Geom Dedicata
DOI 10.1007/s10711-011-9618-8

ORIGINAL PAPER

Invariant frames for vector bundles and applications

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Received: 9 June 2010 / Accepted: 29 April 2011

Abstract This paper completes a proof of the Dirac reduction theorem by involutive tangent subbundles. As a consequence, Dirac reduction by a proper Lie group action having one isotropy type is carried out. The main technical tool in the proof is the notion of partial connections on suitable vector bundles.

Keywords Dirac manifold · Reduction · Connection · Lie group action

Mathematics Subject Classification (2000) 70G65 · 70H45

1 Introduction

Dirac structures provide a unified framework for the study of closed two-forms, Poisson bivectors, foliations, and also a convenient geometric setting for the theory of nonholonomic systems and circuit theory. They also have a wide range of applications in geometry and theoretical physics since they encode constraints in conservative dynamics. Dirac manifolds were introduced in [11] and [8] (see also [9] and [12]) as a means to simultaneously generalize presymplectic and Poisson structures. In this note we present two Dirac reduction theorems, that is, methods by which quotients naturally inherit a Dirac structure.

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Published online: 12 May 2011

The first reduction statement, Theorem 4.1, induces a Dirac structure on the quotient of the given manifold by a foliation. In Theorem 4.3, this foliation is given by the orbits of a proper Lie group action admitting a single orbit type. These results are extended to the singular case in [17].

Theorem 4.1 already appeared in [22]. However, one step of the proof is given without details: the well-definedness of the reduced Dirac structure. A first goal of this note is to provide a simple proof of this, using as tool the notion of partial connections on a vector bundle. A second goal is to improve an analogous result in [4] (which is also a special case of the Dirac reduction result in [2]) by weakening its assumptions; this is achieved in Theorem 4.3.

Conventions and notations If M is a smooth manifold, $C^\infty(M)$ denotes the sheaf of *local* functions on M, that is, an element $f \in C^\infty(M)$ is, by definition, a smooth function $f : U \to \mathbb{R}$, where the domain of definition U of f is an open subset of M.

Similarly, if E is a vector bundle over M, or a generalized distribution on M, $\Gamma(E)$ denotes the sheaf of smooth *local* sections of E. In particular, the sheaves of smooth local vector fields and local one-forms on M are denoted by $\mathfrak{X}(M)$ and $\Omega^1(M)$, respectively. The open domain of definition of the local section σ of E is denoted by $\mathrm{Dom}(\sigma)$. As $\Gamma(E)$ is a sheaf, for any open set U of M we denote by $\Gamma(E)_U$ the vector space of sections of $E|_U \to U$. In particular, $\Gamma(E)_M$ is the vector space of *global* sections of E.

2 Generalities on exact Courant algebroids

Courant algebroids were introduced in [18] as vector bundles E equipped with a fiberwise nondegenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$, a bilinear skew-symmetric bracket $[\cdot\,,\cdot]_c$ on the smooth sections $\Gamma(\mathsf{E})$, and a vector bundle map $\rho : \mathsf{E} \to TM$ satisfying some compatibility conditions. The bracket $[\cdot\,,\cdot]_c$ doesn't satisfy the Jacobi identity in general. We adopt an equivalent definition of Courant algebroids introduced by Ševera [21], in which the bracket is not skew-symmetric but satisfies the Jacobi identity. It has the nice property that any section provides an infinitesimal automorphism of the Courant algebroid via the adjoint action. The equivalence is given by the bijection that assigns to a Courant algebroid in the sense of [18] a quadruple consisting of the same vector bundle E, the same bilinear form $\langle\cdot\,,\cdot\rangle$, the same vector bundle map ρ, and the bracket $[e_1, e_2] := [e_1, e_2]_c + \frac{1}{2}\rho^* \circ \mathbf{d}\langle e_1, e_2\rangle$ (see [20, Prop 2.6.5]).

Definition 2.1 A Courant algebroid over a manifold M is a vector bundle $\mathsf{E} \to M$ equipped with a fiberwise nondegenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$, a bilinear bracket $[\cdot\,,\cdot]$ on the smooth sections $\Gamma(\mathsf{E})$, and a vector bundle map $\rho : \mathsf{E} \to TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$:

1. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
3. $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2$,
4. $\rho(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,
5. $[e_1, e_1] = D\langle e_1, e_1\rangle$,

where $D = \frac{1}{2}\rho^* \circ \mathbf{d} : C^\infty(M) \to \Gamma(\mathsf{E})$. Here we identify E with E^* using $\langle\cdot\,,\cdot\rangle$, denote by $\rho^* : T^*M \to \mathsf{E}$ the dual vector bundle map, and use the same notation $\rho^* : \Omega^1(M) \to \Gamma(\mathsf{E})$ for the associated map on local sections. It follows from axiom (5) that the bracket is not skew-symmetric, but rather satisfies $[e_1, e_2] = -[e_2, e_1] + 2D\langle e_1, e_2\rangle$ for all $e_1, e_2 \in \Gamma(\mathsf{E})$.

A Courant algebroid is *exact* if the following sequence of vector bundles is exact:

$$0 \to T^*M \xrightarrow{\rho^*} \mathsf{E} \xrightarrow{\rho} TM \to 0.$$

Given an exact Courant algebroid, there always exists a section $\sigma : TM \to \mathsf{E}$ of ρ whose range $\sigma(TM) \subseteq \mathsf{E}$ is isotropic. We shall often call such a section a *right splitting* since it defines a splitting $\rho^*(T^*M) \oplus \sigma(TM) = \mathsf{E}$. Associated to such a section, there is a curvature 3-form $H \in \Omega^3_{cl}(M)$ defined as follows: for vector fields $X, Y, Z \in \mathfrak{X}(M)$, $H(X, Y, Z) := 2\langle[\sigma(X), \sigma(Y)], \sigma(Z)\rangle$. Using the bundle isomorphism $\sigma + \frac{1}{2}\rho^* : TM \oplus T^*M \to \mathsf{E}$, we transport the Courant algebroid structure onto the *Pontryagin bundle* $\mathsf{P}_M := TM \oplus T^*M$. Thus the symmetric pairing and the bracket on $\Gamma(\mathsf{P}_M)$ are

