^{**}$ if and only if the 2-representation

$$(\partial_Q\colon Q^*\to Q,\nabla^Q,\nabla^{Q^*},R\in\Omega^2(B,\operatorname{Hom}(Q,Q^*)))$$

is self-dual. □

Corollary 4.7. *Consider a metric double vector bundle $(\mathbb{E},B,Q,M)$ endowed with a linear Lie algebroid structure on $\mathbb{E}\to Q$ over $B\to M$. Then the Lie algebroid structure is metric if and only if $\mathcal{C}(\mathbb{E})$ is closed under the Lie bracket.*

Proof. Fix a Lagrangian splitting of $\mathbb{E}$ and consider the corresponding 2-representation $(\nabla^Q,\nabla^{Q^*},R\in\Omega^2(B,\operatorname{Hom}(Q,Q^*)))$ of B on $\partial_Q\colon Q^*\to Q$. The equation $[\sigma_B(b_1),\sigma_B(b_2)]=\sigma_B[b_1,b_2]-R\widetilde{(b_1,b_2)}$ shows that $[\sigma_B(b_1),\sigma_B(b_2)]\in\mathcal{C}(\mathbb{E})$ for all $b_1,b_2\in\Gamma(B)$ if and only if $R\in\Omega^2(B,Q^*\wedge Q^*)$.

An easy computation shows that $\left[\sigma_B(b),\widetilde{\phi}\right]=\widetilde{\nabla^{\operatorname{Hom}}_b\phi}$ is an element of $\mathcal{C}(\mathbb{E})$ for all $b\in\Gamma(B)$ and $\phi\in\Gamma(Q^*\wedge Q^*)$ if and only if $(\nabla^Q)^*=\nabla^{Q^*}$.

Finally, $\left[\widetilde{\phi_1},\widetilde{\phi_2}\right]=\widetilde{\phi_2\circ\partial_Q\circ\phi_1-\phi_1\circ\partial_Q\circ\phi_2}\in\mathcal{C}(\mathbb{E})$ if and only if $\phi_2\circ\partial_Q\circ\phi_1-\phi_1\circ\partial_Q\circ\phi_2\in\Gamma(Q^*\wedge Q^*)$. This is the case for all $\phi_1,\phi_2\in\Gamma(Q^*\wedge Q^*)$ if and only if $\partial_Q=\partial_Q^*$. □

Corollary 4.8. *Let $(D;Q,Q;M)$ be an involutive double vector bundle and consider the dual metric double vector bundle $(\mathbb{E}=D^*_{\pi_1},B,Q,M)$. A linear Lie algebroid structure on $\mathbb{E}\to Q$ is metric if and only if the dual linear Poisson structure on D is involutive.*

Proof. We need to find

$$\{\mathcal{I}^*(F_1),\mathcal{I}^*(F_2)\}=-\mathcal{I}^*\{F_1,F_2\}\tag{21}$$

for all $F_1,F_2\in C^\infty(D)$. Since $\mathcal{C}(\mathbb{E})$ and $\Gamma^c_Q(\mathbb{E})$ span pointwise $\mathbb{E}$, it is sufficient to check (21) on functions ℓ_χ, $\ell_{\tau^\dagger}=\pi_2^*\ell_\tau$, $\pi_1^*\ell_\tau$ and $\pi_1^*q_Q^*f$ for $\chi\in\mathcal{C}(\mathbb{E})$, $\tau\in\Gamma(Q^*)$ and $f\in C^\infty(M)$. Using Lemma 3.16, it is easy to see that (21) is trivially satisfied on $\pi_1^*q_Q^*f_1$ and $\pi_1^*q_Q^*f_2$, on $\pi_1^*q_Q^*f$ and $\pi_1^*\ell_\tau$, and equivalently on $\pi_1^*q_Q^*f$ and $\ell_{\tau^\dagger}$, on $\pi_1^*\ell_{\tau_1}$ and $\pi_1^*\ell_{\tau_2}$ and equivalently on $\ell_{\tau_1^\dagger}$ and $\ell_{\tau_2^\dagger}$ for $f,f_1,f_2\in C^\infty(M)$ and $\tau,\tau_1,\tau_2\in\Gamma(Q^*)$.

