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SINGULAR REDUCTION OF DIRAC STRUCTURES

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Abstract. The regular reduction of a Dirac manifold acted upon freely and properly by a Lie group is generalized to a nonfree action. For this, several facts about G-invariant vector fields and one-forms are shown.

1. Introduction

Dirac structures generalize Poisson and symplectic manifolds. They also provide a convenient geometric setting for the theory of nonholonomic systems. This concept was introduced in [9] and [10] and has seen a significant development in the recent past both from the geometric point of view as well as in applications to mechanical systems and circuit theory. In the presence of symmetry, one can perform a reduction to eliminate variables. The modern global formulation of reduction of Hamiltonian systems with symmetry is due to Marsden and Weinstein [23] who treat free and proper symplectic actions admitting an equivariant momentum map. This was generalized to Poisson manifolds in [22]. When dealing with implicit Hamiltonian systems, which can be seen as sets of algebraic and differential equations, the geometric description is based on Dirac structures. Hence it is natural to ask if a symmetric Dirac manifold can be reduced. This was carried out for a free and proper Dirac action in [6] and [4] within the context of generalized Poisson structures and can be derived as an easy case of the results in the paper [8] about reduction of Courant algebroids. The methods of [6] and [8] in the particular case of interest to us are equivalent up to a small difference in assumptions (see [16]). It is shown in [19] that the assumptions in [6] can be weakened to the hypotheses of [8] in the case of a free and proper action on the underlying manifold M. Singular Dirac reduction was treated in [5] using the following setup: the symmetric Dirac structure is viewed as a generalized Poisson structure with a momentum map and a reduction of implicit Hamiltonian systems is performed at all values of the momentum map, including singular ones. It turns out that each stratum of the reduced space (which is a Whitney stratified cone space since the action is proper) inherits a Dirac structure. In addition, the Hamiltonian dynamics on the original manifold descend to each stratum of the quotient.

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In this paper, we study the reduction of a smooth Dirac manifold (M, D) by a proper Dirac action of a Lie group G completely within the Dirac category; that is, certain nontrivial technical hypotheses on various distributions present in [6], [4] and [8] (in the case of interest to us) are eliminated. This is achieved by working directly with smooth structures on stratified spaces. This approach, known as singular reduction, was initiated in [12] and formalized in [1]. In [11], singular reduction was shown to be an application of the theory of differential spaces.

The concepts of vector fields and one-forms on Whitney stratified spaces are reviewed and applied to the quotient space of the manifold by the action. We show in Theorem 6.4 that the descending sections of the Dirac structure push forward to a subset of the pairs formed by local vector fields and one-forms on the reduced space that is, in a sense, self-orthogonal. This leads to the following natural question: do the strata of the reduced space inherit Dirac structures induced by D? We show in Theorem 6.5 that this is true if one assumes that the set of descending sections generates a certain subdistribution of the Pontryagin bundle of M. To achieve this, we employ several new techniques. Using the existence of G-invariant Riemannian metrics for proper actions on paracompact manifolds and the tube theorem, we introduce averages of vector fields and one-forms on G-invariant open subsets of M. In a crucial step of the proof, we use the fact that, in certain situations, G-invariant averages of vector fields and one-forms vanish. Also, we study the relationship between the pointwise and the smooth orthogonal distribution of a smooth generalized subdistribution of a vector bundle endowed with a symmetric nondegenerate pairing. This allows us to describe the smooth annihilator of an intersection of smooth generalized distributions in certain cases of interest to us.

The paper is organized as follows. Dirac structures are reviewed in Section 2 and vector fields on differential spaces, stratifications, and orbit type manifolds in Section 3. Generalized distributions and the integrability of tangent distributions as well as pointwise and smooth annihilators are introduced and discussed in Section 4. The averaging method is presented in Section 5. Using this technique, we show that the strata of the quotient $\bar{M}$ correspond to the quotients of the orbit type strata on M and that the local one-forms on the manifold M descend to analogous objects on the reduced stratified space $\bar{M}$. Then we study the properties of descending sections of the Pontryagin bundle and get many technical results needed in the final reduction proof. Section 6 is devoted to the main result of the paper, namely singular Dirac reduction. First we recall the reduction procedure in the case of conjugated isotropy subgroups. Then the two main theorems of the paper (Theorems 6.4 and 6.5) are proved and the reduced dynamics of implicit Hamiltonian systems is constructed. Several examples are also given.

Conventions, definitions, and notation. In this paper we are working in the *smooth category*. All sets considered here are smooth *subcartesian spaces*; see Section 2. In particular, all manifolds and maps are assumed to be smooth. Moreover, the manifold M is *paracompact* and the Lie group G acting on it is *connected*. If not mentioned in the text, the action of G on M is always assumed to be *proper*.

We will write $C^\infty(M)$ for the sheaf of local functions $C^\infty_{\mathrm{loc}}(M)$ on M. That is, an element $f \in C^\infty(M)$ is a smooth function $f : U \to \mathbb{R}$, with U an open subset of M. In the same manner, if E is a vector bundle over M, or a generalized distribution on M, we will denote by $\Gamma(E)$ the set of local sections of E. In particular, the sets of local vector fields and one-forms on M will be denoted by $\mathfrak{X}(M)$ and $\Omega^1(M)$, respectively. We will write $\mathrm{Dom}(\sigma)$ for the open domain of definition of the section σ of E.