$$\begin{aligned} \langle (X, \alpha), (Y, \beta)\rangle &:= \alpha(Y) + \beta(X), \\ [(X, \alpha), (Y, \beta)]_H &:= ([X, Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha + \mathbf{i}_Y\mathbf{i}_X H), \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, $\alpha, \beta, \in \Omega^1(M)$. The bracket $[\cdot, \cdot]_H$ is the *H-twisted Courant bracket* on P_M. If σ, σ' are right splittings, then $\sigma - \sigma' : TM \to \rho^*(T^*M) \cong T^*M$ is skew-symmetric and hence corresponds to a 2-form $b \in \Omega^2(M)$. Explicitly, the 2-form is given by $b(X, Y) = \langle\sigma(X), \sigma'(Y)\rangle$ for all $X, Y \in \mathfrak{X}(M)$. The curvature of σ and curvature of σ' differ by the exact form $\mathbf{d}b$. Hence the cohomology class $[H] \in H^3(M, \mathbb{R})$, called the *Ševera class*, is independent of the splitting and determines the exact Courant algebroid structure on E completely [21][1, Sect. 4.2].

Example 2.2 The easiest example of a Courant algebroid is the Pontryagin bundle $\mathsf{P}_M = TM \oplus TM^*$ of a smooth manifold M [8]. The right splitting is $\sigma : TM \to \mathsf{P}_M, \sigma(X) = (X, 0)$ for all $X \in \mathfrak{X}(M)$. Thus, the curvature is given by $H = 0$ and the Courant bracket by $[(X, \alpha), (Y, \beta)] = ([X, Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)$.

A E-*Dirac structure* on M, simply called *Dirac structure* on M if it is clear from the context, is a subbundle of E that is maximal isotropic with respect to the pairing on E. The Dirac structure is *integrable* if it is closed under the H-twisted Courant bracket.

We recall a statement about reduction of exact Courant algebroids from [22, Theorem 3.7], which is modeled on [2, Theorem 3.3].

Theorem 2.3 *Let E be an exact Courant algebroid over M, C a submanifold of M, and K an isotropic vector subbundle of E over C such that $\rho(\mathsf{K}^\perp) = TC$. Assume that the space $\Gamma_{\mathrm{bas}}(\mathsf{K}^\perp)_C$ of global basic sections over C spans pointwise $\mathsf{K}^\perp$ (i.e., that $\mathrm{span}\{e_p \mid e \in \Gamma_{\mathrm{bas}}(\mathsf{K}^\perp)_C\} = \mathsf{K}^\perp{}_p$ for every $p \in C$) and that the quotient $\underline{C}$ of C by the foliation integrating $\rho(K)$ is a smooth manifold. Then there is an exact Courant algebroid $\underline{\mathsf{E}}$ over $\underline{C}$ that fits in the following pull back diagram of vector bundles:*

$$\begin{array}{ccc} \mathsf{K}^\perp/\mathsf{K} & \longrightarrow & \underline{\mathsf{E}} \\ \downarrow & & \downarrow \\ C & \longrightarrow & \underline{C}. \end{array}$$

Recall from [22, Definition 3.3] that

$$\Gamma_{\mathrm{bas}}\left(\mathsf{K}^\perp\right) := \left\{\theta \in \Gamma\left(\mathsf{K}^\perp\right) \middle| [\Gamma(\mathsf{K}), \theta] \subseteq \Gamma(\mathsf{K})\right\}$$

denotes the sheaf (over C) of sections which are *basic* with respect to K.

3 Connections and invariant subbundles

In this section we develop the technical tools for the proof of Theorem 2.3 given in Sect. 4. We begin with two general remarks.

Remark 3.1 Let $\mathsf{E} \to M$ be an exact Courant algebroid over a smooth manifold M with anchor $\rho : \mathsf{E} \to TM$ and C a submanifold of M. Let $\mathsf{K} \subseteq \mathsf{E}|_C$ be a vector subbundle of E over C such that $\rho(\mathsf{K}^\perp) = TC$.

(1) Lemma 3.1 and Remark 3.2 in [22] imply that if e_1, e_2 are sections of $\mathsf{K}^\perp$, then $[\tilde{e}_1, \tilde{e}_2]|_C$ depends on the extensions $\tilde{e}_1$ and $\tilde{e}_2 \in \Gamma(\mathsf{E})$ of e_1 and e_2 only up to sections of K. Hence, a statement like $[e_1, e_2] \in \Gamma(\mathsf{K})$ for $e_1, e_2 \in \Gamma\left(\mathsf{K}^\perp\right)$ makes sense: it means that $[\tilde{e}_1, \tilde{e}_2]|_C \in \Gamma(\mathsf{K})$ for some (equivalently, for all) extensions $\tilde{e}_1, \tilde{e}_2 \in \Gamma(\mathsf{E})$ on M. Similarly, we take $\left[\Gamma\left(\mathsf{K}^\perp\right), \Gamma\left(\mathsf{K}^\perp\right)\right] \subseteq \Gamma(\mathsf{K})$ to mean $[e_1, e_2] \in \Gamma(\mathsf{K})$ for all $e_1, e_2 \in \Gamma\left(\mathsf{K}^\perp\right)$.

(2) The conditions

$$\left[\Gamma(\mathsf{K}), \Gamma\left(\mathsf{K}^\perp\right)\right] \subseteq \Gamma\left(\mathsf{K}^\perp\right) \tag{1}$$

and

$$[\Gamma(\mathsf{K}), \Gamma(\mathsf{K})] \subseteq \Gamma(\mathsf{K}) \tag{2}$$

are equivalent, by using axiom (4) in the definition of Courant algebroid.

Definition 3.2 ([5]) Let M be a smooth manifold and $F \subseteq TM$ a smooth involutive vector subbundle of the tangent bundle. Let $E \to M$ be a vector bundle over M. A F-partial connection is a map $\nabla : \Gamma(F) \times \Gamma(E) \to \Gamma(E)$, written $\nabla(X, e) =: \nabla_X(e)$ for $X \in \Gamma(F)$ and $e \in \Gamma(E)$, satisfying:

1. ∇ is tensorial in the F-argument, that is, for all $X, X_1, X_2 \in \Gamma(F), e \in \Gamma(E), f \in C^\infty(M)$, we have

$$\nabla_{X_1+X_2}e = \nabla_{X_1}e + \nabla_{X_2}e$$

and

$$\nabla_{fX}e = f\nabla_X e.$$

2. ∇ is $\mathbb{R}$-linear in the E-argument, that is, for all $\alpha, \beta \in \mathbb{R}$, $X \in \Gamma(F)$, and $e_1, e_2 \in \Gamma(E)$, we have

$$\nabla_X(\alpha e_1 + \beta e_2) = \alpha\nabla_X e_1 + \beta\nabla_X e_2.$$

3. ∇ satisfies the Leibniz rule

$$\nabla_X(fe) = X(f)e + f\nabla_X e$$

for all $X \in \Gamma(F), e \in \Gamma(E), f \in C^\infty(M)$.