The equalities

$$\{\mathcal{I}^*\ell_{\tau_1^\dagger},\mathcal{I}^*\pi_1^*\ell_{\tau_2}\}=\{\pi_1^*\ell_{\tau_1},\ell_{\tau_2^\dagger}\}=-\{\ell_{\tau_2^\dagger},\pi_1^*\ell_{\tau_1}\}=-\pi_1^*(\partial_Q\tau_2)^\dagger(\ell_{\tau_1})=-\pi_1^*q_Q^*\langle\partial_Q\tau_2,\tau_1\rangle$$

and

$$\mathcal{I}^*\{\ell_{\tau_1^\dagger}, \pi_1^*\ell_{\tau_2}\} = \mathcal{I}^*\pi_1^*q_Q^*\langle\partial_Q\tau_1,\tau_2\rangle = \pi_1^*q_Q^*\langle\partial_Q\tau_1,\tau_2\rangle$$

show that $\{\mathcal{I}^*\ell_{\tau_1^\dagger}, \mathcal{I}^*\pi_1^*\ell_{\tau_2}\} = -\mathcal{I}^*\{\ell_{\tau_1^\dagger}, \pi_1^*\ell_{\tau_2}\}$ for all $\tau_1,\tau_2\in\Gamma(Q^*)$ if and only if $\partial_Q = \partial_Q^*$. Further, we find $\{\mathcal{I}^*\ell_\chi, \mathcal{I}^*\pi_1^*q_Q^*f\} = -\{\ell_\chi, \pi_1^*q_Q^*f\} = -\pi_1^*q_Q^*\rho_B(b)(f) = -\mathcal{I}^*\pi_1^*q_Q^*\rho_B(b)(f) = -\mathcal{I}^*\{\ell_\chi, \pi_1^*q_Q^*f\}$ for $f\in C^\infty(M)$ and $\chi\in\mathcal{C}(\mathbb{E})$.

Finally, we have $\{\mathcal{I}^*\ell_{\chi_1}, \mathcal{I}^*\ell_{\chi_2}\} = \{\ell_{\chi_1},\ell_{\chi_2}\} = \ell_{[\chi_1,\chi_2]}$ and $-\mathcal{I}^*\{\ell_{\chi_1},\ell_{\chi_2}\} = -\mathcal{I}^*\ell_{[\chi_1,\chi_2]} = \ell_{[\chi_1,\chi_2]}$ if and only if $[\chi_1,\chi_2]\in\mathcal{C}(\mathbb{E})$. Hence, we can conclude using Corollary 4.7. □

Remark 4.9. *Note that a linear Poisson structure on D is involutive if and only if $\mathcal{P}(D)$ defined in §3.4.5 is closed under the Poisson bracket: by* (21)*, if $\mathcal{I}^*F_1 = -F_1$ and $\mathcal{I}^*F_2 = -F_2$, then $\mathcal{I}^*\{F_1,F_2\} = -\{\mathcal{I}^*F_1, \mathcal{I}^*F_2\} = -\{F_1,F_2\}$. The Poisson bracket of $\pi_2^*\ell_{\tau_1} - \pi_1^*\ell_{\tau_1}$ with $\pi_2^*\ell_{\tau_2} - \pi_1^*\ell_{\tau_2}$ vanishes and the Poisson bracket of $\ell_{\sigma_B(b)+\widetilde{\omega}}$ with $\pi_2^*\ell_\tau - \pi_1^*\ell_\tau$ is $\pi_2^*\ell_{\nabla_b\tau - \mathbf{i}_{\partial_Q\tau}\omega} - \pi_1^*\ell_{\nabla_b\tau-\mathbf{i}_{\partial_Q\tau}\omega}$. The rest follows as in the proof of the preceding theorem.*

4.4. **Equivalence of Poisson [2]-manifolds with Poisson involutive double vector bundles.** The functors found in Section 3.4 between the category of metric double vector bundles and the category of [2]-manifolds induce functors between the category of metric VB-algebroids and the category of Poisson [2]-manifolds.

