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THE LEAF SPACE OF A MULTIPLICATIVE FOLIATION

M. Jotz

Section de Mathématiques
Ecole Polytechnique Fédérale de Lausanne
CH-1015 Lausanne, Switzerland

Dedicated to Tudor S. Ratiu for his 60th birthday.

Abstract. We show that if a smooth multiplicative subbundle $S \subseteq TG$ on a groupoid $G \rightrightarrows P$ is involutive and satisfies completeness conditions, then its leaf space G/S inherits a groupoid structure over the space of leaves of $TP \cap S$ in P.

As an application, a special class of Dirac groupoids is shown to project by forward Dirac maps to Poisson groupoids.

Contents

1. **Introduction.** Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$ and multiplication map $\mathsf{m} : G \times G \to G$. Then the tangent space TG of G is also a Lie group with unit $0_e \in \mathfrak{g}$ and multiplication map $T\mathsf{m} : TG \times TG \to TG$. A multiplicative distribution $S \subseteq TG$ is a distribution on G, i.e., $S(g) := S \cap T_gG$ is a vector subspace of T_gG for all $g \in G$, that is in addition a subgroup of TG. Since at each $g \in G$, $S(g)$ is a vector subspace of T_gG, the zero section of TG is contained in S. Thus, using $T_{(g,h)}\mathsf{m}(0_g, v_h) = T_hL_gv_h$ for any $g, h \in G$ and $v_h \in T_hG$, where $L_g : G \to G$ is the left translation by g, we find that the distribution S is left invariant. It follows that S is a smooth left invariant subbundle of TG defined by $S(g) = \mathfrak{s}^L(g)$, where $\mathfrak{s}$ the vector subspace $S(e) = S \cap \mathfrak{g}$ of $\mathfrak{g}$. In the same manner, S is right invariant and we find thus that $\mathfrak{s}$ is invariant under the adjoint action of G on $\mathfrak{g}$. Hence, $\mathfrak{s}$ is an ideal in $\mathfrak{g}$ and the subbundle $S \subseteq TG$ is completely integrable in the sense of Frobenius. Its leaf N through the unit element e of G is a normal subgroup of G and since the leaf space G/S of S is equal to G/N, it inherits a group structure from G such that the projection $G \to G/N$ is a homomorphism of groups. If N is closed in G, the

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foliation by S is simple and the leaf space $G/S = G/N$ is a Lie group such that the projection is a smooth surjective submersion. Similar discussions leading to this result can be found in [18, 9].

If $G \rightrightarrows P$ is a Lie groupoid, its tangent space TG is, in the same manner as in the Lie group case, a Lie groupoid over the tangent space TP of the units. A distribution $S \subseteq TG$ on G is multiplicative if it is a subgroupoid of $TG \rightrightarrows TP$. The problem studied in this paper is hence natural: we ask how the simple fact stated above about the leaf space of a multiplicative distribution on a Lie group generalizes to Lie groupoids. We consider smooth multiplicative and involutive *subbundles* $S \subseteq TG$ (that is, distributions of constant rank). The foliation of G by the leaves of S is then said to be *multiplicative* and the intersection $TP \cap S$ is automatically an involutive subbundle of TP. We show that the leaf space G/S inherits the source, target, unit inclusion and inversion maps of a groupoid over the space of leaves in P of $TP \cap S$, that are all compatible with the projections $G \to G/S$ and $P \to P/(TP \cap S)$. To be able to define a multiplication on the leaf space, we have to assume that a special family of vector fields spanning S on the Lie groupoid is complete in a sense that will be explained, and that a compatibility condition on the intersections of the leaves with the s-fibers is satisfied.

If the foliations S and $S \cap TP$ are simple, we get hence, under these conditions, the structure of a *Lie* groupoid on the leaf space $G/S \rightrightarrows P/(S \cap TP)$, such that the projection is a Lie groupoid morphism.

Using a partial $S \cap TP$-connection on $AG/(AG \cap S)$ that is naturally induced by the multiplicative, involutive subbundle $S \subseteq TG$ [6], we show that $\Gamma(AG)$ is spanned as a $C^\infty(P)$-module by sections which left invariant images leave S invariant. We show that under (rather strong) conditions on these special left invariant vector fields that leave the foliation invariant, the completeness condition on S is not necessary and the compatibility condition that is necessary for the multiplication is satisfied.

We illustrate the theory by four examples. The first two examples (Examples 3.2 and 3.3) illustrate the result of the main theorem, and the third example (Example 3.16) shows that we recover as a special case the quotient of a Lie groupoid by a Lie group action by groupoid morphisms. This example suggests that our main theorem (Theorem 3.14) could be useful for discrete mechanics on groupoids (see Remark 3.17). The fourth example (Example 3.19, which is taken from [27]) is an example of a multiplicative foliation on a Lie groupoid that doesn't satisfy the necessary condition and which space of leaves fails hence to inherit a groupoid multiplication.

As an application, we show that if $(G \rightrightarrows P, \mathsf{D}_G)$ is a Dirac groupoid such that the characteristic distribution $TG \cap \mathsf{D}_G$ has constant rank and its leaf space is simple and satisfies the required conditions, then the quotient Lie groupoid inherits the structure of a Poisson groupoid such that the quotient map is a forward Dirac map and a morphism of Lie groupoids. Again, this is a fact that is automatically true for an integrable Dirac Lie group if the leaf N through the unit e of the characteristic foliation is closed in the group ([9]).

Notations and conventions. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle of M, respectively. For an arbitrary vector bundle $\mathsf{E} \to M$, the space of (local) sections of E will be written $\Gamma(\mathsf{E})$. We will write $\mathrm{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(\mathsf{E})$ is defined.

A distribution F on a smooth manifold M is a subset $F \subseteq TM$ such that $F(m) := F \cap T_mM$ is a vector subspace of T_mM for all $m \in M$. The distribution is smooth if for all $v_m \in F(m)$, there exists a smooth vector field $X \in \mathfrak{X}(M)$ with values in F such that $X(m) = v_m$. Note that in general, the rank of $F(m)$ depends on m. If not, then F is simply a subbundle of TM.

The *pullback* or *restriction* of a distribution $F \subseteq TM$ to an embedded submanifold N of M will be written $F|_N$. We say that the distribution F has constant rank on N if the pullback $F|_N$ is a vector bundle over N, i.e., if all its fibers have the same dimension.

Assume that F is an involutive *subbundle* of TM, hence completely integrable in the sense of Frobenius. We say that F, or the foliation defined by F, is *simple* if the leaf space M/F of the foliation is a smooth manifold such that the projection $M \to M/F$ is a smooth surjective submersion.

We write $\mathsf{E} \times_M \mathsf{F}$ for the direct sum of two vector bundles $\mathsf{E} \to M$ and $\mathsf{F} \to M$. The *Pontryagin bundle* of M is the direct sum $TM \times_M T^*M \to M$. [1]

Assume finally that M, N are smooth manifolds and $f : M \to N$ is a smooth surjective submersion. The tangent space to the f-fibers will be written $T^fM = \ker(Tf : TM \to TN) \subseteq TM$.

2. The Pontryagin groupoid of a Lie groupoid.

2. The Pontryagin groupoid of a Lie groupoid. The general theory of Lie groupoids and their Lie algebroids can be found in [14], [15]. We fix here first of all some notations and conventions.

A groupoid G with base P will be written $G \rightrightarrows P$. The set P will be considered most of the time as a subset of G, that is, the unity 1_p will be identified with p for all $p \in P$. A *Lie groupoid* is a groupoid G on base P together with the structures of *smooth Hausdorff manifolds* on G and P such that the maps $\mathsf{s}, \mathsf{t} : G \to P$ are *surjective submersions*, and such that the object inclusion map $\epsilon : P \hookrightarrow G$, $p \mapsto 1_p$ and the partial multiplication $\mathsf{m} : G \times_P G \to G$ are all smooth.

Let $g \in G$, then the *right translation by* g is

$$R_g : \mathsf{s}^{-1}(\mathsf{t}(g)) \to \mathsf{s}^{-1}(\mathsf{s}(g)), \qquad h \mapsto R_g(h) = h \star g.$$

The *left translation* L_g is defined in an analogous manner. Given a (local) bisection K of $G \rightrightarrows P$, we write also R_K for the right-translation.

In this paper, the Lie algebroid of the Lie groupoid $G \rightrightarrows P$ is $AG := T^{\mathsf{t}}_P G$, equipped with the anchor map $T\mathsf{s}|_{AG}$ and the Lie bracket defined by the left invariant vector fields.

We give in the following the induced Lie groupoid structures on the tangent, cotangent and Pontryagin bundle of a Lie groupoid.

The tangent prolongation of a Lie groupoid. Let $G \rightrightarrows P$ be a Lie groupoid. Applying the tangent functor to each of the maps defining G yields a Lie groupoid structure on TG with base TP, source $T\mathsf{s}$, target $T\mathsf{t}$ and multiplication $T\mathsf{m} : T(G \times_P G) \to TG$. The identity at $v_p \in T_pP$ is $1_{v_p} = T_p\epsilon v_p$. This defines the *tangent prolongation* $TG \rightrightarrows TP$ *of* $G \rightrightarrows P$ or the *tangent groupoid associated to* $G \rightrightarrows P$.

[1] Note that this vector bundle is also sometimes called the *generalized tangent bundle* in the literature.

The cotangent Lie groupoid defined by a Lie groupoid. If $G \rightrightarrows P$ is a Lie groupoid, then there is also an induced Lie groupoid structure on $T^*G \rightrightarrows A^*G = (TP)^\circ$. The source map $\hat{\mathsf{s}} : T^*G \to A^*G$ is given by

$$\hat{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}G \text{ for } \alpha_g \in T^*_gG, \qquad \hat{\mathsf{s}}(\alpha_g)(u_{\mathsf{s}(g)}) = \alpha_g(T_{\mathsf{s}(g)}L_g u_{\mathsf{s}(g)})$$

for all $u_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)}G$, and the target map $\hat{\mathsf{t}} : T^*G \to A^*G$ is given by

$$\hat{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}G, \qquad \hat{\mathsf{t}}(\alpha_g)(u_{\mathsf{t}(g)}) = \alpha_g\big(T_{\mathsf{t}(g)}R_g(u_{\mathsf{t}(g)} - T_{\mathsf{t}(g)}\mathsf{s}u_{\mathsf{t}(g)})\big)$$

for all $u_{\mathsf{t}(g)} \in A_{\mathsf{t}(g)}G$. If $\hat{\mathsf{s}}(\alpha_g) = \hat{\mathsf{t}}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by

$$(\alpha_g \star \alpha_h)(v_g \star v_h) = \alpha_g(v_g) + \alpha_h(v_h)$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(G \times_P G)$. This Lie groupoid structure was introduced in [1] and is explained for instance in [1], [20] and [14].