Lemma 3.3 *Let* $\mathsf{E} \to M$ *be an exact Courant algebroid,* $C \subset M$ *a submanifold, and* $\mathsf{K} \subseteq \mathsf{E}|_C$ *a subbundle satisfying*

- K *is isotropic*
- $[\Gamma(\mathsf{K}), \Gamma(\mathsf{K})] \subseteq \Gamma(\mathsf{K})$ (*the inclusion* (2))
- $\rho(\mathsf{K}^\perp) = TC$.

Then the map $\nabla : \Gamma(\rho(\mathsf{K})) \times \Gamma(\mathsf{K}^{\perp}/\mathsf{K}) \to \Gamma(\mathsf{K}^{\perp}/\mathsf{K})$ *defined by*

$$\nabla_{\rho(k)}(\bar{\theta}) = \overline{[k, \theta]},$$

where $\bar{\theta} \in \Gamma(\mathsf{K}^{\perp}/\mathsf{K})$ *is the projection of* $\theta \in \Gamma(\mathsf{K}^{\perp})$*, is a flat* $\rho(\mathsf{K})$*- partial connection on* $\mathsf{K}^{\perp}/\mathsf{K} \to C$.

Remark 3.4 The partial connection ∇ arises naturally when one considers coisotropic submanifolds of the degree 2 graded symplectic manifold associated to the Courant algebroid E [3].

Proof Note first that $\mathsf{K}^{\perp} \cap T^*M = \ker(\rho|_{\mathsf{K}^{\perp}} : \mathsf{K}^{\perp} \to TC)$ has constant rank. Hence $\rho(\mathsf{K}) = \left(\mathsf{K}^{\perp} \cap T^*M\right)^{\circ} \subseteq \rho(\mathsf{K}^{\perp}) = TC$ (where the upper small circle denotes the annihilator of the subbundle in the dual of the ambient vector bundle) is a vector subbundle, and it is involutive because of the inclusion (2) and axiom 2 in the definition of Courant algebroid.

Using (1) and (2), it is easy to check that ∇ doesn't depend on the choice of the representative θ and that it has image in $\mathsf{K}^{\perp}/\mathsf{K}$. To check that it is also independent on the choice of $k \in \Gamma(\mathsf{K})$, we have to show that if $\rho(k) = 0$, then $\overline{[k, \theta]} = 0$. For this, assume that k lies in $\Gamma(\mathsf{K} \cap T^*M) = \Gamma\left((\rho(\mathsf{K}^{\perp}))^{\circ}\right) = \Gamma\left((TC)^{\circ}\right)$. Since C is a smooth submanifold, we can find closed 1-forms ξ_i on M such that $\{\xi_i|_C\}$ are a local frame for $\mathsf{K} \cap T^*M$ and we write $k = f^i \xi_i$. So, using the "Leibniz rule for the first entry" of the Courant bracket

$$[fe_1, e_2] = f[e_1, e_2] - (\rho(e_2)f)e_1 + 2\langle e_1, e_2\rangle \mathbf{d}f \tag{3}$$

and the fact that for closed 1-forms $[\xi_i, \cdot] = 0$, we see that $[k, \Gamma(\mathsf{K}^{\perp})] \subset \Gamma(\mathsf{K})$.

We check that ∇ is tensorial in $\rho(\mathsf{K})$. Indeed, for all $\theta \in \Gamma(\mathsf{K}^{\perp})$, $k, k_1, k_2 \in \Gamma(\mathsf{K})$, $f \in C^{\infty}(C)$, taking into account (3), implies

$$\nabla_{\rho(k_1)+\rho(k_2)}\bar{\theta} = \overline{[k_1 + k_2, \theta]} = \overline{[k_1, \theta]} + \overline{[k_2, \theta]} = \nabla_{\rho(k_1)}\bar{\theta} + \nabla_{\rho(k_2)}\bar{\theta}$$

and

$$\nabla_{f\rho(k)}\bar{\theta} = \overline{[fk, \theta]} = \overline{f[k, \theta] - \rho(\theta)(f)k + 0} = f\overline{[k, \theta]} = f\nabla_{\rho(k)}\bar{\theta}.$$

It is easy to see that ∇ is $\mathbb{R}$-linear in $\Gamma(\mathsf{K}^{\perp}/\mathsf{K})$ and that for all $f \in C^{\infty}(C)$, $\theta \in \Gamma(\mathsf{K}^{\perp})$, $k \in \Gamma(\mathsf{K})$, we have

$$\nabla_{\rho(k)}(f\bar{\theta}) = \overline{[k, f\theta]} = \overline{f[k, \theta] + \rho(k)(f)\theta} = f\nabla_{\rho(k)}(\bar{\theta}) + \rho(k)(f)\bar{\theta}.$$

Finally, we verify that ∇ is flat. Indeed, for all $k_1, k_2 \in \Gamma(\mathsf{K})$, $\theta \in \Gamma(\mathsf{K}^{\perp})$, the Jacobi identity yields

$$\nabla_{\rho(k_1)}\nabla_{\rho(k_2)}\bar{\theta} - \nabla_{\rho(k_2)}\nabla_{\rho(k_1)}\bar{\theta} - \nabla_{\rho([k_1,k_2])}\bar{\theta} = \overline{[k_1, [k_2, \theta]] - [k_2, [k_1, \theta]] - [[k_1, k_2], \theta]} = 0.$$

□

Lemma 3.5 *Let A be a vector bundle over a manifold M and* $B \subset A$ *a vector subbundle of A over M. Let* ∇ *be a partial F-connection on A, where* $F \subseteq TM$ *is an involutive subbundle. Assume that*

$$\nabla_X b \in \Gamma(B) \tag{4}$$

for all vector fields $X \in \Gamma(F)$ *and* $b \in \Gamma(B)$*. For any curve* $c\colon [0, 1] \to M$ *with image in a leaf of F and* $\sigma(0) \in A_{c(0)}$*, denote by* $\sigma(t)\colon [0, 1] \to A$ *the parallel translation of* $\sigma(0)$ *along c. If* $\sigma(0) \in B_{c(0)}$ *then* $\sigma(t) \in B_{c(t)}$ *for all* $t \in [0, 1]$.