Let $(\mathcal{M},\{\cdot\,,\cdot\})$ be a Poisson [2]-manifold and consider the involutive double vector bundle $(\mathcal{G}(\mathcal{M}), E_1, E_1, M)$ with core E_2 corresponding to $\mathcal{M}$ as in §3.4.2. Then the isotropic linear sections $\mathcal{C}(\mathbb{E})$ can be identified with the degree 2 functions on $\mathcal{M}$ and the core sections $\Gamma_Q^c(\mathbb{E})$ can be identified with the degree 1 functions on $\mathcal{M}$. Since the sections $\mathcal{C}(\mathbb{E})\cup\Gamma_Q^c(\mathbb{E})$ span $\mathbb{E}\to Q$, the Poisson bracket on $C^\infty(\mathcal{M})$ defines a linear Poisson bracket on $\mathcal{G}(\mathcal{M})$:

(22)

$$\begin{gathered}
\{\ell_{\chi_1},\ell_{\chi_2}\} = \ell_{\{\chi_1,\chi_2\}}, \qquad \{\ell_\chi, \pi_1^*q_Q^*f\} = \pi_1^*q_Q^*\{\chi,f\}, \qquad \{\ell_\chi,\pi_1^*\ell_\tau\} = \pi_1^*\ell_{\{\chi,\tau\}}\\
\{\ell_\chi,\ell_{\tau^\dagger}\} = \ell_{\{\chi,\tau\}^\dagger}, \qquad \{\ell_{\tau_1^\dagger},\ell_{\tau_2^\dagger}\} = 0, \qquad \{\ell_{\tau_1^\dagger},\pi_1^*\ell_{\tau_2}\} = \pi_1^*q_Q^*\{\tau_1,\tau_2\}\\
\{\pi_1^*\ell_{\tau_1},\pi_1^*\ell_{\tau_2}\} = 0, \qquad \{\ell_{\tau^\dagger},\pi_1^*q_Q^*f\} = 0, \qquad \{\pi_1^*\ell_\tau,\pi_1^*q_Q^*f\} = 0, \text{ and } \{\pi_1^*q_Q^*f_1,\pi_1^*q_Q^*f_2\} = 0
\end{gathered}$$

for all $f,f_1,f_2\in C^\infty(M)$, $\tau,\tau_1,\tau_2\in\Gamma(E_1^*)$ and $\chi,\chi_1,\chi_2\in\mathcal{C}(\mathbb{E})$. Per definition, the involution $\mathcal{I}$ on $\mathcal{G}(\mathcal{M})$ is anti-Poisson, and so $(\mathcal{G}(\mathcal{M}),\mathcal{I},\{\cdot\,,\cdot\})$ is a Poisson involutive double vector bundle.

Let now $\mu\colon\mathcal{M}_1\to\mathcal{M}_2$ be a morphism of Poisson [2]-manifolds. By the definition of the Poisson bracket in (22), the morphism $\mathcal{G}(\mu)\colon\mathcal{G}(\mathcal{M}_1)\to\mathcal{G}(\mathcal{M}_2)$ is automatically a Poisson morphism. For example, for functions $\chi_1,\chi_2\in C^\infty(\mathcal{M}_2)$ of degree 2, we have

$$\begin{aligned}
\{\mathcal{G}(\mu)^*\ell_{\chi_1},\mathcal{G}(\mu)^*\ell_{\chi_1}\} &= \{\ell_{\mu^\star\chi_1},\ell_{\mu^\star\chi_2}\} = \ell_{\{\mu^\star\chi_1,\mu^\star\chi_2\}} = \ell_{\mu^\star\{\chi_1,\chi_2\}}\\
&= \mathcal{G}(\mu)^*\ell_{\{\chi_1,\chi_2\}} = \mathcal{G}(\mu)^*\{\ell_{\chi_1},\ell_{\chi_2}\}.
\end{aligned}$$

We let the reader check the other cases. Hence, the functor $\mathcal{G}$ induces a functor $\mathcal{G}_P$ from the category of Poisson [2]-manifolds to the category of Poisson involutive double vector bundles.