The "Pontryagin groupoid" of a Lie groupoid. If $G \rightrightarrows P$ is a Lie groupoid, there is hence an induced Lie groupoid structure on $\mathsf{P}_G = TG \times_G T^*G$ over $TP \times_P A^*G$. We will write $\mathbb{T}\mathsf{t}$ for the target map

$$\begin{array}{rcl} \mathbb{T}\mathsf{t} : TG \times_G T^*G & \to & TP \times_P A^*G \\ (v_g, \alpha_g) & \mapsto & \big(T\mathsf{t}(v_g), \hat{\mathsf{t}}(\alpha_g)\big) \end{array},$$

$\mathbb{T}\mathsf{s}$ for the source map

$$\mathbb{T}\mathsf{s} : TG \times_G T^*G \to TP \times_P A^*G$$

and $\mathbb{T}\epsilon$, $\mathbb{T}\mathsf{i}$, $\mathbb{T}\mathsf{m}$ for the embedding of the units, the inversion map and the multiplication of this Lie groupoid.

3. Multiplicative subbundles of the tangent space TG.

We start with the definition of a multiplicative distribution on a Lie groupoid $G \rightrightarrows P$. Later, we will mostly be interested in multiplicative *subbundles* of the tangent bundle TG. Yet, as we will see in Examples 3.2 and 3.3, our main result can already be valid in the more general setting of smooth integrable multiplicative distributions. Since the discussion about the Lie group case in the introduction also starts from (even non necessarily smooth) distributions, we prefer to give the general definition here.

Definition 3.1. Let $G \rightrightarrows P$ be a Lie groupoid and $TG \rightrightarrows TP$ its tangent prolongation. A distribution $S \subseteq TG$ is *multiplicative* if S is a (set) subgroupoid of $TG \rightrightarrows TP$.

Note that this means in particular that $T_g\mathsf{s}(v_g) \in S(\mathsf{s}(g))$ and $T_g\mathsf{t}(v_g) \in S(\mathsf{t}(g))$ for all $g \in G$ and $v_g \in S(g)$.

3.1. Examples.

The two examples in this subsection illustrate the general theory in the following sections. We study two completely integrable, multiplicative smooth distributions on special classes of Lie groupoids and show that their spaces of leaves inherit groupoid structures. In the general theory, we will have to assume that the distributions have constant rank, that the elements of a special family of vector fields spanning S are *complete*, and that a compatibility condition on the leaves and the multiplication is satisfied.

We will see in both examples that the leaf spaces inherit groupoid structures, although the studied distributions are here not necessarily *subbundles* of the tangent space. This shows that our main theorem can also hold for leaf spaces of singular distributions.

Example 3.2. Let M be a smooth manifold and $M \times M \rightrightarrows M$ the pair groupoid associated to it. If F is a smooth distribution on M, then $S := F \times F \subseteq TM \times TM \simeq T(M \times M)$ is a smooth multiplicative distribution in $TM \times TM \rightrightarrows TM$. Its intersection with $T\Delta_M$ is Δ_F, where $\Delta_F(m,m) = \{(v_m, v_m) \mid v_m \in F(m)\}$, which is also a smooth distribution on Δ_M.

If F is completely integrable in the sense of Stefan and Sussmann, then S and Δ_F are also completely integrable. Let $\mathrm{pr}_S : M \times M \to (M \times M)/S$ and $\mathrm{pr}_F : M \to M/F$ be the quotient maps. If L_m, respectively L_n is the leaf of F through m, respectively n, then the leaf of S through $(m,n) \in M \times M$ is $L_{(m,n)} = L_m \times L_n$. Thus, the space $(M \times M)/S$ of leaves of S coincides with $M/F \times M/F$ via the map $\Phi : (M \times M)/S \to M/F \times M/F$, $\mathrm{pr}_S(m,n) \mapsto (\mathrm{pr}_F(m), \mathrm{pr}_F(n))$.

Since the following diagram commutes,

$$\begin{array}{ccc} M \times M & & \\ {\scriptstyle \mathrm{pr}_S}\downarrow & \searrow^{\mathrm{pr}_F \times \mathrm{pr}_F} & \\ (M \times M)/S & \xrightarrow[\Phi]{} & M/F \times M/F \end{array}$$

it is easy to check that Φ is a homeomorphism.

The groupoid structure on $(M \times M)/S \simeq M/F \times M/F$ is just the pair groupoid structure $M/F \times M/F \rightrightarrows M/F$ and $(\mathrm{pr}_S, \mathrm{pr}_F)$ is a groupoid morphism. That is, the following diagram commutes

$$\begin{array}{ccc} M \times M & \xrightarrow{\mathrm{pr}_S} & (M \times M)/S \\ {\scriptstyle \mathsf{t}}\downarrow\downarrow{\scriptstyle \mathsf{s}} & & {\scriptstyle \mathsf{t}}\downarrow\downarrow{\scriptstyle \mathsf{s}} \\ M & \xrightarrow[\mathrm{pr}_F]{} & M/F \end{array}$$

and we have $\mathrm{pr}_S((m,n) \star (n,p)) = \mathrm{pr}_S(m,n) \star \mathrm{pr}_S(n,p)$ for all $m,n,p \in M$.

Note that in general, $(M \times M)/S \to M/F$ is not a *Lie* groupoid since the quotients $(M \times M)/S$ and M/F are not necessarily smooth manifolds. If F is an involutive subbundle of TM such that the leaf space M/F is a smooth manifold and $\mathrm{pr}_F : M \to M/F$ a smooth surjective submersion, then the induced groupoid $(M \times M)/S \rightrightarrows M/F$ is a Lie groupoid.

Example 3.3. Let M be a smooth manifold and $\mathsf{p} : \mathbb{R}^k \times M \to M$ the trivial vector bundle over M. Then $\mathbb{R}^k \times M \rightrightarrows M$ has the structure of a Lie groupoid over the base M.

We identify in the following $T(\mathbb{R}^k \times M)$ with $\mathbb{R}^k \times \mathbb{R}^k \times TM$, and we write $T_{(x,m)}(\mathbb{R}^k \times M) = \{x\} \times \mathbb{R}^k \times T_m M$. The source and target maps in $T(\mathbb{R}^k \times M) \rightrightarrows TM$ are then $T\mathsf{s}(x, v, v_m) = T\mathsf{t}(x, v, v_m) = v_m$ and the partial multiplication is given by $(x, v, v_m) \star (y, w, v_m) = (x + y, v + w, v_m)$. Hence, the groupoid $T(\mathbb{R}^k \times M) \rightrightarrows TM$ is the vector bundle groupoid $\mathbb{R}^{2k} \times TM \rightrightarrows TM$.

Consider a completely integrable, smooth distribution F on M and a vector subspace $W \subseteq \mathbb{R}^k$. It is easy to see that $S := \mathbb{R}^k \times W \times F$ is an integrable multiplicative distribution in $T(\mathbb{R}^k \times M)$.

The leaf of S through (x,m) is the set $(x + W) \times L_m$, where L_m is the leaf of F through m. The intersection $TM \cap S$ equals F and we have a (singular) foliation M/F. It is easy to see that the quotient $(\mathbb{R}^k \times M)/S \simeq (\mathbb{R}^k/W) \times (M/F)$ inherits the structure of a groupoid over M/F such that the diagram

$$\begin{array}{ccc} \mathbb{R}^k \times M & \xrightarrow{\mathrm{pr}_S} & (\mathbb{R}^k/W) \times (M/F) \\ \mathsf{t}\downarrow\downarrow\mathsf{s} & & \mathsf{t}\downarrow\downarrow\mathsf{s} \\ M & \xrightarrow[\mathrm{pr}_F]{} & M/F \end{array}$$

commutes and $(\mathrm{pr}_S, \mathrm{pr}_F)$ is a morphism of groupoids. Here also, we get a *Lie* groupoid if and only if F is a *simple involutive subbundle* of TM, i.e., its leaf space is a smooth manifold and the projection is a smooth surjective submersion.

Remark 3.4. Note that in Example 3.2, the quotient groupoid $(M\times M)/S \rightrightarrows M/F$ is exactly the quotient of $M \times M \rightrightarrows M$ by the kernel of $(\mathrm{pr}_S, \mathrm{pr}_F)$, i.e., the normal subgroupoid

$$N := \ker(\mathrm{pr}_S, \mathrm{pr}_F) = \cup_{m\in M}(L_m \times L_m).$$

In Example 3.3, the groupoid structure on the leaf space does not coincide with the quotient by the kernel N but rather with the quotient by the *normal subgroupoid system* $(N, R(\mathrm{pr}_F), \theta)$, where

$$N = \ker(\mathrm{pr}_S, \mathrm{pr}_F) = \cup_{m\in M} L_{(0,m)} = W \times M,$$

$$R(\mathrm{pr}_F) = \{(m, n) \in M \times M \mid m \sim_F n\}$$

are the subgroupoids of $\mathbb{R}^k \times M \rightrightarrows M$ and of $M \times M \rightrightarrows M$, respectively, and θ is the action of $R(\mathrm{pr}_F)$ on $G/N := (\mathbb{R}^k/W) \times M \to M$ given by $\theta(m, n)(x + W, n) = (x + W, m)$ for all $x \in \mathbb{R}^k$ and $(m, n) \in R(\mathrm{pr}_F)$. These two quotients are explained in more details in [8].

This illustrates hence the different definitions of normal subgroupoids objects in [12] and [14], and the fact that the notion of *normal subgroupoid systems of groupoids* is needed to generalize to groupoids the relation between the kernel of surjective homomorphisms and normal subgroups of a group (see [14] for more details).

3.2. **Properties of multiplicative subbundles of TG.** In the following, the subbundle $S \subseteq TG$ is *always assumed to be smooth*, even if this is not stated explicitely.

Lemma 3.5. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative* subbundle. *Then the intersection $S \cap TP$ has constant rank on P. Since this is the set of units of S viewed as a subgroupoid of TG, the pair $S \rightrightarrows (S \cap TP)$ is a Lie groupoid.*

The bundle $S|_P$ splitts as $S|_P = (S\cap TP)\oplus(S\cap AG)$. Furthermore, if we denote by S^{t} the intersection $S\cap T^{\mathsf{t}}G$ of vector bundles over G, we have $S^{\mathsf{t}}(g) = 0_g \star S^{\mathsf{t}}(\mathsf{s}(g)) = T_{\mathsf{s}(g)}L_g\left(S^{\mathsf{t}}(\mathsf{s}(g))\right)$ for all $g \in G$. In the same manner, $S^{\mathsf{s}}(g) = S^{\mathsf{s}}(\mathsf{t}(g)) \star 0_g$ for all $g \in G$.