Proof Set $m := \dim M, n := \dim A_p, k := \dim F_p, r := \dim B_p$, where $p \in M$. Since the vector subbundle F is involutive, it is integrable by the Frobenius Theorem and thus any $p \in M$ lies in a foliated chart domain U described by coordinates $(x^1, \dots, x^m)$ such that the first k among them define the local integral submanifold containing p. Thus, for any $p' \in U$, the basis vector fields $\partial_{x^1}, \dots, \partial_{x^k}$ evaluated at p' span $F(p')$.

Take $p = c(0)$, choose a local basis frame $\{b_1, \dots, b_r\}$ for B defined on a neighborhood U of p. Assume, without loss of generality, that U is a coordinate neighborhood of M that is adapted to the foliation by the leaves of F and that the image of c lies in U (if not, divide c into curves lying in such coordinate neighborhoods).

Choose smooth sections $b_{r+1}, \dots, b_n \in \Gamma(A)$ such that $\{b_1, \dots, b_n\}$ is a frame for A on U (if necessary, we can shrink again U). We define the Christoffel symbols $\Gamma^j_{\alpha i}$ by

$$\nabla_{\partial_{x^\alpha}} b_i = \sum_{j=1}^{n} \Gamma^j_{\alpha i} b_j \qquad \text{for } \alpha = 1, \dots, k, \; i = 1, \dots, n.$$

Dividing the curve c in smaller pieces, if necessary, we may assume that there is a section a of A defined in a neighborhood of p such that $a \circ c = \sigma : [0, 1] \to A$. Writing $a = \sum_{j=1}^{n} f^j b_j$ for some $f^1, \dots, f^n \in C^\infty(M)$, we conclude that

$$\sigma(t) = a(c(t)) = \sum_{j=1}^{n} \sigma^j(t) b_j(c(t)), \text{ where } \sigma^j := f^j \circ c \text{ for } 1 \leq j \leq n$$

Furthermore, we can write $\dot{c}(t) = \sum_{\alpha=1}^{k} \dot{c}^\alpha(t) \frac{\partial}{\partial x_\alpha}$ since c is tangent to a leaf of F. If $\sigma : [0, 1] \to A$ is parallel along c, as in [15, Eq. (3.1.6)], it satisfies

$$\begin{aligned} 0 =&\nabla_{\dot{c}(t)}\sigma = \sum_{j=1}^{n} \left(\dot{\sigma}^j(t) b_j(c(t)) + \sigma^j(t) \sum_{\alpha=1}^{k} \dot{c}^\alpha(t) \sum_{i=1}^{n} \Gamma^i_{\alpha j}(c(t)) b_i(c(t)) \right) \\ &= \sum_{j=1}^{n} \left(\dot{\sigma}^j(t) + \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \dot{c}^\alpha(t) \sigma^i(t) \Gamma^j_{\alpha i}(c(t)) \right) b_j(c(t)). \end{aligned}$$

Hence, we get the system of ordinary differential equations

$$0 = \dot{\sigma}^j(t) + \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \dot{c}^\alpha(t) \sigma^i(t) \Gamma^j_{\alpha i}(c(t)) \quad \text{for all } j = 1, \dots, n. \tag{5}$$

Condition (4) on U means that

$$\Gamma^j_{\alpha i} = 0 \qquad \text{for } i \leq r, \quad j > r, \quad \alpha = 1, \dots, k.$$

Hence the equations in the system (5) for $j > r$ read

$$0 = \dot{\sigma}^j(t) + \sum_{\alpha=1}^{k} \sum_{i=r+1}^{n} \dot{c}^\alpha(t) \sigma^i(t) \Gamma^j_{\alpha i}(c(t)). \tag{6}$$

Since $a(0) \in B_{c(0)}$, we have $a^j(0) = 0$ for $j > r$. By the uniqueness of the solution to (6) with prescribed initial value, we conclude that

$$\sigma^j(t) \equiv 0 \quad \text{for } j > r,$$

which means that $\sigma(t) \in B_{c(t)}$ for all $t \in [0, 1]$. □

Thus, we recover the following corollary, which is a generalization of a result already proven in [19] and [14]; see Remark 3.7.

Corollary 3.6 *Let* $\mathsf{E} \to M$ *be an exact Courant algebroid and* C *a smooth submanifold of* M*. Let* $\mathsf{K} \subseteq \mathsf{E}|_C$ *be a vector subbundle of* E *over* C *satisfying the assumptions of Lemma 3.3, i.e.*

- K *is isotropic*
- $[\Gamma(\mathsf{K}), \Gamma(\mathsf{K})] \subseteq \Gamma(\mathsf{K})$ (*the inclusion* (2))
- $\rho(\mathsf{K}^\perp) = TC$.

Let $\mathsf{D} \subseteq \mathsf{K}^\perp$ *be a rank* r *subbundle of* $\mathsf{K}^\perp$ *on* C *such that* $\mathsf{D} \cap \mathsf{K}$ *has constant rank. Assume that*

$$[\Gamma(\mathsf{K}), \Gamma(\mathsf{D})] \subseteq \Gamma(\mathsf{K} + \mathsf{D}). \tag{7}$$

Then for each $p \in C$ *there exist an open set* $U \subseteq C$ *with* $p \in U$ *and a basis frame of smooth sections* $d_1, \ldots, d_r$ *of* D*, defined on* U *and satisfying*

$$[\Gamma(\mathsf{K}), d_i] \subseteq \Gamma(\mathsf{K}) \quad \text{on } U \quad \text{for all } i = 1, \ldots, r.$$

In other words, $\mathsf{D}_x = \operatorname{span}\{d_x | d \in \Gamma_{\text{bas}}(\mathsf{K}^\perp)_U \cap \Gamma(\mathsf{D})_U\}$ *for all* $x \in U$.