Conversely, we consider a Poisson involutive double vector bundle $(D,Q,Q,M;\{\cdot\,,\cdot\})$ with core B^*, or equivalently, a metric VB-algebroid $(D^*_{\pi_1} =: \mathbb{E}\to Q, B\to M)$. Consider the image $\mathcal{M}(D)$ of D under the functor $\mathcal{M}$. Then $\mathcal{M}(D)$ is the [2]-manifold which degree 0 functions are the elements of $C^\infty(M)$, which degree 1 functions are the elements of $\Gamma(Q^*)$ and which degree 2-functions are the elements of $\mathcal{C}(\mathbb{E})$, with $\tau_1\wedge\tau_2 = \widetilde{\tau_1\wedge\tau_2}\in\mathcal{C}(\mathbb{E})$ for $\tau_1,\tau_2\in\Gamma(Q^*)$.

We define a Poisson bracket on $\mathcal{M}(D)$ by

$$\begin{aligned} \{\chi_1, \chi_2\} = [\chi_1, \chi_2], \quad \{\chi, \tau\}^\dagger = [\chi, \tau^\dagger], \\ \{\tau_1, \tau_2\} = \langle \tau_1, \partial_Q \tau_2 \rangle, \quad \{\chi, f\} = \rho_B(b)(f), \quad \{\tau, f\} = \{f_1, f_2\} = 0 \end{aligned} \tag{23}$$

on the generators $\chi, \chi_1, \chi_2 \in \mathcal{C}(\mathbb{E})$, with χ linear over $b \in \Gamma(B)$, $f, f_1, f_2 \in C^\infty(M)$ and $\tau, \tau_1, \tau_2 \in \Gamma(Q^*)$, and by graded symmetry and graded Leibniz extension to all functions on $\mathcal{M}$. Clearly, this graded bracket has degree -2 and is well-defined. Let $\Omega\colon D_1 \to D_2$ be a morphism of Poisson involutive double vector bundle. In a similar manner as above, it is easy to check that $\mathcal{M}(\Omega)\colon \mathcal{M}(D_1) \to \mathcal{M}(D_2)$ is a morphism of Poisson [2]-manifolds. Hence, $\mathcal{M}$ induces a functor $\mathcal{M}_P$ from the category of Poisson involutive double vector bundles to the category of Poisson [2]-manifolds.

The two functors $\mathcal{M}_P$ and $\mathcal{G}_P$ define together an equivalence of the category of Poisson involutive double vector bundle with the category of Poisson [2]-manifolds. Hence, we have proved the following theorem.

Theorem 4.10. *The functors $\mathcal{M}_P$ and $\mathcal{G}_P$ define together an equivalence of the category of Poisson involutive double vector bundle with the category of Poisson [2]-manifolds.*

4.5. **Examples.** We conclude by discussing three important classes of examples.

4.5.1. *Tangent doubles of metric vector bundles vs symplectic [2]-manifolds.* Consider a metric vector bundle $E \to M$ and a metric connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. The double tangent

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

has a VB-algebroid structure $(TE \to E; TM \to M)$ and a linear metric $\langle\cdot\,,\cdot\rangle\colon TE \times_{TM} TE \to \mathbb{R}$ defined as in Example 3.11.

Recall that Lagrangian linear splittings of TE are equivalent to metric connections $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. In other words, $\nabla = \nabla^*$ when E^* is identified with E via the non-degenerate pairing. The 2-representation $(\mathrm{Id}_E\colon E \to E, \nabla, \nabla, R_\nabla)$ defined by the Lagrangian splitting $\Sigma^\nabla\colon E \times_M TM \to TE$ and the VB-algebroid $(TE \to E, TM \to M)$ is then self-dual (see also §4.2).

The Poisson [2]-manifold $\mathcal{M}(TE)$ associated to TE is given as follows. The functions of degree 0 are elements of $C^\infty(M)$, the functions of degree 1 are sections of E (E is identified with E^* via the isomorphism $\boldsymbol{\beta}\colon E \to E^*$ defined by the pairing) and the functions of degree 2 are the vector fields $\widehat{\delta} \in \mathfrak{X}(E)$ for a derivation δ of E over $X \in \mathfrak{X}(M)$, that preserves the pairing. The Poisson bracket is given by $\{\widehat{\delta_1}, \widehat{\delta_2}\} = \widehat{[\delta_1, \delta_2]}$, $\{\widehat{\delta}, e\} = \delta(e)$ and $\{\widehat{\delta}, f\} = X(f)$, $\{e_1, e_2\} = \langle e_1, e_2 \rangle$, and $\{e, f\} = \{f_1, f_2\} = 0$ for all $e, e_1, e_2 \in \Gamma(E)$, $f, f_1, f_2 \in C^\infty(M)$ and $\widehat{\delta}, \widehat{\delta_1}, \widehat{\delta_2} \in \mathfrak{X}^{\langle\cdot\,,\cdot\rangle, l}(E)$. The Poisson [2]-manifold $\mathcal{M}(TE)$ splits as the split Poisson [2]-manifold described in §4.2. It is hence symplectic.