As a consequence, the intersections $S \cap T^{\mathsf{t}}G$ and $S \cap T^{\mathsf{s}}G$ have constant rank on G.

Proof. We start by showing that the intersection $S|_P \cap TP$ is smooth. If $p \in P$ and $v_p \in S(p) \cap T_pP$, then we find a smooth section X of S defined at p such that $X(p) = v_p$. The restriction of X to $\mathrm{Dom}(X) \cap P$ is then a smooth section of $S|_P$, and, since s is a smooth surjective submersion, and S is a subgroupoid of TG, the image $T\mathsf{s}(X|_S)$ is a smooth section of $S|_P \cap TP$. Futhermore, we have $T\mathsf{s}(X(p)) = T\mathsf{s}(v_p) = v_p$ since $v_p \in T_pP$.

Since the intersection $S \cap TP$ is a smooth intersection of vector bundles over P, we know (for instance by Proposition 4.4 in [10]) that it is a vector bundle on P. In particular, it is the set of units of S viewed as a subgroupoid of TG.

Since S is a vector bundle on G, it has constant rank on G and in particular on P. For each $p \in P$, we can write $S(p) = (S(p) \cap T_pP) \times (S(p) \cap T_p^{\mathsf{t}}G) = (S(p) \cap T_pP) \times (S(p) \cap T_p^{\mathsf{s}}G)$. Indeed, if $v_p \in S(p)$, then we can write $v_p = T_p\mathsf{t}v_p + (v_p - T_p\mathsf{t}v_p) = T_p\mathsf{s}v_p + (v_p - T_p\mathsf{s}v_p)$.

From this follows the fact that $S|_P \cap T_P^{\mathsf{t}}G$ and $S|_P \cap T_P^{\mathsf{s}}G$ have constant rank on P. If $v_g \in S^{\mathsf{t}}(g)$, then we have $T\mathsf{s}(0_{g^{-1}}) = 0_{\mathsf{t}(g)} = T\mathsf{t}(v_g)$, and since $S(g^{-1})$ is a vector subspace of $T_{g^{-1}}G$, we have $0_{g^{-1}} \in S(g^{-1})$. Since S is multiplicative, we find hence $0_{g^{-1}} \star v_g \in S(\mathsf{s}(g))$. The equality $0_{g^{-1}} \star v_g = T_gL_{g^{-1}}v_g$ is easy to check. We show the other inclusion and the equality $S^{\mathsf{s}}(g) = T_{\mathsf{t}(g)}R_g\left(S^{\mathsf{s}}(\mathsf{t}(g))\right)$ in the same manner.

Since $S|_P \cap AG$ and $S|_P \cap T_P^{\mathsf{s}}G$ have constant rank on P, we get from this that $S \cap T^{\mathsf{t}}G$ and $S \cap T^{\mathsf{s}}G$ have constant rank on G. □

Corollary 3.6. *Let $G \rightrightarrows P$ be a Lie groupoid and S a multiplicative* subbundle *of TG. The induced maps $T_g\mathsf{s} : S(g) \to S(\mathsf{s}(g)) \cap T_{\mathsf{s}(g)}P$ and $T_g\mathsf{t} : S(g) \to S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$ are surjective for each $g \in G$.*

Proof. The map $T\mathsf{s} : S/(S \cap T^{\mathsf{s}}G) \to S \cap TP$ is a well-defined injective vector bundle homomorphism over $\mathsf{s} : G \to P$. Since

$$\begin{aligned}\operatorname{rank}(S/(S \cap T^{\mathsf{s}}G)) &= \dim((S/(S \cap T^{\mathsf{s}}G))(g)) = \dim(S(g)) - \dim(S(g) \cap T_g^{\mathsf{s}}G)\\ &= \dim(S(\mathsf{t}(g))) - \dim(S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}^{\mathsf{s}}G)\\ &= \dim(S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P) = \operatorname{rank}(S \cap TP)\end{aligned}$$

for any $g \in G$, both vector bundles have the same rank, and the map is an isomorphism in every fiber. Thus, the claim follows. □

The following corollary (see also a result in [13] about *star-sections*) will be used often in the following.

Corollary 3.7. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative subbundle. Let $\bar{X}$ be a section of $S \cap TP$ defined on $\operatorname{Dom}(\bar{X}) =: \bar{U} \subseteq P$. Then there exist sections $X, Y \in \Gamma(S)$ defined on $U := \mathsf{s}^{-1}(\bar{U})$, respectively $V = \mathsf{t}^{-1}(\bar{U})$ such that $X \sim_{\mathsf{s}} \bar{X}$ and $Y \sim_{\mathsf{t}} \bar{X}$.*

Proof. Since the induced map $T\mathsf{s} : S/(S \cap T^{\mathsf{s}}G) \to S \cap TP$ is a smooth isomorphism in every fiber, there exists a unique smooth section σ of $S/(S \cap T^{\mathsf{s}}G)$ defined on $\mathsf{s}^{-1}(\bar{U})$ such that $\mathsf{s}(\sigma(g)) = \bar{X}(\mathsf{s}(g))$ for all $g \in U$. Choose a representative $X \in \Gamma(S)$ for σ, then we have $T_g\mathsf{s}X(g) = \bar{X}(\mathsf{s}(g))$ for all $g \in U$. □

We say that a vector field $X \in \mathfrak{X}(G)$ is t- (respectively s-) descending if there exists $\bar{X} \in \mathfrak{X}(P)$ such that $X \sim_{\mathsf{t}} \bar{X}$ (respectively $X \sim_{\mathsf{s}} \bar{X}$), that is, for all $g \in \operatorname{Dom}(X)$, we have $T_g\mathsf{t}X(g) = \bar{X}(\mathsf{t}(g))$.

Corollary 3.8. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative subbundle. Then*

1. *S is spanned by local t-descending sections and*
2. *S is spanned by local s-descending sections.*

Proof. Choose $g \in G$ and smooth sections $\bar{X}_1, \ldots, \bar{X}_k$ of $S \cap TP$ spanning $S \cap TP$ on a neighborhood U_1 of $\mathsf{t}(g)$. Choose also $Y_1, \ldots, Y_m$ spanning $(S \cap T^{\mathsf{t}}G)|_P$ in a neighborhood U_2 of $\mathsf{s}(g)$. The vector fields $Y_1^l, \ldots, Y_m^l$ span $S \cap T^{\mathsf{t}}G$ on the neighborhood $\mathsf{s}^{-1}(U_2)$ of g and we find smooth t-descending sections $X_1, \ldots, X_k$ of S such that $X_i \sim_{\mathsf{t}} \bar{X}_i$ on $\mathsf{t}^{-1}(U_1)$. The sections $Y_1^l, \ldots, Y_m^l, X_1, \ldots, X_k$ are t-descending and span S on the neighborhood $U := \mathsf{s}^{-1}(U_2) \cap \mathsf{t}^{-1}(U_1)$ of g. □

Let M be a smooth manifold and $F \subseteq TM$ a subbundle spanned by a family $\mathcal{F}$ of vector fields. If F is involutive, it is integrable in the sense of Frobenius and each of its leaves is an *accessible set* of $\mathcal{F}$, i.e., the leaf L_m of F through $m \in M$ is the set

$$L_m = \left\{ \phi_{t_1}^{X_1} \circ \ldots \circ \phi_{t_k}^{X_k}(m) \,\middle|\, \begin{array}{c} k \in \mathbb{N}, X_1, \ldots, X_k \in \mathcal{F}, t_1, \ldots, t_k \in \mathbb{R} \\ \text{and } \phi^{X_i} \text{ is a local flow of } X_i \end{array} \right\}$$

(see [17], [21], [23], [22]). If the multiplicative subbundle $S \subseteq TG$ is involutive, we get the following corollary which will be very useful in the next section.

Corollary 3.9. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then S is completely integrable in the sense of Frobenius and its leaves are the accessible sets of each of the two following families of vector fields.*

$$\mathcal{F}_S^{\mathsf{t}} = \{X \in \Gamma(S) \mid \exists \bar{X} \in \mathfrak{X}(P) \text{ such that } X \sim_{\mathsf{t}} \bar{X}\} \tag{1}$$

$$\mathcal{F}_S^{\mathsf{s}} = \{X \in \Gamma(S) \mid \exists \bar{X} \in \mathfrak{X}(P) \text{ such that } X \sim_{\mathsf{s}} \bar{X}\}. \tag{2}$$

By Corollary 3.7, there exists for each section $\bar{X}$ of $TP \cap S$ defined on $U \subseteq P$ a section of S that is defined on $\mathsf{t}^{-1}(U)$ and t-related to $\bar{X}$. We can find a family of spanning sections of $S \cap TP$ that are all complete, but the corresponding sections of S are then not necessarily complete. For the proof of our main theorem, we will have to assume that $S \cap TP$ is spanned by the following families of vector fields:

$$\bar{\mathcal{F}}_S^{\mathsf{t}} := \left\{ \bar{X} \in \Gamma(S \cap TP) \,\middle|\, \begin{array}{c} \exists X \in \Gamma(S) \text{ such that } X \sim_{\mathsf{t}} \bar{X} \\ \text{and } X, \bar{X} \text{ are complete,} \\ \operatorname{Dom}(X) = \mathsf{t}^{-1}(\operatorname{Dom}(\bar{X})) \end{array} \right\} \tag{3}$$

$$\bar{\mathcal{F}}_S^{\mathsf{s}} := \left\{ \bar{X} \in \Gamma(S \cap TP) \,\middle|\, \begin{array}{c} \exists X \in \Gamma(S) \text{ such that } X \sim_{\mathsf{s}} \bar{X} \\ \text{and } X, \bar{X} \text{ are complete,} \\ \operatorname{Dom}(X) = \mathsf{s}^{-1}(\operatorname{Dom}(\bar{X})) \end{array} \right\}. \tag{4}$$

Note that there exists a complete family $\bar{\mathcal{F}}_S^{\mathsf{t}}$ if and only if there exists a complete family $\bar{\mathcal{F}}_S^{\mathsf{s}}$, since the vector fields in $\mathcal{F}_S^{\mathsf{t}}$ are the inverses of the vector fields in $\mathcal{F}_S^{\mathsf{s}}$ and vice versa. We say that S is *complete* if it has this property.

For instance, if $G \rightrightarrows P$ is such that the target map t (and equivalently the source map s) is proper, then it is easy to see that any multiplicative subbundle $S \subseteq TG$ is complete. Also, note that the multiplicative distributions in Examples 3.2 and 3.3 are complete.

We study now the left invariant vector fields that leave the multiplicative subbundle S invariant. The following lemma is a result that is shown in [6].