Proof Note first that since K is isotropic, we have $\mathsf{K} \subseteq \mathsf{K}^\perp$ and hence $\rho(\mathsf{K}) \subseteq \rho(\mathsf{K}^\perp) = TC$. As in the proof of Lemma 3.3, we get that $\rho(\mathsf{K})$ is a smooth involutive subbundle of TC, which is consequently integrable in the sense of Frobenius. Let $n := \dim(C)$. Choose $p \in C$ and a foliated chart domain U centered at p and described by coordinates $(x^1, \ldots, x^n)$ such that the first k among them define the local integral submanifold of $\rho(\mathsf{K})$ containing p. Let $S \subseteq U$ be the slice $\phi^{-1}(\{0\} \times \mathbb{R}^{n-k})$, where $\phi : U \to \mathbb{R}^n$ is the chart adapted to the foliation.

Denote $l := \operatorname{rank}((\mathsf{D}+\mathsf{K})/\mathsf{K})$. Choose $e_1, \ldots, e_l \in \Gamma(\mathsf{D})$ such that $\bar{e}_1, \ldots, \bar{e}_l \in \Gamma(\mathsf{K}^\perp/\mathsf{K})$ is a basis frame for $(\mathsf{D}+\mathsf{K})/\mathsf{K}$ on U. We consider this frame at points of $S \cap U$ and construct $\bar{d}_1, \ldots, \bar{d}_l \in \Gamma((\mathsf{D}+\mathsf{K})/\mathsf{K})$ as follows. If $q \in U$, $\phi(q) = (x_1, \ldots, x_n)$, then we find a path $c : [0, 1] \to \phi^{-1}(\mathbb{R}^r \times \{(x_{r+1}, \ldots, x_n)\})$ (the leaf of $\rho(\mathsf{K})$ through q) with $c(1) = q$ and $c(0) = q' \in S$ satisfying $\phi(q') = (0, \ldots, 0, x_{r+1}, \ldots, x_n)$. Define $\bar{d}_i(q) := P_c^1(\bar{e}_i(q'))$, where $P_c^1(\bar{e}_i(q'))$ is the parallel translate of $\bar{e}_i(q')$ along c at time 1 by the $\rho(\mathsf{K})$-partial connection ∇ defined in Lemma 3.3. Since U is simply connected and the connection ∇ is flat, parallel translation is independent of the chosen path (see, for example, [13]), hence the $\bar{d}_i$ are ∇-parallel sections of $\mathsf{K}^\perp/\mathsf{K}$.

Since, by hypothesis, $[\Gamma(\mathsf{K}), e_i] \subseteq \Gamma(\mathsf{K} + \mathsf{D})$ for $i = 1, \ldots, l$, we have $\nabla_{\rho(k)}\bar{e}_i \in \Gamma((\mathsf{D}+\mathsf{K})/\mathsf{K})$ for all $k \in \Gamma(\mathsf{K})$. Lemma 3.5, applied to $(\mathsf{D}+\mathsf{K})/\mathsf{K} \subset \mathsf{K}^\perp/\mathsf{K}$, implies that the sections $\bar{d}_i$ lie in $(\mathsf{D}+\mathsf{K})/\mathsf{K}$. Hence we get l parallel sections $\bar{d}_1, \ldots, \bar{d}_l \in \Gamma((\mathsf{D}+\mathsf{K})/\mathsf{K})$ that form a point-wise basis of $(\mathsf{D} + \mathsf{K})/\mathsf{K}$ on U. Choose representatives $d_1, \ldots, d_l \in \Gamma(\mathsf{D})$ for $\bar{d}_1, \ldots, \bar{d}_l$. Since $\nabla_{\rho(k)}\bar{d}_i = 0$ for $i = 1, \ldots, l$ and all $k \in \Gamma(\mathsf{K})$, we have $[\Gamma(\mathsf{K}), d_i] \subseteq \Gamma(\mathsf{K})$ for $i = 1, \ldots, l$. Take $d_{l+1}, \ldots, d_r$ to be a frame of $\mathsf{D} \cap \mathsf{K}$ over U. Then $d_1, \ldots, d_r$ is a frame of D over U composed of basic sections. □

Remark 3.7 Choose the Pontryagin bundle P_M as the ambient Courant algebroid. Let $\mathcal{V} \subset TM$ be an involutive rank k vector subbundle of TM and $\mathcal{D}$ a rank r subbundle of TM such that $\mathcal{D} \cap \mathcal{V}$ has constant rank on M. Then $\mathcal{V} \oplus \{0\} \subseteq \mathsf{P}_M$ is isotropic and its orthogonal $TM \oplus \mathcal{V}^\circ$ satisfies $\operatorname{pr}_{TM}(TM \oplus \mathcal{V}^\circ) = TM$. Assume that

$$[\Gamma(\mathcal{D}), \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V} + \mathcal{D}).$$

Then the preceding corollary yields the following, which was already shown in [19], see also [14].

For each $p \in M$ there is an open set $U \subseteq M$ with $p \in U$ and smooth $\mathcal{D}$-valued vector fields $Z_1, \ldots, Z_r$ on U satisfying

(i) $\mathcal{D}(q) = \operatorname{span}\{Z_1(q), \ldots, Z_r(q)\}$ for all $q \in U$,
(ii) $[Z_i, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ on U for all $i = 1, \ldots, r$.

In the appendix of [16] it is shown, following [10], that it is possible to drop the hypothesis on the constant rank of $\mathcal{D} \cap \mathcal{V}$.

Remark 3.8 Let K be an isotropic subbundle over C such that $\rho(\mathsf{K}^{\perp}) = TC$. The following three conditions are equivalent:

$\mathsf{K}^{\perp}$ is spanned by its (local) sections that are basic relative to K

$$[\Gamma(\mathsf{K}), \Gamma(\mathsf{K}^{\perp})] \subseteq \Gamma(\mathsf{K}^{\perp}), \tag{1}$$

$$[\Gamma(\mathsf{K}), \Gamma(\mathsf{K})] \subseteq \Gamma(\mathsf{K}). \tag{2}$$

Indeed (1) and (2) are equivalent by part 2 of Remark 3.1. If we assume (2) and apply Corollary 3.6 to $\mathsf{D} := \mathsf{K}^{\perp}$, we conclude that the vector bundle $\mathsf{K}^{\perp}$ is spanned by its (local) basic sections. Conversely, the latter condition implies (1) by using axiom 3) in the definition of Courant algebroid. The stronger condition that $\mathsf{K}^{\perp}$ is spanned by its global sections that are basic (needed in Theorem 2.3) is equivalent to saying that the flat partial connection defined in Lemma 3.3 has no holonomy.