Let N be a smooth manifold. Then T^*N carries the canonical symplectic structure ω_{can} given by $\omega_{\mathrm{can}} = -\mathbf{d}\theta_{\mathrm{can}}$ with $\theta_{\mathrm{can}} \in \Omega^1(T^*N)$ given by $\theta_{\mathrm{can}}(v_{\alpha_n}) = \langle \eta_n, T_{\eta_n} c_N v_{\alpha_n} \rangle$, where $c_N\colon T^*N \to N$ is the canonical projection. Each vector field $X \in \mathfrak{X}(N)$, the derivation $\pounds_X$ of

T^*N defines a linear vector field $\widehat{\mathcal{L}_X} \in \mathfrak{X}^l(T^*N)$. Further, each 1-form $\eta \in \Omega^1(N)$ defines a vertical vector field $\eta^\uparrow \in \mathfrak{X}^c(T^*N)$. It is easy to see that

$$\mathbf{i}_{\widehat{\mathcal{L}_{X_2}}}\mathbf{i}_{\widehat{\mathcal{L}_{X_1}}}\omega_{\rm can} = -\ell_{[X_1,X_2]}, \quad \mathbf{i}_{\eta^\uparrow}\mathbf{i}_{\widehat{\mathcal{L}_X}}\omega_{\rm can} = c_N^*\langle \eta, X\rangle, \quad \mathbf{i}_{\eta_1^\uparrow}\mathbf{i}_{\eta_2^\uparrow}\omega_{\rm can} = 0$$

for $X, X_1, X_2 \in \mathfrak{X}(N)$ and $\eta, \eta_1, \eta_2 \in \Omega^1(N)$. Next we apply this to the smooth manifold E to compute the canonical symplectic form on T^*E.

The involutive double vector bundle that is dual to TE is T^*E with sides E and $E \simeq E^*$ and core T^*M (see Example 3.17). Recall that $\mathcal{C}(TE) \subseteq \mathfrak{X}^l(E)$ is here the set of linear vector fields on E defined by derivations of E that preserve the pairing. We consider $\widehat{\mathcal{L}_{\widehat{\delta}}} \in \mathfrak{X}(T^*E)$ and $\widehat{\mathcal{L}_{e^\uparrow}} \in \mathfrak{X}(T^*E)$ for $\widehat{\delta} \in \mathcal{C}(TE)$ and $e \in \Gamma(E)$. We consider the 1-forms $\mathbf{d}\ell_e, q_e^*\mathbf{d}f \in \Omega^1(E)$ for $e \in \Gamma(E)$ and $f \in C^\infty(M)$, and so the induced vector fields $(\mathbf{d}\ell_e)^\uparrow, (q_E^*\mathbf{d}f)^\uparrow \in \mathfrak{X}(T^*E)$. Recall also that $\widehat{\delta} \in \mathcal{C}(TE)$ and $e^\uparrow \in \mathfrak{X}^c(E)$ define linear functions on T^*E, and that $c_E^*\ell_e$ and $c_E^*q_E^*f \in C^\infty(T^*E)$. The considerations above show that

$$\omega_{\rm can}\left(\widehat{\mathcal{L}_{\widehat{\delta_1}}}, \widehat{\mathcal{L}_{\widehat{\delta_2}}}\right) = \ell_{[\widehat{\delta_2}, \widehat{\delta_1}]} = \left\{\ell_{\widehat{\delta_2}}, \ell_{\widehat{\delta_1}}\right\}, \qquad \omega_{\rm can}\left(\widehat{\mathcal{L}_{\widehat{\delta}}}, \widehat{\mathcal{L}_{e^\uparrow}}\right) = -\ell_{\delta(e)^\uparrow} = \{\ell_{e^\uparrow}, \ell_{\widehat{\delta}}\},$$