Lemma 3.10. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then the Bott connection, i.e., the partial S-connection on TG/S, induces a flat partial $S \cap TP$-connection ∇ on $AG/(S \cap AG) \to P$ with the following property. If $a \in \Gamma(AG)$ is such that its class in $\Gamma(AG/(S \cap AG))$ is ∇-parallel, then the left invariant vector field a^l satisfies*

$$\left[a^l, \Gamma(S)\right] \subseteq \Gamma(S). \tag{5}$$

Note that (5) is automatically satisfied for any $a \in \Gamma(AG\cap S)$ since S is involutive and a^l is a section of $S \cap T^{\mathsf{t}}G$ (see Lemma 3.5). We can hence say that a section $a \in \Gamma(AG)$ is ∇-parallel if its class $\bar{a}$ in $\Gamma(AG/(AG \cap S))$ is ∇-parallel.

Note also that (5) implies that the flow of a^l leaves the subbundle $S \subseteq TG$ invariant. This fact is well-known, see for instance the appendix of [6] for a proof of it.

Corollary 3.11. *Let $G\rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then AG is spanned by the following family of (local) sections*

$$\mathcal{A}^S := \left\{ a \in \Gamma(AG) \,\middle|\, \left[a^l, \Gamma(S)\right] \subseteq \Gamma(S) \right\}.$$

Proof. We show that there exists for each point $p \in P$ a local frame of ∇-parallel sections for AG defined on a neighborhood of p, where ∇ is the flat partial $S\cap TP$-connection as in the preceding lemma. Let $m := \dim(P)$ and $r := \operatorname{rank}(AG)$. Choose a foliated chart domain U centered at p and described by coordinates $(x^1, \ldots, x^m)$ such that the first k among them define the local integral submanifold of $TP \cap S$ containing p. Let $\Sigma \subseteq U$ be the slice $\phi^{-1}(\{0\} \times \mathbb{R}^{m-k})$, where $\phi : U \to \mathbb{R}^m$ is the chart adapted to the foliation.

We denote by l the rank of $AG/(AG \cap S)$. Choose $a_1, \ldots, a_l \in \Gamma(AG)$ such that $\bar{a}_1, \ldots, \bar{a}_l \in \Gamma(AG/(AG \cap S))$ is a basis frame for $AG/(AG \cap S)$ on U. We consider this frame at points of $\Sigma \cap U$ and construct $\bar{\alpha}_1, \ldots, \bar{\alpha}_l \in \Gamma(AG/(AG \cap S))$ as follows. If $q \in U$, $\phi(q) = (x_1, \ldots, x_m)$, then we find a path $c : [0,1] \to \phi^{-1}(\mathbb{R}^k \times \{(x_{k+1}, \ldots, x_n)\})$ (the leaf of $TP \cap S$ through q) with $c(1) = q$ and $c(0) = q' \in \Sigma$ satisfying $\phi(q') = (0, \ldots, 0, x_{k+1}, \ldots, x_n)$. Define $\bar{\alpha}_i(q) := P_c^1(\bar{a}_i(q'))$, where $P_c^1(\bar{a}_i(q'))$ is the parallel translate of $\bar{a}_i(q')$ along c at time 1 by the $TP \cap S$-partial connection ∇. Since U is simply connected and ∇ is flat, parallel translation is independent of the chosen path (see, for example, [4]), hence the sections $\bar{\alpha}_i$ are ∇-parallel sections of $AG/(AG \cap S)$ and form a pointwise basis of $AG/(AG \cap S)$ on U. Choose representatives $\alpha_1, \ldots, \alpha_l \in \Gamma(AG)$ for $\bar{\alpha}_1, \ldots, \bar{\alpha}_l$. Take $\alpha_{l+1}, \ldots, \alpha_r$ to be a frame of $AG \cap S$ over U. Then $\alpha_1, \ldots, \alpha_r$ is a frame of AG over U composed of ∇-parallel sections. □

3.3. **The leaf space of an involutive multiplicative subbundle of TG.** Let $G\rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then S is completely integrable in the sense of Frobenius. Let $\operatorname{pr} : G \to G/S$ be the projection to the space of leaves of S.

It is easy to check that the intersection $S \cap TP$ is an involutive subbundle of TP and hence itself also completely integrable. Let P/S be *the space of leaves of* $S \cap TP$ in P and $\operatorname{pr}_\circ$ the projection $\operatorname{pr}_\circ : P \to P/S$. Note that by a theorem in [7], S is involutive if and only if $S \cap TP$ is involutive and $S \cap AG$ is a subalgebroid of AG. Let P/S be *the space of leaves of* $S \cap TP$ in P and $\operatorname{pr}_\circ$ the projection $\operatorname{pr}_\circ : P \to P/S$.

For $g, h \in G$, we will write $g \sim_S h$ if g and h lie in the same leaf of S and $[g] := \{h \in G \mid h \sim_S g\}$ for the leaf of S through $g \in G$. By the following proposition, we can use the same notation for the equivalence relation defined by the foliation by $S \cap TP$ on P. We will write $[p]_\circ$ for the leaf of $S \cap TP$ through $p \in P$.

Note that g and $h \in G$ lie in the same leaf of S if they can be joined by finitely many flow curves of vector fields lying in $\mathcal{F}_S^{\mathsf{t}}$ or $\mathcal{F}_S^{\mathsf{s}}$ (see Corollary 3.9). For simplicity, if $g \sim_S h$, we will often assume without loss of generality that g and h can be joined by *one* such integral curve.

We will see that if S is complete with some additional properties, then the space G/S inherits a groupoid structure over P/S. If the foliations are simple, we will get a *Lie* groupoid $G/S \rightrightarrows P/S$. We start by showing that the structure maps ϵ, s, t, i *always* induce maps on G/S.

Theorem 3.12. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive, multiplicative subbundle.*

1. *If p and q lie in the same leaf of $S \cap TP$, then p and q seen as elements of G lie in the same leaf of S. Hence, the map $[\epsilon] : P/S \to G/S$, $\mathrm{pr} \circ \epsilon = [\epsilon] \circ \mathrm{pr}_\circ$, is well-defined.*
2. *Choose $g, h \in G$ such that $g \sim_S h$. Then $\mathsf{s}(g) \sim_S \mathsf{s}(h)$ and $\mathsf{t}(g) \sim_S \mathsf{t}(h)$ and the maps $[\mathsf{s}], [\mathsf{t}] : G/S \to P/S$ defined by $[\mathsf{s}] \circ \mathrm{pr} = \mathrm{pr}_\circ \circ \mathsf{s}$, $[\mathsf{t}] \circ \mathrm{pr} = \mathrm{pr}_\circ \circ \mathsf{t}$ are well-defined.*
3. *Choose $g, h \in G$ such that $g \sim_S h$. Then $g^{-1} \sim_S h^{-1}$. Hence, $[\mathsf{i}] : G/S \to G/S$, $[\mathsf{i}] \circ \mathrm{pr} = \mathrm{pr} \circ \mathsf{i}$, is well-defined.*

Proof. 1. This is a standard fact about foliations that are compatible with submanifolds.

2. If g and $h \in G$ are in the same leaf of S, we find by Corollary 3.9 smooth s-descending vector fields $X_1, \ldots, X_k \in \Gamma(S)$ and $t_1, \ldots, t_k \in \mathbb{R}$ such that $h = \phi^k_{t_k} \circ \ldots \circ \phi^1_{t_1}(g)$, where ϕ^i is the flow of the vector field X_i for each $i = 1, \ldots, k$. There exist then smooth vector fields $\bar{X}_1, \ldots, \bar{X}_k \in \Gamma(S \cap TP)$ such that $X_i \sim_{\mathsf{s}} \bar{X}_i$, and hence, if $\bar{\phi}^i$ is the flow of the vector field $\bar{X}_i$, $\mathsf{s} \circ \phi^i = \bar{\phi}^i \circ \mathsf{s}$ for $i = 1, \ldots, k$. We compute then

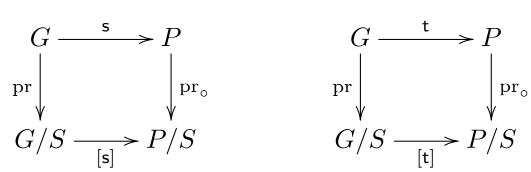
$$\mathsf{s}(h) = \mathsf{s}\left(\phi^k_{t_k} \circ \ldots \circ \phi^1_{t_1}(g)\right) = \bar{\phi}^k_{t_k} \circ \ldots \circ \bar{\phi}^1_{t_1}(\mathsf{s}(g)),$$

which shows that $\mathsf{s}(h)$ and $\mathsf{s}(g)$ lie in the same leaf of $S \cap TP$. The map $[\mathsf{s}] : G/S \to P/S$, $[g] \mapsto [\mathsf{s}(g)]_\circ$ is consequently well-defined. We show in the same manner, but using this time the family (1), that $[\mathsf{t}] : G/S \to P/S$, $[g] \mapsto [\mathsf{t}(g)]_\circ$ is well-defined (note that we don't use here the completeness of $\mathcal{F}^{\mathsf{t}}_S$).

3. If $g \sim_S h$, then there exists without loss of generality one smooth section $X \in \Gamma(S)$ and $\sigma \in \mathbb{R}$ such that $g = \phi^X_\sigma(h)$. Since $X(\phi^X_\tau(h)) \in S(\phi^X_\tau(h))$ for all $\tau \in [0, \sigma]$, the curve $c : [0, \sigma] \to G$, $c(\tau) = (\phi^X_\tau(h))^{-1}$ satisfies $\dot{c}(\tau) = T_{\phi^X_\tau(h)}\mathsf{i}\left(X(\phi^X_\tau(h))\right) \in S(\mathsf{i}(\phi^X_\tau(h)))$ for all $\tau \in [0, \sigma]$. The image of c lies hence in the leaf of S through $c(0) = h^{-1}$. Since $c(\sigma) = g^{-1}$, we have shown that $h^{-1} \sim_S g^{-1}$.