4 Application to Dirac reduction

In this section we recall a statement about Dirac reduction by distributions (Theorem 4.1) that appeared in [22] and fill a gap in the proof. Then we infer a Dirac reduction theorem by Lie group symmetries (Theorem 4.3) that is more general than similar statements found in the literature.

Dirac reduction by distributions Let E be an exact Courant algebroid over M. We saw in Sect. 2 that an (integrable) Dirac structure is a maximal isotropic subbundle of E which is closed under the Courant bracket.

Theorem 4.1 (Dirac reduction by distributions) *Assume that the vector bundles* $\mathsf{E} \to M$ *and* $\mathsf{K} \to C$ *satisfy the assumptions of Theorem 2.3, so that we have an exact Courant algebroid* $\underline{\mathsf{E}} \to \underline{C}$. *Let* L *be a maximal isotropic subbundle of* $\mathsf{E}|_C$ *such that* $\mathsf{L} \cap \mathsf{K}^{\perp}$ *has constant rank, and assume that*

$$\left[\Gamma(\mathsf{K}), \Gamma\left(\mathsf{L} \cap \mathsf{K}^{\perp}\right)\right] \subset \Gamma(\mathsf{L} + \mathsf{K}). \tag{8}$$

Then L *descends to a maximal isotropic subbundle* $\underline{\mathsf{L}}$ *of* $\underline{\mathsf{E}} \to \underline{C}$. *If furthermore*

$$\left[\Gamma_{\text{bas}}\left(\mathsf{L} \cap \mathsf{K}^{\perp}\right), \Gamma_{\text{bas}}\left(\mathsf{L} \cap \mathsf{K}^{\perp}\right)\right] \subset \Gamma(\mathsf{L} + \mathsf{K}). \tag{9}$$

then $\underline{\mathsf{L}}$ *is an integrable Dirac structure. Here,* $\Gamma_{\text{bas}}\left(\mathsf{L} \cap \mathsf{K}^{\perp}\right) := \Gamma(\mathsf{L}) \cap \Gamma_{\text{bas}}(\mathsf{K}^{\perp})$

Remark 4.2 In [22] conditions (8) and (9) are required for *global* sections. Requiring them for global sections is equivalent to requiring them for local sections.

The above theorem is given in [22, Prop 4.1]. One step of the proof – the well-definedness of $\underline{\mathsf{L}}$ – needs to be added. We present below the complete proof.

Proof At every $p \in C$ we have a Lagrangian relation between E_p and $(\mathsf{K}^\perp/\mathsf{K})_p$ given by $\{(e, e+\mathsf{K}_p) : e \in \mathsf{K}^\perp{}_p\}$. The image $\underline{\mathsf{L}}_p$ of L_p under this relation is maximal isotropic because L_p has these properties. Thus we obtain a maximally isotropic subbundle $\underline{\mathsf{L}}$ of $\mathsf{K}^\perp/\mathsf{K}$. This subbundle $\underline{\mathsf{L}}$ is smooth because L_p is the image of $(\mathsf{L}\cap\mathsf{K}^\perp)_p$ by the projection $\mathsf{K}^\perp{}_p \to (\mathsf{K}^\perp/\mathsf{K})_p$, which has constant rank by assumption. (Note that since L is Lagrangian and $\mathsf{L}\cap\mathsf{K}^\perp$ has constant rank, we get easily the fact that $\mathsf{L}\cap\mathsf{K}$ has constant rank.)

The assumptions of Lemma 3.3 hold (see Remark 3.8) and hence

$$\nabla_{\rho(k)}\overline{\theta} := \overline{[k,\theta]} \qquad \text{for } k \in \Gamma(\mathsf{K}),\ \theta \in \Gamma(\mathsf{K}^\perp)$$

defines a flat, $\rho(\mathsf{K})$-connection on $\mathsf{K}^\perp/\mathsf{K} \to C$. Here $\overline{\theta}$ is the image of θ under the projection $\mathsf{K}^\perp \to \mathsf{K}^\perp/\mathsf{K}$.

Denote by $\kappa : \mathsf{K}^\perp \to \mathsf{K}^\perp/\mathsf{K}$ the projection. Let p and q lie in the same leaf of $\rho(\mathsf{K})$. In Theorem 2.3 we identified the fibers $(\mathsf{K}^\perp/\mathsf{K})_p$ and $(\mathsf{K}^\perp/\mathsf{K})_q$ in the following manner (see the proof of Theorem 3.7 of [22]): $\hat{e}(p) \in (\mathsf{K}^\perp/\mathsf{K})_p$ and $\hat{e}(q) \in (\mathsf{K}^\perp/\mathsf{K})_q$ are identified if and only if there is a global section $e \in \Gamma_{\text{bas}}(\mathsf{K}^\perp)$ such that $(\kappa\circ e)(p) = \hat{e}(p)$ and $(\kappa\circ e)(q) = \hat{e}(q)$. Since $e \in \Gamma_{\text{bas}}(\mathsf{K}^\perp)$, it follows that $\overline{e}$ is a ∇-parallel section of $\mathsf{K}^\perp/\mathsf{K}$. Hence $\hat{e}(q)$ is the parallel transport of $\hat{e}(p)$ by ∇ along any curve c from p to q lying in the leaf integrating $\rho(\mathsf{K})$.

Now assume that $\hat{e}(p) \in \underline{\mathsf{L}}_p = (\mathsf{L}\cap\mathsf{K}^\perp)_p/(\mathsf{L}\cap\mathsf{K})_p$. We can apply Lemma 3.5 to $A := \mathsf{K}^\perp/\mathsf{K}$, $B := (\mathsf{L}\cap\mathsf{K}^\perp)/(\mathsf{L}\cap\mathsf{K})$, and ∇, because (8) implies that condition (4) is satisfied. Lemma 3.5 implies that $\hat{e}(q)$, the parallel transport by ∇ along c of $\hat{e}(p)$, lies in $\underline{\mathsf{L}}_q$. Hence the identification $\mathsf{K}^\perp/\mathsf{K}_p \cong \mathsf{K}^\perp/\mathsf{K}_q$ maps $\underline{\mathsf{L}}_p$ to $\underline{\mathsf{L}}_q$. Consequently, we obtain a well-defined smooth maximally isotropic subbundle $\underline{\mathsf{L}}$ of the reduced Courant algebroid $\underline{\mathsf{E}}$, i.e., an $\underline{\mathsf{E}}$-almost Dirac structure.