$$\omega_{\rm can}\left(\widehat{\mathcal{L}_{\widehat{\delta}}}, (\mathbf{d}\ell_e)^\uparrow\right) = c_E^*\ell_{\delta(e)} = \{\ell_{\widehat{\delta}}, c_E^*\ell_e\}, \ \omega_{\rm can}\left(\widehat{\mathcal{L}_{\widehat{\delta}}}, (q_E^*\mathbf{d}f)^\uparrow\right) = c_E^*q_E^*X(f) = \left\{\ell_{\widehat{\delta}}, c_E^*q_E^*f\right\},$$

$$\omega_{\rm can}\left(\widehat{\mathcal{L}_{e_1^\uparrow}}, \widehat{\mathcal{L}_{e_2^\uparrow}}\right) = 0 = \{\ell_{e_1^\uparrow}, \ell_{e_2^\uparrow}\}, \qquad \omega_{\rm can}\left(\widehat{\mathcal{L}_{e_1^\uparrow}}, (\mathbf{d}\ell_e)^\uparrow\right) = c_E^*q_E^*\langle e_1, e_2\rangle = \left\{\ell_{e_1^\uparrow}, c_E^*\ell_{e_2}\right\},$$

and that $\omega_{\rm can}$ vanish on any other combincation of two vector fields of the four type. This shows that $\omega_{\rm can}^\flat(\widehat{\mathcal{L}_{\widehat{\delta}}}) = \mathbf{d}\ell_{\widehat{\delta}}$, $\omega_{\rm can}^\flat(\widehat{\mathcal{L}_{e^\uparrow}}) = \mathbf{d}\ell_{e^\uparrow}$, $\omega_{\rm can}^\flat((\mathbf{d}\ell_e)^\uparrow) = -\mathbf{d}(c_E^*\ell_e)$ and $\omega_{\rm can}^\flat((q_E^*\mathbf{d}f)^\uparrow) = -\mathbf{d}(c_E^*q_E^*f)$. An easy computation shows then that the Poisson structure constructed in (22) equals the Poisson structure on $C^\infty(T^*E)$ that is induced by $-\omega_{\rm can}$.

Thus, we have found that the equivalence found in Theorem 4.10 restricts to an equivalence of symplectic [2]-manifolds with symplectic cotangent doubles of metric vector bundles (see also [23]).

4.5.2. *The metric double of a VB-algebroid.* Take a VB-algebroid $(D \to A, B \to M)$ with core C and a linear splitting $\Sigma\colon A \times_M B \to D$. Let $(\partial_A\colon C \to A, \nabla^A, \nabla^C, R \in \Omega^2(B, \operatorname{Hom}(A, C)))$ be the 2-representation of the Lie algebroid B that is induced by Σ. Recall from (8) that the splitting Σ induces a splitting $\Sigma^\star\colon B \times_M C^* \to D_B^*$, and from §2.5.1 that $(D_B^* \to C^*, B \to M)$ has an induced VB-algebroid structure given in this splitting by the 2-representation

$$(\partial_A^*\colon A^* \to C^*, \nabla^{A^*}, \nabla^{C^*}, -R^* \in \Omega^2(B, \operatorname{Hom}(C^*, A^*))).$$

The direct sum $D \oplus_B D_B^*$ over B

$$\begin{array}{ccc} D \oplus_B D_B^* & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \oplus C^* & \longrightarrow & M \end{array}$$

has then a VB-algebroid structure $(D \oplus_B D_B^* \to A \oplus C^*, B \to M)$ with core $C \oplus A^*$. It is easy to see that Σ and $\Sigma^\star$ define a linear splitting $\tilde{\Sigma}\colon B \times_M (A \oplus C^*) \to D \oplus_B D_B^*$, $\tilde{\Sigma}(b_m, (a_m, \gamma_m)) = (\Sigma(a_m, b_m), \Sigma^\star(b_m, \gamma_m))$. The induced 2-representation is

$$(\partial_A \oplus \partial_A^*\colon C \oplus A^* \to A \oplus C^*, \nabla^A \oplus \nabla^{C^*}, \nabla^C \oplus \nabla^{A^*}, R \oplus (-R^*)),$$

a self-dual 2-representation of the Lie algebroid B. This gives us a new class of examples of (split) Poisson 2-manifolds induced from ordinary 2-representations or VB-algebroids. Note that the splittings of $D \oplus D_B^*$ obtained as above are not the only Lagrangian splittings, and that the example of $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ discussed in the next example and in [10] is a special case.