$\square$

Hence, we have shown that the structure maps ϵ, s, t and i project to well-defined maps on P/S and G/S and the following diagrams commute.



$$\begin{array}{ccc} P & \xrightarrow{\epsilon} & G \\ {\scriptstyle \mathrm{pr}_\circ}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ P/S & \xrightarrow[{[\epsilon]}]{} & G/S \end{array} \qquad\qquad \begin{array}{ccc} G & \xrightarrow{\mathsf{i}} & G \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ G/S & \xrightarrow[{[\mathsf{i}]}]{} & G/S \end{array}$$

On the existence of an induced multiplication. Assume that the leaf space G/S has a groupoid structure over P/S such that the projection $(\mathrm{pr}, \mathrm{pr}_\circ)$ is a groupoid morphism. Then:

$$\text{If } g, h \in G \text{ are such that } \mathsf{s}(g) = \mathsf{t}(h), \\ \text{then } [g] \star [h] = [g \star h].$$

Thus, the multiplication on $G/S \rightrightarrows P/S$ has to be defined as follows.

$$\begin{gathered}\text{If } [g], [h] \in G/S \text{ are such that } [\mathsf{s}]([g]) = [\mathsf{t}]([h]), \text{ then the product of } [g] \text{ and } [h] \\ \text{is given by } [g] \star [h] = [g' \star h], \text{ where } g' \in G \text{ is such that} \\ g \sim_S g' \text{ and } \mathsf{s}(g') = \mathsf{t}(h).\end{gathered} \tag{6}$$

For this to be well-defined, we will have to assume that the following conditions are satisfied.

$$\text{For all } g \in G,\ p \in P \text{ such that } p \sim_S \mathsf{s}(g), \text{ there exists } h \in G \\ \text{such that } h \sim_S g \text{ and } \mathsf{s}(h) = p. \tag{7}$$

$$\text{For all } g \in G \text{ and } p := \mathsf{t}(g) : \left([p] \cap \mathsf{s}^{-1}(p)\right) \star g = [g] \cap \mathsf{s}^{-1}(\mathsf{s}(g)). \tag{8}$$

In Example 3.19 (taken from [27, Example 6]), we have a Lie groupoid with a multiplicative foliation such that (8) is *not* satisfied, and the leaf space does not inherit a multiplication.

Condition (8) is satisfied for instance if all the intersections $[g] \cap \mathsf{t}^{-1}(\mathsf{t}(g))$ have one connected component and are hence the leaves of the involutive vector bundle $T^{\mathsf{t}}G \cap S$. This is the case if the Lie groupoid is a Lie group, and also in Examples 3.2 and 3.3.

We will show later that, under some (rather strong) regularity conditions on the family $\mathcal{A}^S$, (7) and (8) hold (see Proposition 3.18).

In the next lemma, we prove that (7) is always satisfied if S is complete.

Lemma 3.13. *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ be a* complete *multiplicative involutive subbundle of TG. If $g \in G$ and $\mathsf{t}(g) \sim_S p \in P$, then there exists $h \in G$ such that $g \sim_S h$ and $\mathsf{t}(h) = p$. In the same manner, if $\mathsf{s}(g) \sim_S p \in P$, then there exists $h \in G$ such that $g \sim_S h$ and $\mathsf{s}(h) = p$. That is, condition (7) is satisfied.*

Proof. Choose a vector field $\bar{X} \in \bar{\mathfrak{F}}^{\mathsf{t}}_S$ and $\sigma \in \mathbb{R}$ such that $p = \phi^{\bar{X}}_\sigma(\mathsf{t}(g))$. We find then a t-descending vector field $X \sim_{\mathsf{t}} \bar{X}$ defined at g. Since $\bar{X}$ and X can be taken complete by hypothesis, the integral curve of X starting at g is defined at σ and we have $\mathsf{t}(\phi^X_\tau(g)) = \phi^{\bar{X}}_\tau(\mathsf{t}(g))$ for all $\tau \in [0, \sigma]$. Set $h := \phi^X_\sigma(g)$, then $h \sim_S g$ and $\mathsf{t}(h) = \phi^{\bar{X}}_\sigma(\mathsf{t}(g)) = p$. □

We can now formulate our main theorem and complete its proof.

Theorem 3.14. *Let $G \rightrightarrows P$ be a Lie groupoid and S a complete, multiplicative, involutive subbundle of TG. Then there is an induced groupoid structure on the leaf* *space* $G/S \rightrightarrows P/S$*, such that* $(\mathrm{pr}, \mathrm{pr}_\circ)$ *is a groupoid morphism, if and only if* (8) *is satisfied.*

Remark 3.15. Note that in general, the induced groupoid $G/S \rightrightarrows P/S$ is not a Lie groupoid. As in the easier Lie group case, we get a Lie groupoid if the involutive subbundles $S \subseteq TG$ and $S \cap TP \subseteq TP$ are simple.

Note that if a *Lie group* G with Lie algebra $\mathfrak{g}$ is simply connected, then any ideal $\mathfrak{s} \subseteq \mathfrak{g}$ integrates automatically to a closed, normal subgroup ([3]). Thus, the space of leaves of a multiplicative subbundle $S := \mathfrak{s}^L = \mathfrak{s}^R \subseteq TG$ is always a *Lie* group. In the general case of a t-simply connected Lie groupoid, it is easy to see that the leaf space of a multiplicative foliation is not necessarily a manifold. Take for instance any manifold as a groupoid over itself (and hence t-simply connected) and any non-simple foliation (that is automatically multiplicative) on the manifold.

Proof of Theorem 3.14. Assume that $G/S \rightrightarrows P/S$ is a groupoid such that $(\mathrm{pr}, \mathrm{pr}_\circ)$ is a groupoid morphism and choose $g \in G$. If $h \in [g] \cap \mathsf{s}^{-1}(\mathsf{s}(g))$, then we have $[h \star g^{-1}] = [h] \star [g^{-1}] = [g] \star [g^{-1}] = [\mathsf{t}(g)]$ and hence

$$h = (h \star g^{-1}) \star g \in ([\mathsf{t}(g)] \cap \mathsf{s}^{-1}(\mathsf{t}(g))) \star g.$$

The converse inclusion in (8) can be checked in the same manner.

Conversely, assume that (8) holds. We show that the multiplication (6) is well-defined, i.e., that is doesn't depend on the choice of the representatives g, h. Using Theorem 3.12, the groupoid axioms are then easily verified to hold and the projection $(\mathrm{pr}, \mathrm{pr}_\circ)$ is a groupoid morphism by definition of the structure maps of $G/S \rightrightarrows P/S$.

Choose $[g]$ and $[h] \in G/S$ such that $[\mathsf{s}]([g]) = [\mathsf{t}]([h])$. Then we have $[\mathsf{s}(g)] = [\mathsf{t}(h)]$ by definition of $[\mathsf{s}]$ and $[\mathsf{t}]$ and, by Lemma 3.13, there exists $g' \in [g]$ such that $\mathsf{s}(g') = \mathsf{t}(h)$. For simplicity, assume that the chosen representative g already satisfies this condition.

Assume that g, g' are two such choices, i.e., $g' \in [g]$ and $\mathsf{s}(g') = \mathsf{t}(h) = \mathsf{s}(g)$. By (8), we have

$$[g \star h] \cap \mathsf{s}^{-1}(\mathsf{s}(h)) = \big([\mathsf{t}(g)] \cap \mathsf{s}^{-1}(\mathsf{t}(g))\big) \star (g \star h) = \big([g] \cap \mathsf{s}^{-1}(\mathsf{s}(g))\big) \star h,$$

and hence $g \star h \sim_S g' \star h$. That is, (6) doesn't depend on the choice of the representative g of $[g]$. By symmetry, (6) doesn't depend on the choice of h. □

Example 3.16. Consider a t-connected Lie groupoid $G \rightrightarrows P$ and let a connected Lie group H act freely and properly on $G \rightrightarrows P$ by Lie groupoid morphisms. Let $\Phi : H \times G \to G$ be the action. That is, for all $h \in H$, the map $\Phi_h : G \to G$ is a groupoid morphism over the map $\phi_h := \Phi_h|_P : P \to P$. Let $\mathcal{V} \subseteq TG$ be the vertical space of the action, i.e., $\mathcal{V}(g) = \{\xi_G(g) \mid \xi \in \mathfrak{h}\}$ for all $g \in G$, where $\mathfrak{h}$ is the Lie algebra of H.

We check that $\mathcal{V} \subseteq TG$ is multiplicative. Choose $\xi_G(g) \in \mathcal{V}(g)$. Then we have

$$\begin{aligned} T_g\mathsf{t}\left(\xi_G(g)\right) &= \left.\frac{d}{dt}\right|_{t=0} \mathsf{t}(\Phi_{\exp(t\xi)}(g)) = \left.\frac{d}{dt}\right|_{t=0} \phi_{\exp(t\xi)}(\mathsf{t}(g)) \\ &= \xi_P(\mathsf{t}(g)) \quad \in \quad \mathcal{V}(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P \end{aligned}$$

and in the same manner

$$T_g\mathsf{s}\left(\xi_G(g)\right) \in \mathcal{V}(\mathsf{s}(g)) \cap T_{\mathsf{s}(g)}P.$$

This shows that $\mathcal{V} \cap T^{\mathsf{s}}G = \mathcal{V} \cap T^{\mathsf{t}}G = \{0\}$ in this example.

If $\xi_G(g) \in \mathcal{V}(g)$ and $\eta_G(g') \in \mathcal{V}(g')$ are such that

$$T_g\mathsf{s}\left(\xi_G(g)\right) = T_g\mathsf{t}\left(\eta_G(g')\right),$$

then we have $\mathsf{s}(g) = \mathsf{t}(g') =: p$ and

$$\xi_P(p) = \eta_P(p),$$

which implies $\xi = \eta$ since the action is free. We get then

$$\begin{aligned}\xi_G(g) \star \eta_G(g') = \xi_G(g) \star \xi_G(g') &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(g) \star \Phi_{\exp(t\xi)}(g') \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(g \star g') = \xi_G(g \star g') \in \mathcal{V}(g \star g').\end{aligned}$$

The inverse of $\xi_G(g)$ is then $\xi_G(g^{-1})$ for all $\xi \in \mathfrak{h}$ and $g \in G$.

Choose $g \in G$. Then we have

$$[g] \cap \mathsf{s}^{-1}(\mathsf{s}(g)) = \{g' \in G \mid \mathsf{s}(g) = \mathsf{s}(g'), \exists\, x \in H : x \cdot g = g'\} = \{g\},$$

since $x \cdot g = g'$ implies $x \cdot \mathsf{s}(g) = \mathsf{s}(g') = \mathsf{s}(g)$ and hence $x = e$ because the action of H on P is assumed to be free. As a consequence, Condition (8)

$$[g] \cap \mathsf{s}^{-1}(\mathsf{s}(g)) = \left([\mathsf{t}(g)] \cap \mathsf{s}^{-1}(\mathsf{t}(g))\right) \star g$$

is satisfied in a trivial manner.

It follows also from the considerations above that the vector fields ξ_G are t- and s-descending to ξ_P. Hence, $\mathcal{V}$ is spanned by complete vector fields, that are t-related to complete vector fields.

Hence, we recover from Theorem 3.14 the fact that the quotient $G/H \rightrightarrows P/H$ has the structure of a Lie groupoid such that the projections $\mathrm{pr} : G \to G/H$, $\mathrm{pr}_\circ : P \to P/H$ form a Lie groupoid morphism.