Now assume that (9) holds and take two sections $\underline{e}_1, \underline{e}_2$ of $\underline{\mathsf{L}}$. Since $\mathsf{L}\cap\mathsf{K}^\perp$ has constant rank we can lift them to sections e_1, e_2 of $\Gamma_{\text{bas}}(\mathsf{L}\cap\mathsf{K}^\perp)$. As for all elements of $\Gamma_{\text{bas}}(\mathsf{K}^\perp)$, their bracket lies in $\Gamma_{\text{bas}}(\mathsf{K}^\perp)$. Since, by assumption, it also lies in $\mathsf{L}+\mathsf{K}$, it follows that $[e_1, e_2]$ is a basic section of $(\mathsf{L}+\mathsf{K})\cap\mathsf{K}^\perp = (\mathsf{L}\cap\mathsf{K}^\perp)+\mathsf{K}$. Its projection under $\mathsf{K}^\perp/\mathsf{K} \to \underline{\mathsf{E}}$ which is, by definition, the bracket of $\underline{e}_1$ and $\underline{e}_2$, lies then in $\underline{\mathsf{L}}$. □

Regular Dirac reduction by a Lie group action We consider a smooth manifold M and the exact Courant algebroid

$$\mathsf{P}_M = TM \oplus T^*M$$

over M. Let $\Phi : G \times M \to M$ be a proper action of the *connected* Lie group G and assume that all isotropy subgroups are conjugated. Let $\mathcal{V}$ be the vertical bundle of the action, that is, the subbundle of TM spanned at every point by the values of the fundamental vector fields ξ_M for all $\xi \in \mathfrak{g}$. Define $\mathsf{K} = \mathcal{V} \oplus \{0\}$, so its orthogonal is $\mathsf{K}^\perp = TM \oplus \mathcal{V}^\circ$. Then K is isotropic and $\rho(\mathsf{K}^\perp) = TM$. Since G is connected, the orbit space $\bar{M} := M/G$ is equal to the space of leaves of the Frobenius integrable subbundle $\mathcal{V} \subseteq TM$, $\bar{M} = M/G = M/\mathcal{V}$. Since all the isotropy subgroups are conjugated, it inherits a smooth manifold structure such that the quotient map $\pi : M \to \bar{M}$ is a regular submersion.

In addition, $\pi : M \to \bar{M}$ is a locally trivial fiber bundle with fiber the orbit of $G\cdot m$ and structure group $N(H)/H$, where $H = G_m$ is the isotropy group at $m \in M$ and $N(H)$ is its normalizer. Since all isotropy groups are conjugated, the previous statement is independent of m.

A section $\theta = (X, \alpha) \in \Gamma(\mathsf{K}^\perp) \simeq \mathfrak{X}(M) \times \Gamma(\mathcal{V}^\circ)$ is basic (here, we say also *descending*) if it satisfies $[(\xi_M, 0), (X, \alpha)] \in \Gamma(\mathsf{K})$ for all $\xi \in \mathfrak{g}$. Hence, if $(X, \alpha) \in \Gamma(\mathsf{K}^\perp)$ is descending, we have

$$[(\xi_M, 0), (X, \alpha)] = ([\xi_M, X], \pounds_{\xi_M}\alpha) \in \Gamma(\mathsf{K}) = \{(V, 0) \in \mathfrak{X}(M) \times \Omega^1(M) \mid V \in \Gamma(\mathcal{V})\}.$$

Therefore, $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ and $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ by the Leibniz identity since $\{\xi_M \mid \xi \in \mathfrak{g}\}$ spans $\Gamma(\mathcal{V})$ as a $C^\infty(M)$-module.

If $X \in \mathfrak{X}(M)$ is a smooth vector field satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$, then there exists $\bar{X} \in \mathfrak{X}(\bar{M})$ such that $X \sim_\pi \bar{X}$ and hence X can be written as a sum $X = X^G + X^{\mathcal{V}}$ with $X^G \in \mathfrak{X}(M)^G$ and $X^{\mathcal{V}} \in \Gamma(\mathcal{V})$ (see [17]); $\mathfrak{X}(M)^G := \{X \in \mathfrak{X}(M) \mid \Phi_g^* X = X \text{ for all } g \in G\}$. If α is a G-invariant local section of T^*M annihilating the vertical spaces $\mathcal{V}(m)$ for all $m \in \mathrm{Dom}(\alpha)$, then there exists a unique $\bar{\alpha} \in \Omega^1(\bar{M})$ such that $\alpha = \pi^*\bar{\alpha}$. Hence, a descending section of $\mathsf{K}^\perp$ *pushes forward* to the quotient M/G. Using Example 3.9 in [22], we know that $\mathsf{K}^\perp$ is spanned by its global basic sections.

Let D be a Dirac structure on M. The Lie group G is called a *symmetry Lie group of* (M, D) if for every $g \in G$ the condition $(X, \alpha) \in \Gamma(\mathsf{D})$ implies that $\left(\Phi_g^* X, \Phi_g^* \alpha\right) \in \Gamma(\mathsf{D})$ (here we use the convention $\Phi_g^* X = T\Phi_{g^{-1}} \circ X \circ \Phi_g$). We say then that the action of G on (M, D) is *Dirac* or *canonical*.

Let $\mathfrak{g}$ be the Lie algebra of G and $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ be the smooth left Lie algebra action induced by the (Dirac) action of G on M. Then the Lie algebra $\mathfrak{g}$ is a *symmetry Lie algebra of* (M, D): for every $\xi \in \mathfrak{g}$ the condition $(X, \alpha) \in \Gamma(\mathsf{D})$ implies that $\left(\pounds_{\xi_M} X, \pounds_{\xi_M}\alpha\right) \in \Gamma(\mathsf{D})$.