4.5.3. *The Pontryagin algebroid over a Lie algebroid.* If A is a Lie algebroid, then $(TA \to TM, A \to M)$ is a VB-algebroid and since $TA^*_A = T^*A$, the double vector bundle T^*A has a VB-algebroid structure $(T^*A \to A^*, A \to M)$ with core T^*M. As a consequence, the direct sum $TA \oplus T^*A$ over A has a VB-algebroid structure $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$. Recall from Example 3.12 that $(TA \oplus T^*A; TM \oplus A^*, A; M)$ has also a natural linear metric, which is given by (17).

Recall from Example 3.12 that linear splittings of $TA \oplus T^*A$ are in bijection with dull brackets on sections of $TM \oplus A^*$, and so also with Dorfman connections $\Delta\colon \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$. We give in [10] the 2-representation $((\rho, \rho^*)\colon A \oplus T^*M \to TM \oplus A^*, \nabla^{\text{bas}}, \nabla^{\text{bas}}, R^{\text{bas}}_\Delta)$ of A that is defined by the VB-algebroid $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ and any such Dorfman connection: The connections $\nabla^{\text{bas}}\colon \Gamma(A) \times \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M)$ and $\nabla^{\text{bas}}\colon \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*)$ are

$$\nabla^{\text{bas}}_a(X, \alpha) = (\rho, \rho^*)(\Omega_{(X,\alpha)}a) + \pounds_a(X, \alpha) \quad \text{and} \quad \nabla^{\text{bas}}_a(b, \theta) = \Omega_{(\rho,\rho^*)(b,\theta)}a + \pounds_a(b, \theta),$$

where $\Omega\colon \Gamma(TM \oplus A^*) \times \Gamma(A) \to \Gamma(A \oplus T^*M)$ is defined by

$$\Omega_{(X,\alpha)}a = \Delta_{(X,\alpha)}(a, 0) - (0, \mathbf{d}\langle \alpha, a\rangle)$$

and for $a \in \Gamma(A)$, the derivations $\pounds_a$ over $\rho(a)$ are defined by:

$$\pounds_a\colon \Gamma(A \oplus T^*M) \to \Gamma(A \oplus T^*M), \quad \pounds_a(b, \theta) = ([a, b], \pounds_{\rho(a)}\theta)$$

and

$$\pounds_a\colon \Gamma(TM \oplus A^*) \to \Gamma(TM \oplus A^*), \quad \pounds_a(X, \alpha) = ([\rho(a), X], \pounds_a\alpha).$$

We prove in [10] that the two connections above are dual to each other if and only if the dull bracket dual to Δ is skew-symmetric. Hence, the two connections are dual to each other if and only if the chosen linear splitting is Lagrangian (see Example 3.12). The basic curvature $R^{\text{bas}}_\Delta\colon \Gamma(A) \times \Gamma(A) \times \Gamma(TM \oplus A^*) \to \Gamma(A \oplus T^*M)$ is given by

$$\begin{aligned} R^{\text{bas}}_\Delta(a, b)(X, \xi) = &- \Omega_{(X,\xi)}[a, b] + \pounds_a\left(\Omega_{(X,\xi)}b\right) - \pounds_b\left(\Omega_{(X,\xi)}a\right) \\ &+ \Omega_{\nabla^{\text{bas}}_b(X,\xi)}a - \Omega_{\nabla^{\text{bas}}_a(X,\xi)}b. \end{aligned}$$

Assume that the linear splitting is Lagrangian. A relatively long but straightforward computation shows that ${R^{\text{bas}}_\Delta}^* = -R^{\text{bas}}_\Delta$, and so that the 2-representation is self-dual. Hence $(TA \oplus T^*A \to TM \oplus A^*, A \to M)$ is a metric VB-algebroid.