Remark 3.17. In discrete mechanics, the velocity space TQ of a configuration space Q is replaced by $Q \times Q$. The lift of a Lie group action by a Lie group H on the configuration space is then naturally replaced by the diagonal action of H on $Q \times Q$. It is observed by [25] that the Lagrangian formalism for discrete mechanics can be generalized using groupoids. This idea has been developped by [16] and [5]. In this setting, Lie group actions preserving the geometric structure should be Lie group actions by Lie groupoid morphisms as in the previous example. This suggests that Theorem 3.14 could be very useful for reduction by distributions in the context of discrete mechanics and discrete mechanics on Lie groupoids.

We show now that under additional conditions on the family $\mathcal{A}^S$, (7) and (8) are satisfied if the Lie groupoid is t-connected. We will assume that:

- The elements of $\mathcal{A}^S$ are globally defined.
- The images of the elements of $\mathcal{A}^S$ under the anchor are complete. This implies that the corresponding left invariant vector fields on G are also complete (see [14, Theorem 3.6.4]).

Proposition 3.18. *Let $G \rightrightarrows P$ be a t-connected Lie groupoid and $S \subseteq TG$ an involutive, multiplicative subbundle. Assume that $\mathcal{A}^S$ is a family of* global *sections of AG, with* complete *images under the anchor. Then* (7) *and* (8) *are satisfied and there is hence an induced groupoid structure on the leaf space $G/S \rightrightarrows P/S$, such that* $(\mathrm{pr}, \mathrm{pr}_\circ)$ *is a groupoid morphism.*

Proof. Since AG is spanned by $\mathcal{A}^S$ and $G\rightrightarrows P$ is t-connected, each element h of G can be written

$$h = R_{\mathrm{Exp}(t_1 a_1)} \circ \ldots \circ R_{\mathrm{Exp}(t_n a_n)}(\mathsf{t}(h))$$

with $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathcal{A}^S$ and $t_1, \ldots, t_n \in \mathbb{R}$. Without loss of generality, we can assume in the following that $n = 1$ and we set $a_1 =: a$, $t_1 =: t$. That is, $h = \mathrm{Exp}(ta)(\mathsf{t}(h))$. By hypothesis, for all $t \in \mathbb{R}$, the bisection $\mathrm{Exp}(ta)$ is globally defined on P.

Equation (5) implies that the flow of a^l, which is equal to $R_{\mathrm{Exp}(\cdot a)}$, leaves S invariant. As a consequence, if $g' \sim_S g$, then $R_{\mathrm{Exp}(\tau a)}(g') \sim_S R_{\mathrm{Exp}(\tau a)}(g)$ for all τ.

In particular, if $q \in P$ is $S \cap TP$-related to $\mathsf{t}(h)$, then $h = R_{\mathrm{Exp}(ta)}(\mathsf{t}(h)) \sim_S R_{\mathrm{Exp}(ta)}(q) = \mathrm{Exp}(ta)(q)$ and $\mathrm{Exp}(ta)(q)$ is such that $\mathsf{t}\left(\mathrm{Exp}(ta)(q)\right) = q$. This implies (7), by using the inversion.

Furthermore, if $\mathsf{s}(g') = \mathsf{s}(g) = \mathsf{t}(h)$, we get $g' \star h = R_h(g') = R_{\mathrm{Exp}(ta)}(g') \sim_S R_{\mathrm{Exp}(ta)}(g) = R_h(g) = g \star h$. □

We end this paragraph with an example of a multiplicative foliation which leaf space is not a groupoid.

Example 3.19 ([27])**.** Let P be the trivial circle bundle over the open 2-disk D, but with one point removed in the fiber over $0 \in D$. We write $P = \hat{\mathbb{S}}^1 \to D$, where $\hat{\mathbb{S}}^1_p$ denotes the circle for all non-zero $p \in D$, while $\hat{\mathbb{S}}^1_0$ is the circle with a point removed. Note that $\pi_1(P) = \mathbb{Z}$, generated by any of the circle fibers. It is easy to see that the universal cover of P is $\tilde{P} = (D \times \mathbb{R}) \setminus (\{0\} \times \mathbb{Z})$. We write $\tilde{P} = \hat{\mathbb{R}} \to D$ as a bundle over D, where $\hat{\mathbb{R}}_p = \mathbb{R}$ for non-zero $p \in D$ and $\hat{\mathbb{R}}_0 = \mathbb{R} \setminus \mathbb{Z}$.

Consider the Lie groupoid $G := \hat{\mathbb{R}} \times_{\mathbb{Z}} \hat{\mathbb{R}} \rightrightarrows P$, where the action of $\mathbb{Z} = \pi_1(P)$ is the diagonal action by deck transformations. Let π be the projection $\hat{\mathbb{R}} \times \hat{\mathbb{R}} \to \hat{\mathbb{R}} \times_{\mathbb{Z}} \hat{\mathbb{R}}$, (p_1, x_1, p_2, x_2) coordinates on $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$ and (p, x) coordinates on P. The subbundle of $T(\hat{\mathbb{R}} \times \hat{\mathbb{R}})$ spanned by ∂_{x_1} and ∂_{x_2} is easily seen to project under π to a multiplicative subbundle S of TG. The intersection $S_P := S \cap TP$ is the span of ∂_x and we find $P/S_P \simeq D$.

The leaves of S almost coincide with the fibers of the natural projection onto $D \times D$: for $(p_1, p_2) \in D \times D$, the leaf $L_{[p_1, x_1, p_2, x_2]}$ of S through $g = [p_1, x_1, p_2, x_2] \in G$ is:

$$\begin{cases} L_g = \{[(p_1, t_1, p_2, t_2)] \mid t_1, t_2 \in \mathbb{R}\} \quad \text{if } p_1 \neq 0 \text{ and } p_2 \neq 0, \\ L_g = \{[(p_1, t_1, p_2, t_2)] \mid t_1 \in \mathbb{R} \setminus \mathbb{Z}, t_2 \in \mathbb{R}\} \quad \text{if } p_1 = 0 \text{ and } p_2 \neq 0, \\ L_g = \{[(p_1, t_1, p_2, t_2)] \mid t_1 \in \mathbb{R}, t_2 \in \mathbb{R} \setminus \mathbb{Z}\} \quad \text{if } p_1 \neq 0 \text{ and } p_2 = 0, \\ L_g = \{[(p_1, t_1, p_2, t_2)] \mid t_1 \in ([x_1], [x_1]+1), t_2 \in ([x_2], [x_2]+1)\} \quad \text{if } p_1 = p_2 = 0. \end{cases}$$

Thereby, for $x \in \mathbb{R}$, $[x]$ is the biggest integer smaller than x. In other words, the leaf through $[p_1, x_1, p_2, x_2]$ is a cylinder if $p_1 \neq 0$ and $p_2 \neq 0$, a rectangle if either p_1 or p_2 vanishes, and, over $(0,0) \in D \times D$, we have the quotient of $(\mathbb{R} \setminus \mathbb{Z}) \times (\mathbb{R} \setminus \mathbb{Z})$ by the diagonal $\mathbb{Z}$-action, which consists of countably many leaves. Hence the leaf space is $\widehat{D \times D}$, where the latter denotes the non-Hausdorff manifold obtained from $D \times D$ replacing $(0,0)$ with a copy of $\mathbb{Z}$.

Is it easy to see that, as shown in Theorem 3.12, there are well-defined "source" and "target" maps $\widehat{D \times D} \to D$ such that

$$\begin{array}{ccc} G = \hat{\mathbb{R}} \times_{\mathbb{Z}} \hat{\mathbb{R}} & \xrightarrow{\mathrm{pr}} & \widehat{D \times D} \\ {\scriptstyle \mathsf{t}}\downdownarrows{\scriptstyle \mathsf{s}} & & {\scriptstyle [\mathsf{t}]}\downdownarrows{\scriptstyle [\mathsf{s}]} \\ P = \hat{\mathbb{S}} & \xrightarrow[\mathrm{pr}_\circ]{} & D \end{array}$$

commutes, but the multiplication does not descend to the quotient. Indeed, consider $(0, p) \in \widehat{D \times D}$ where p is non-zero. A preimage under pr is $[(0, \mu_1), (p, x)]$ where $\mu_1 \in \mathbb{R} \setminus \mathbb{Z}$ and $x \in \mathbb{R}$ are arbitrary. Similarly, we consider $(p, 0) \in \widehat{D \times D}$ and as a preimage we pick $[(p, x), (0, \mu_2)]$ where again $\mu_2 \in \mathbb{R} \setminus \mathbb{Z}$ is arbitrary. Now multiplying these two elements of G we obtain $[(0, \mu_1), (0, \mu_2)]$. The value of its projection under pr depends on the concrete choice of μ_1 and μ_2. This shows that $\widehat{D \times D}$ does not have an induced groupoid structure. In the same manner, one can see that (8) is not satisfied.

The Lie algebroid of $G \rightrightarrows P$ is the projection under $T\pi$ of the restriction to $\Delta_{\hat{\mathbb{R}}}$ of the subbundle $\{0\} \times T\hat{\mathbb{R}}$ of $T(\hat{\mathbb{R}} \times \hat{\mathbb{R}}) = T\hat{\mathbb{R}} \times T\hat{\mathbb{R}}$. The vector fields on the second copy of D and ∂_{x_2} are π-related in this manner to ∇-parallel sections of AG. It is easy to see using points as above that the conditions for Proposition 3.18 are not satisfied here.

Tangent and cotangent spaces of the quotient. We end this section with the following lemma, which will be useful for the proof of Theorem 4.4. The proof is a straightforward computation that is left to the reader.