Moreover, if (X, α) is a section of $\mathsf{D} \cap \mathsf{K}^\perp$, then $(\pounds_{\xi_M} X, \pounds_{\xi_M}\alpha) = ([\xi_M, X], \pounds_{\xi_M}\alpha) = [(\xi_M, 0), (X, \alpha)] \in \Gamma(\mathsf{D} \cap \mathsf{K}^\perp)$. Since $\Gamma(\mathsf{K})$ is spanned as a $C^\infty(M)$-module by $\{(\xi_M, 0) \mid \xi \in \mathfrak{g}\}$, we get Eq. (8) with the Leibniz identity:

$$[\Gamma(\mathsf{K}), \Gamma(\mathsf{D} \cap \mathsf{K}^\perp)] \subseteq \Gamma(\mathsf{D} \cap \mathsf{K}^\perp + \mathsf{K}).$$

If D is integrable, then the bracket of two descending sections of D is again descending. Indeed, if $(X, \alpha), (Y, \beta) \in \Gamma(\mathsf{D}) \cap \left(\mathfrak{X}(M) \times \Gamma(\mathcal{V}^\circ)^G\right)$ are such that $[\Gamma(\mathcal{V}), X] \subseteq \Gamma(\mathcal{V})$ and $[\Gamma(\mathcal{V}), Y] \subseteq \Gamma(\mathcal{V})$, then it is easy to see that the bracket $[(X, \alpha), (Y, \beta)]$ is a descending section of P_M. Since D is integrable, it is also a section of D. Thus, if D is integrable, since $\Gamma_{\mathrm{bas}}(\mathsf{K}^\perp)$ is closed with respect to the bracket, we have $[\Gamma_{\mathrm{bas}}(\mathsf{D} \cap \mathsf{K}^\perp), \Gamma_{\mathrm{bas}}(\mathsf{D} \cap \mathsf{K}^\perp)] \subseteq \Gamma_{\mathrm{bas}}(\mathsf{D} \cap \mathsf{K}^\perp)$, which implies Eq. (9).

Assume that the G-action on the Dirac manifold (M, D) is canonical, free, and proper. Then both vector bundles D and $\mathsf{K}^\perp$ are G-invariant and it is shown in [16] (following [2]) that, under the assumption that $\mathsf{D} \cap \mathsf{K}^\perp$ is a vector bundle on M, the quotient bundle

$$\mathsf{D}_{\mathrm{red}} = \left. \frac{(\mathsf{D} \cap \mathsf{K}^\perp) + \mathsf{K}}{\mathsf{K}} \middle/ G \right. \tag{10}$$

defines a Dirac structure on M/G, called the *reduced* Dirac structure.

Historically, the first method to reduce Dirac structures is due to [4] and [7] (see [6] for a corresponding singular reduction method). The reduced Dirac structure on M/G is given by

$$\bar{\mathsf{D}}(\bar{m}) = \left\{ (\bar{X}, \bar{\alpha})(\bar{m}) \in \Gamma(T\bar{M} \oplus T^*\bar{M}) \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(M) \text{ such that } X \sim_\pi \bar{X} \\ \text{and } (X, \pi^*\bar{\alpha}) \in \Gamma(\mathsf{D}) \end{array} \right\} \tag{11}$$

for all $\bar{m} \in M/G$. Although this is just the formulation of (10) in terms of smooth sections, the proofs in [4] and [7] use an additional hypothesis in order to guarantee that the construction above yields a Dirac structure: $\mathcal{V} + \mathsf{G}_0$ and G_0 have constant rank on M, where

$\mathsf{G}_0 \subseteq TM$ is the smooth generalized distribution induced by D and

$$\mathsf{G}_0(m) := \{X(m) \mid X \in \mathfrak{X}(M) \text{ is such that } (X, 0) \in \Gamma(\mathsf{D})\}$$

for all $m \in M$. This, together with the involutivity of the vector subbundle $\mathcal{V}$, is needed in their proof in order to be able to use results in [19] and [14].

Theorem 4.1 is the analogue of the reduction theorem of [2] in the context of reduction by smooth distributions. We show now, as a consequence of Theorem 4.1 and the considerations above, that the Dirac reduction theorems of [2] (in the case of a free and proper action on the underlying manifold M) and [4] are valid under weaker assumptions.

Theorem 4.3 *Assume that the Lie group G acts properly and canonically on the Dirac manifold (M, D) with all its isotropy subgroups conjugated. If $\mathsf{D} \cap \mathsf{K}^{\perp}$ has constant rank on M, then the Dirac structure D on M induces a Dirac structure $\bar{\mathsf{D}}$ on $\bar{M}$ whose fiber at every point $\bar{m} \in \bar{M}$ is given by*

$$\bar{\mathsf{D}}(\bar{m}) = \left\{ \left(\bar{X}(\bar{m}), \bar{\alpha}(\bar{m})\right) \in T_{\bar{m}}\bar{M} \times T^*_{\bar{m}}\bar{M} \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(M) \text{ such that } X \sim_\pi \bar{X} \\ \text{and } (X, \pi^*\bar{\alpha}) \in \Gamma(\mathsf{D}) \end{array} \right\}. \tag{12}$$

If D is integrable, then $\bar{\mathsf{D}}$ is also integrable.

Proof By the considerations above, all the hypotheses in Theorem 4.1 applied to $C = M$, $\mathsf{E} = \mathsf{P}_M$, $\mathsf{K} = \mathcal{V} \oplus \{0\}$, and the Dirac structure D, are satisfied. Since the ambient Courant algebroid is the Pontryagin bundle P_M on M, the reduced Courant algebroid is the Pontryagin bundle on $\bar{M}$ and we get a Dirac structure on $\bar{M}$ that is given by (12). □

Remark 4.4 Since $\mathsf{D} \cap \mathsf{K}^{\perp}$ is assumed to have constant rank on M and D is maximal isotropic, the intersection $\mathsf{D} \cap \mathsf{K}$ automatically has constant rank on M. Using Corollary 3.6, it is also possible to prove Theorem 4.3 exactly in the same manner as in [4,7], but without the assumption on G_0 and $\mathsf{G}_0 + \mathcal{V}$ to have constant rank on M. The assumption that the action is free can be weakened to the hypothesis that the isotropy subgroups are all conjugated.

Acknowledgments M. Jotz and T.S. Ratiu are Partially supported by Swiss NSF grant 200021-121512. M. Zambon is Partially supported by CMUP and FCT, by grants PTDC/MAT/098770/2008 and PTDC/MAT/099880/2008 (Portugal) and by grants MICINN RYC-2009-04065 and MTM2009-08166-E (Spain).

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