Appendix A. Proof of Lemma 2.1

Let $\omega\colon A \to B$ be a vector bundle morphism over a smooth map $\omega_0\colon M \to N$. The morphism ω induces then a vector bundle morphism $\omega^!\colon A \to \omega_0^*B$, $\omega^!(a_m) = (m, \omega(a_m))$ over the identity on M. For a section $b \in \Gamma_V(B)$, we get in a similar manner a section $\omega_0^!b \in \Gamma_{\omega_0^{-1}(V)}(\omega_0^*B)$; defined by $(\omega_0^!b)(m) = (m, b(\omega_0(m)))$ for all $m \in \omega_0^{-1}(V)$.

In order to prove Lemma 2.1 we first check that $\omega^\star$ has the specified codomain, that is, that the image under $\omega^\star$ of a smooth section of B^* is again smooth. Consider the pullback of B under ω_0, i.e. the vector bundle $\omega_0^*B \to M$. We have $(\omega_0^*B)^* \simeq \omega_0^*B^*$ and the smoothness of $\omega^\star(\beta)$ follows from the equality $\omega^\star(\beta) = (\omega^!)^*(\omega_0^!\beta)$: for each $m \in M$, and each $a_m \in A_m$, we have

$$\langle ((\omega^!)^*(\omega_0^!\beta))(m), a_m\rangle = \langle (\omega_0^!\beta)(m), \omega^!(a_m)\rangle = \langle (m, \beta(\omega_0(m))), (m, \omega(a_m))\rangle$$

$$= \langle \beta(\omega_0(m)), \omega(a_m)\rangle = \langle \omega^\star(\beta)(m), a_m\rangle.$$

The map $\omega^\star$ is obviously a morphism of modules over ω_0^*: for $\beta \in \Gamma(B^*)$ and $f \in C^\infty(N)$, we easily find $\omega^\star(f\cdot\beta) = \omega_0^* f \cdot \omega^\star(\beta)$.

Next we need to show that a morphism $\mu^\star\colon \Gamma(B^*) \to \Gamma(A^*)$ of modules over $\mu_0^*\colon C^\infty(N) \to C^\infty(M)$, for $\mu_0\colon M \to N$ smooth, induces a morphism $A \to B$ of vector bundles over $\mu_0\colon M \to N$. Choose a_m in the fiber of A over m and define $\mu(a_m) \in B_{\mu_0(m)}$ by

$$\langle \beta(\mu_0(m)), \mu(a_m)\rangle = \langle \mu^\star(\beta)(m), a_m\rangle$$

for all $\beta \in \Gamma(B^*)$.

The smoothness of μ is seen as follows: let $b_1, \ldots, b_n$ be local basis fields for B and let $\beta_1, \ldots, \beta_n$ be the dual basis fields. Then for each $a_m \in A$, $\mu(a_m)$ can be written $\sum_{i=1}^n \langle \mu(a_m), \beta_i(\mu_0(m))\rangle b_i(\mu_0(m))$. Since each $\langle \mu(a_m), \beta_i(\mu_0(m))\rangle$ equals $\ell_{\mu^\star(\beta_i)}(a_m)$, we find that locally, $\mu = \sum_{i=1}^n \ell_{\mu^\star(\beta_i)} \cdot (b_i \circ \mu_0 \circ q_A)$. To prove that μ is a vector bundle morphism, we need to check that $\langle \beta(\mu_0(m)), \mu(a_m)\rangle$ only depends on the value of β at $\mu_0(m)$, or in other words, that if β vanishes at $\mu_0(m)$, then $\langle \beta(\mu_0(m)), \mu(a_m)\rangle = 0$. Without loss of generality, assume that β can be written as $f\cdot\beta'$ with $\beta' \in \Gamma(B^*)$ and $f \in C^\infty(N)$ with $f(\mu_0(m)) = 0$. Then $\langle \beta(\mu_0(m)), \mu(a_m)\rangle = \langle f(\mu_0(m))\mu^\star(\beta')(m), a_m\rangle = 0$. The morphism μ of vector bundles clearly induces $\mu^\star$ on the sets of sections of the duals, and vice-versa.

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