Lemma 3.20. *In the setting of Theorem 3.14, if $G/S \rightrightarrows P/S$ is a Lie groupoid and $(\mathrm{pr}, \mathrm{pr}_\circ)$ a pair of smooth surjective submersions, choose $v_{[g]} \in T_{[g]}(G/S)$ and $v_{[h]} \in T_{[h]}(G/S)$ such that $T_{[g]}[\mathsf{s}]v_{[g]} = T_{[h]}[\mathsf{t}]v_{[h]}$. Then we can assume without loss of generality that $\mathsf{s}(g) = \mathsf{t}(h)$. If $v_g \in T_gG$ and $v_h \in T_hG$ are such that $T_g \,\mathrm{pr}\, v_g = v_{[g]}$ and $T_h \,\mathrm{pr}\, v_h = v_{[h]}$, then there exists $w_g \in S(g)$ such that $T_g\mathsf{s}(v_g - w_g) = T_h\mathsf{t}v_h$. We have then $T_{g\star h}\,\mathrm{pr}((v_g - w_g) \star v_h) = v_{[g]} \star v_{[h]}$.*

If $\alpha_{[g]} \in T^*_{[g]}(G/S)$ and $\alpha_{[h]} \in T^*_{[h]}(G/S)$ are such that $[\hat{\mathsf{s}}](\alpha_{[g]}) = [\hat{\mathsf{t}}](\alpha_{[h]})$, then

$$\hat{\mathsf{s}}((T_g \,\mathrm{pr})^*\alpha_{[g]}) = (T_{\mathsf{s}(g)} \,\mathrm{pr})^*\hat{\mathsf{s}}(\alpha_{[g]}), \qquad \hat{\mathsf{t}}((T_h \,\mathrm{pr})^*\alpha_{[h]}) = (T_{\mathsf{t}(h)} \,\mathrm{pr})^*\hat{\mathsf{t}}(\alpha_{[h]}),$$

hence $\hat{\mathsf{s}}\left((T_g \,\mathrm{pr})^*\alpha_{[g]}\right) = \hat{\mathsf{t}}\left((T_h \,\mathrm{pr})^*\alpha_{[h]}\right)$ and we have

$$((T_g \,\mathrm{pr})^*\alpha_{[g]}) \star ((T_h \,\mathrm{pr})^*\alpha_{[h]}) = (T_{g\star h} \,\mathrm{pr})^*(\alpha_{[g]} \star \alpha_{[h]}).$$

4. **An application.** We use Theorem 3.14 to study how a theorem in [18] (see also [9]) about the shape of *Dirac Lie groups* generalizes to Dirac groupoids.

The Pontryagin bundle $\mathsf{P}_M = TM \times_M T^*M$ of a smooth manifold M is endowed with the non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by

$$\langle (v_m, \alpha_m), (w_m, \beta_m) \rangle = \alpha_m(w_m) + \beta_m(v_m) \tag{9}$$

for all $m \in M$, $v_m, w_m \in T_mM$ and $\alpha_m, \beta_m \in T^*_mM$. An *almost Dirac structure* (see [2]) on M is a Lagrangian vector subbundle $\mathsf{D} \subset \mathsf{P}_M$. That is, D coincides with its orthogonal relative to (9) and so its fibers are necessarily $\dim M$-dimensional.

Let (M, D) be an almost Dirac manifold. For each $m \in M$, the almost Dirac structure D defines a subspace $\mathsf{G}_0(m) \subset T_mM$ by

$$\mathsf{G}_0(m) := \{v_m \in T_mM \mid (v_m, 0) \in \mathsf{D}(m)\}.$$

The distribution $\mathsf{G}_0 = \cup_{m\in M} \mathsf{G}_0(m)$ is not necessarily smooth.

The almost Dirac structure D is *a Dirac structure* if its set of sections is closed under the *Courant-Dorfman bracket*, i.e.,

$$[(X, \alpha), (Y, \beta)] = ([X, Y], \boldsymbol{\pounds}_X\beta - \mathbf{i}_Y\mathbf{d}\alpha) \in \Gamma(\mathsf{D}) \tag{10}$$

for all $(X, \alpha), (Y, \beta) \in \Gamma(\mathsf{D})$. The class of Dirac structures generalizes the one of Poisson manifolds in the sense of the following example.

Example 4.1. Let M be a smooth manifold endowed with a globally defined bivector field $\pi \in \Gamma\left(\bigwedge^2 TM\right)$. Then $\mathsf{D}_\pi \subseteq \mathsf{P}_M$ defined by

$$\mathsf{D}_\pi(m) = \left\{(\pi^\sharp(\alpha_m), \alpha_m) \mid \alpha_m \in T_m^*M\right\} \qquad \text{for all } m \in M,$$

where $\pi^\sharp : T^*M \to TM$ is defined by $\pi^\sharp(\alpha) = \pi(\alpha, \cdot) \in \mathfrak{X}(M)$ for all $\alpha \in \Omega^1(M)$, is an almost Dirac structure on M. It is a Dirac structure if and only if the Schouten bracket of the bivector field with itself vanishes, $[\pi, \pi] = 0$, that is, if and only if (M, π) is a Poisson manifold.

Let (M, D_M) and (N, D_N) be two (almost) Dirac manifolds and $\Phi : M \to N$ a smooth map. Then Φ is a *forward Dirac map* if for all $n \in N$, $m \in \Phi^{-1}(n)$ and $(v_n, \alpha_n) \in \mathsf{D}_N(n)$ there exists $(v_m, \alpha_m) \in \mathsf{D}_M(m)$ such that $T_m\Phi v_m = v_n$ and $\alpha_m = (T_m\Phi)^*\alpha_n$.

Let $G \rightrightarrows P$ be a Lie groupoid. Recall that the Pontryagin bundle of G has then a Lie groupoid structure over $TP \times_P A^*G$.

Definition 4.2 ([19])**.** An (almost) Dirac groupoid is a Lie groupoid $G \rightrightarrows P$ endowed with an (almost) Dirac structure D_G such that $\mathsf{D}_G \subseteq TG \times_G T^*G$ is a Lie subgroupoid. The (almost) Dirac structure is then said to be *multiplicative*.

If G is a Lie group, i.e., with $P = \{e\}$, an (almost) Dirac structure D_G on G is multiplicative if the multiplication $\mathsf{m} : (G \times G, \mathsf{D}_G \oplus \mathsf{D}_G) \to (G, \mathsf{D}_G)$ is a forward Dirac map. It is shown in [18], [9], that the induced distribution G_0 on G is multiplicative, hence left and right invariant and equal to $\mathsf{G}_0 = \mathfrak{g}_0^L = \mathfrak{g}_0^R$ for some ideal $\mathfrak{g}_0 \subseteq \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G. Thus, G_0 is automatically an involutive subbundle of TG for any multiplicative almost Dirac structure on the Lie group G. The leaf N of G_0 through e is then a normal subgroup of G and if it is closed, the quotient $G/\mathsf{G}_0 = G/N$ is a Lie group such that the projection $q : G \to G/N$ is a smooth surjective submersion. If (G, D_G) is a Dirac Lie group and $N \subseteq G$ is a closed subgroup, the quotient G/N inherits a Poisson structure π such that $q : (G, \mathsf{D}_G) \to (G/N, \pi)$ is a forward Dirac map and $(G/N, \pi)$ is a *Poisson Lie group*. Furthermore, $(G/N, \pi)$ is a Poisson homogeneous space of the Dirac Lie group (G, D_G) ([9]).

The fact that the characteristic distribution of an almost Dirac Lie group is always multiplicative is a special case of the fact that the characteristic distribution of an almost Dirac groupoid is always multiplicative, as shows the next proposition.

Proposition 4.3. *Let* $(G \rightrightarrows P, \mathsf{D}_G)$ *be an (almost) Dirac groupoid. Then the subbundle* $\mathsf{G}_0 \subseteq TG$ *is a (set) subgroupoid over* $TP \cap \mathsf{G}_0$*.*

Proof. Choose a composable pair $(g,h)\in G\times_P G$ and $v_g\in \mathsf{G}_0(g)$, $v_h\in\mathsf{G}_0(h)$ such that $T_g\mathsf{s}(v_g)=T_h\mathsf{t}(v_h)$. Then we have $(v_g,0_g)\in\mathsf{D}_G(g)$, $(v_h,0_h)\in\mathsf{D}_G(h)$ such that $\mathbb{T}\mathsf{s}(v_g,0_g)=(T_g\mathsf{s}(v_g),0_{\mathsf{s}(g)})=(T_g\mathsf{t}(v_h),0_{\mathsf{t}(h)})=\mathbb{T}\mathsf{t}(v_h,0_h)$ and hence $\mathbb{T}\mathsf{t}(v_g,0_g)\in\mathsf{D}_G(\mathsf{t}(g))$, $\mathbb{T}\mathsf{s}(v_g,0_g)\in\mathsf{D}_G(\mathsf{s}(g))$, $(v_g,0_g)^{-1}\in\mathsf{D}_G(g^{-1})$ and $(v_g,0_g)\star(v_h,0_h)\in\mathsf{D}_G(g\star h)$. Since $(v_g,0_g)^{-1}=(v_g^{-1},0_{g^{-1}})$ and $(v_g,0_g)\star(v_h,0_h)=(v_g\star v_h,0_{g\star h})$, this shows that $T_g\mathsf{s}(v_g)\in\mathsf{G}_0(\mathsf{s}(g))$, $T_g\mathsf{t}(v_g)\in\mathsf{G}_0(\mathsf{t}(g))$, $v_g^{-1}\in\mathsf{G}_0(g^{-1})$ and $v_g\star v_h\in\mathsf{G}_0(g\star h)$. □

Yet, the situation in the general case of an (almost) Dirac groupoid is more involved than in the Lie groups case. First of all, the induced distribution G_0 has not automatically constant rank on G anymore. If it is smooth, it is in general not completely integrable in the sense of Stefan and Sussmann, unless D_G is a Dirac structure. For instance, each manifold can be seen as a (trivial) groupoid over itself (i.e., with $\mathsf{t}=\mathsf{s}=\mathrm{Id}_M$) and any (almost) Dirac manifold can thus be seen as an (almost) Dirac groupoid, which will, in general not satisfy these conditions. Thus, trivial Dirac groupoids yield already many examples of Dirac groupoids for which G_0 is not an involutive subbundle of TG. (The class of Dirac groupoids described in the next theorem seems hence to be a small class of examples.)

If G_0 associated to a *Dirac groupoid* $(G\rightrightarrows P,\mathsf{D}_G)$ is *assumed* to be a vector bundle on G, then we are in the same situation as in the group case.

We know by Theorem 3.14 that we have to assume that G_0 is *complete* and satisfies (8) to ensure that the leaf space G/G_0 inherits a multiplication map and hence to show that there exists a Poisson *groupoid* structure on the quotient G/G_0.

The next theorem shows that, under these conditions, we obtain the same result as the one that is already known in the group case.

Theorem 4.4. *Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid. Assume that G_0 is a subbundle of TG, that it is complete and that (8) holds. If the leaf spaces G/G_0 and P/G_0 have smooth manifold structures such that the projections are submersions, then there is an induced multiplicative Poisson structure on the Lie groupoid $G/\mathsf{G}_0\rightrightarrows P/\mathsf{G}_0$, such that the projection $\mathrm{pr}:G\to G/\mathsf{G}_0$ is a forward Dirac map.*

Proof. Recall that since D_G is assumed to be a Dirac structure, the subbundle $\mathsf{G}_0\subseteq TG$ is involutive. Since G_0 is multiplicative by Proposition 4.3 and all the hypotheses for Theorem 3.14 are hence satisfied, we get that $G/\mathsf{G}_0\rightrightarrows P/\mathsf{G}_0$ has the structure of a Lie groupoid such that if $\mathrm{pr}:G\to G/\mathsf{G}_0$ and $\mathrm{pr}_\circ:P\to P/\mathsf{G}_0$ are the projections, then $(\mathrm{pr},\mathrm{pr}_\circ)$ is a Lie groupoid morphism.

We have $\mathsf{D}_G\cap(TG\times_G{\mathsf{G}_0}^\circ)=\mathsf{D}_G\cap(TG\times_G\mathsf{P}_1)=\mathsf{D}_G$ and $[\Gamma(\mathsf{G}_0\times_G\{0\}),\Gamma(\mathsf{D}_G)]\subseteq\Gamma(\mathsf{D}_G)$ because D_G is Dirac. Hence, by a result in [26] (see also [11]), we find that the Dirac structure pushes-forward to the quotient G/G_0. The (almost) Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ is given by

$$\mathrm{pr}(\mathsf{D}_G)([g])=\left\{(v_{[g]},\alpha_{[g]})\in\mathsf{P}_{G/\mathsf{G}_0}([g])\;\middle|\;\begin{array}{c}\exists v_g\in T_gG\text{ such that}\\(v_g,(T_g\,\mathrm{pr})^*\alpha_{[g]})\in\mathsf{D}_G(g)\\\text{and }T_g\,\mathrm{pr}\,v_g=v_{[g]}\end{array}\right\}$$

for all $g\in G$. The fact that $\mathrm{pr}(\mathsf{D}_G)$ is a Dirac structure follows D_G being a Dirac structure. If $(v_{[g]},0)\in\mathrm{pr}(\mathsf{D}_G)([g])$, there exists $v_g\in T_gG$ such that $T_g\,\mathrm{pr}\,v_g=v_{[g]}$ and $(v_g,0)\in\mathsf{D}_G(g)$. But then we get $v_g\in\mathsf{G}_0(g)$ and hence $v_{[g]}=T_g\,\mathrm{pr}\,v_g=0_{[g]}$. This shows that the characteristic distribution G_0 associated to the Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ is trivial, and since it is integrable, $\mathrm{pr}(\mathsf{D}_G)$ is the graph of the vector bundle homomorphism $T^*(G/\mathsf{G}_0)\to T(G/\mathsf{G}_0)$ associated to a Poisson bivector on G/G_0.

We have then to show that the Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ on G/G_0 is multiplicative. Choose $(v_{[g]},\alpha_{[g]}) \in \mathrm{pr}(\mathsf{D}_G)([g])$ and $(v_{[h]},\alpha_{[h]}) \in \mathrm{pr}(\mathsf{D}_G)([h])$ such that $\mathbb{T}\mathsf{s}(v_{[g]},\alpha_{[g]}) = \mathbb{T}\mathsf{t}(v_{[h]},\alpha_{[h]})$. We can then assume without loss of generality that $\mathsf{s}(g)=\mathsf{t}(h)$ (see Lemma 3.20). By the definition of $\mathrm{pr}(\mathsf{D}_G)$, we find then $v_g \in T_gG$ and $v_h \in T_hG$ such that $T_g\,\mathrm{pr}\, v_g = v_{[g]}$, $T_h\,\mathrm{pr}\, v_h = v_{[h]}$ and $(v_g,(T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(g)$, $(v_h,(T_h\,\mathrm{pr})^*\alpha_{[h]}) \in \mathsf{D}_G(h)$. Since

$$\mathbb{T}\mathsf{s}(v_g,(T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(\mathsf{s}(g))$$

and

$$T_{\mathsf{s}(g)}\,\mathrm{pr}(T_g\mathsf{s}v_g) = T_{[g]}\mathsf{s}v_{[g]},$$
$$(T_{\mathsf{s}(g)}\,\mathrm{pr})^*(\mathsf{s}(\alpha_{[g]})) = \hat{\mathsf{s}}((T_g\,\mathrm{pr})^*\alpha_{[g]}),$$

we find that

$$\mathbb{T}\mathsf{s}(v_{[g]},\alpha_{[g]}) \in \mathrm{pr}(\mathsf{D}_G)(\mathsf{s}[g]).$$

In the same manner, we get that $\mathbb{T}\mathsf{t}(v_{[g]},\alpha_{[g]}) \in \mathrm{pr}(\mathsf{D}_G)(\mathsf{t}[g])$ and the Dirac structure is closed under the source and target maps on $\mathsf{P}_{G/\mathsf{G}_0}$.

By Lemma 3.20, we find $w_g \in \mathsf{G}_0(g)$ such that $T_g\mathsf{s}(v_g-w_g) = T_h\mathsf{t}v_h$ and $T_{g\star h}\,\mathrm{pr}((v_g-w_g)\star v_h) = v_{[g]}\star v_{[h]}$. By the same Lemma, we have $\hat{\mathsf{s}}((T_g\,\mathrm{pr})^*\alpha_{[g]}) = \hat{\mathsf{t}}((T_h\,\mathrm{pr})^*\alpha_{[h]})$ and $(T_g\,\mathrm{pr})^*\alpha_{[g]}\star(T_h\,\mathrm{pr})^*\alpha_{[h]} = (T_{g\star h}\,\mathrm{pr})^*(\alpha_{[g]}\star\alpha_{[h]})$. The pairs $(v_g-w_g,(T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(g)$ and $(v_h,(T_h\,\mathrm{pr})^*\alpha_{[h]}) \in \mathsf{D}_G(h)$ are hence compatible and their product,

$$((v_g-w_g)\star v_h,(T_{g\star h}\,\mathrm{pr})^*(\alpha_{[g]}\star\alpha_{[h]}))$$

is an element of $\mathsf{D}_G(g\star h)$. Since it pushes forward to $(v_{[g]}\star v_{[h]},\alpha_{[g]}\star\alpha_{[h]})$, we find that $(v_{[g]}\star v_{[h]},\alpha_{[g]}\star\alpha_{[h]}) \in \mathrm{pr}(\mathsf{D}_G)([g]\star[h])$.

It remains to show that the inverse $(v_{[g]},\alpha_{[g]})^{-1}$ is an element of $\mathrm{pr}(\mathsf{D}_G)([g]^{-1})$. Recall that $[g]^{-1}=[g^{-1}]$. We have $(v_g,(T_g\,\mathrm{pr})^*\alpha_{[g]})^{-1} \in \mathsf{D}_G(g^{-1})$. Since

$$\left((T_g\,\mathrm{pr})^*\alpha_{[g]}\right)\star\left((T_{g^{-1}}\,\mathrm{pr})^*\alpha_{[g]}^{-1}\right) = \hat{\mathsf{t}}((T_g\,\mathrm{pr})^*\alpha_{[g]})$$

and in the same manner

$$\left((T_{g^{-1}}\,\mathrm{pr})^*\alpha_{[g]}^{-1}\right)\star\left((T_g\,\mathrm{pr})^*\alpha_{[g]}\right) = \hat{\mathsf{s}}((T_g\,\mathrm{pr})^*\alpha_{[g]}),$$

we find that $(T_{g^{-1}}\,\mathrm{pr})^*\alpha_{[g]}^{-1} = ((T_g\,\mathrm{pr})^*\alpha_{[g]})^{-1}$. Since $(\mathrm{pr},\mathrm{pr}_\circ)$ is a morphism of Lie groupoids, we find also that $T_{g^{-1}}\,\mathrm{pr}(v_g^{-1}) = (T_g\,\mathrm{pr}\,v_g)^{-1} = v_{[g]}^{-1}$. Thus, $(v_g,(T_g\,\mathrm{pr})^*\alpha_{[g]})^{-1} \in \mathsf{D}_G(g^{-1})$ pushes forward to $(v_{[g]},\alpha_{[g]})^{-1}$, which is consequently an element of $\mathrm{pr}(\mathsf{D}_G)([g]^{-1})$. □

Example 4.5 (Trivial example)**.** Let $G\rightrightarrows P$ be a t-connected Lie groupoid and $S\subseteq TG$ a multiplicative, involutive subbundle. Then, if $S^\circ\subseteq T^*G$ is the annihilator of S, the Dirac structure $\mathsf{D}=S\oplus S^\circ$ is multiplicative with characteristic distribution equal to S. If S is complete and simple, then the reduced Poisson groupoid is just the trivial Poisson groupoid $(G/S\rightrightarrows P/S,\pi=0)$.

Remark 4.6. In the situation of the previous theorem, the multiplicative subbundle G_0 of TG has constant rank on G. In particular, the intersection $TP\cap\mathsf{G}_0$ is a smooth vector bundle over P and for each $g\in G$, the restriction to $\mathsf{G}_0(g)$ of the target map, $T_g\mathsf{t}:\mathsf{G}_0(g)\to\mathsf{G}_0(\mathsf{t}(g))\cap T_{\mathsf{t}(g)}P$, is surjective (see Lemma 3.5). By a theorem in [7], there exists then a Dirac structure D_P on P such that $\mathsf{t}:(G,\mathsf{D}_G)\to(P,\mathsf{D}_P)$ is a forward Dirac map. Since $(G/\mathsf{G}_0\rightrightarrows P/\mathsf{G}_0,\mathrm{pr}(\mathsf{D}_G))$ is a Poisson groupoid, we know also by a theorem in [24] that there is a Poisson structure $\{\cdot\,,\cdot\}_{P/\mathsf{G}_0}$ on P/G_0 such that $[\mathsf{t}] : (G/\mathsf{G}_0, \mathrm{pr}(\mathsf{D}_G)) \to (P/\mathsf{G}_0, \{\cdot\,,\cdot\}_{P/\mathsf{G}_0})$ is a forward Dirac map. It is easy to check that the map $\mathrm{pr}_\circ : (P, \mathsf{D}_P) \to (P/\mathsf{G}_0, \{\cdot\,,\cdot\}_{P/\mathsf{G}_0})$ is then also a forward Dirac map, i.e., $\mathsf{D}_{\{\cdot\,,\cdot\}_{P/\mathsf{G}_0}}$ is the forward Dirac image of D_P under $\mathrm{pr}_\circ$.

Remark 4.7. In the Lie group case, the Poisson Lie group $(G/N, q(\mathsf{D}_G))$ associated to a Dirac Lie group (G, D_G) satisfying the necessary regularity assumptions was also a *Poisson homogeneous space* of the Dirac Lie group. In general, the Poisson groupoid associated to the Dirac groupoid is *not* a Poisson homogeneous space of the Dirac groupoid since the quotient G/G_0 is not a homogeneous space of the Lie groupoid $G \rightrightarrows P$.

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E-mail address: madeleine.jotz@a3.epfl.ch