A section X (respectively α) of TM (respectively T^*M) will be called *G-invariant* if $\Phi_g^* X = X$ (respectively $\Phi_g^*\alpha = \alpha$) for all $g \in G$, where $\Phi : G \times M \to M$ is the action of G on M. Here, the vector field $\Phi_g^* X$ is defined by $\Phi_g^* X = T\Phi_{g^{-1}} \circ X \circ \Phi_g$, that is, $(\Phi_g^* X)(m) = T_{gm}\Phi_{g^{-1}} X(gm)$ for all $m \in M$.

Recall that a subset $N \subset M$ is an *initial* submanifold of M if N carries a manifold structure such that the inclusion $\iota : N \hookrightarrow M$ is a smooth immersion and satisfies the following condition: for any smooth manifold P, an arbitrary map $g : P \to N$ is smooth if and only if $\iota \circ g : P \to M$ is smooth. The notion of initial submanifold lies strictly between those of injectively immersed and embedded submanifolds.

In the following, we write $TM \ominus TM^*$ for the sum of the vector bundles TM and T^*M and use the same notation for the sum of a tangent (that is, a subdistribution of TM) and cotangent distribution (a subdistribution of T^*M; see Section 4 for the definitions of those objects). We choose this notation because we want to distinguish these direct sums from direct sums of subdistributions of a bundle, which will be written as usual with $\oplus$.

2. Generalities on Dirac structures

Dirac structures. The *Pontryagin bundle* $TM \ominus T^*M$ of a smooth manifold M is endowed with a nondegenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by

$$\langle\langle (u_m, \alpha_m), (v_m, \beta_m) \rangle\rangle := \langle \beta_m, u_m \rangle + \langle \alpha_m, v_m \rangle \tag{2.1}$$

for all $u_m, v_m \in T_m M$ and $\alpha_m, \beta_m \in T_m^* M$. A *Dirac structure* (see [9]) on M is a Lagrangian subbundle $D \subset TM \ominus T^*M$. That is, D coincides with its orthogonal relative to (2.1) and so its fibers are necessarily $\dim M$-dimensional.

The space $\Gamma(TM \ominus T^*M)$ of local sections of the Pontryagin bundle is also endowed with an $\mathbb{R}$-bilinear skew-symmetric bracket (which does not satisfy the Jacobi identity) given by

$$\begin{aligned}[][(X,\alpha),(Y,\beta)] &:= \left([X,Y], \pounds_X \beta - \pounds_Y \alpha + \frac{1}{2}\mathbf{d}\left(\alpha(Y) - \beta(X)\right) \right) \\ &= \left([X,Y], \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha - \frac{1}{2}\mathbf{d}\left\langle\left\langle (X,\alpha),(Y,\beta)\right\rangle\right\rangle \right) \end{aligned} \tag{2.2}$$

(see [9]). The Dirac structure is *integrable* or *closed* if $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$. Since $\langle\langle (X,\alpha),(Y,\beta)\rangle\rangle = 0$ if $(X,\alpha),(Y,\beta) \in \Gamma(D)$, integrability of the Dirac structure is often expressed in the literature relative to a non-skew-symmetric bracket that differs from (2.2) by eliminating in the second line the third term of the second component. This truncated expression which satisfies the Jacobi identity but is no longer skew-symmetric is called the *Courant bracket*:

$$[(X,\alpha),(Y,\beta)] := \left([X,Y], \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha \right). \tag{2.3}$$

Symmetries of Dirac manifolds. Let G be a Lie group and $\Phi : G \times M \to M$ a smooth left action. Then G is called a *symmetry Lie group of* D if for every $g \in G$ the condition $(X,\alpha) \in \Gamma(D)$ implies that $\left(\Phi_g^* X, \Phi_g^*\alpha\right) \in \Gamma(D)$. We say then that the Lie group G acts *canonically* or *by Dirac actions* on M.

Let $\mathfrak{g}$ be a Lie algebra and $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ be a smooth left Lie algebra action; that is, the map $(x,\xi) \in M \times \mathfrak{g} \mapsto \xi_M(x) \in TM$ is smooth and $\xi \in \mathfrak{g} \mapsto$

$\xi_M \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism. The Lie algebra $\mathfrak{g}$ is said to be a *symmetry Lie algebra of* D if for every $\xi \in \mathfrak{g}$ the condition $(X, \alpha) \in \Gamma(D)$ implies that $(\pounds_{\xi_M} X, \pounds_{\xi_M} \alpha) \in \Gamma(D)$. Of course, if $\mathfrak{g}$ is the Lie algebra of G and $\xi \mapsto \xi_M$ the Lie algebra anti-homomorphism, then if G is a symmetry Lie group of D it follows that $\mathfrak{g}$ is a symmetry Lie algebra of D.

3. Differential spaces

3.1. **Subcartesian spaces.** A differential structure on a topological space S is a family $C^\infty_{\text{glob}}(S)$ of real-valued functions on S such that:

A1. The family

$$\{f^{-1}((a,b)) \mid f \in C^\infty_{\text{glob}}(S),\ a, b \in \mathbb{R}\}$$

is a subbasis for the topology on S.

A2. If $f_1, ..., f_n \in C^\infty_{\text{glob}}(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, ..., f_n) \in C^\infty_{\text{glob}}(S)$.

A3. If $f : S \to \mathbb{R}$ is such that, for every $x \in S$, there exist an open neighborhood U_x of x and a function $f_x \in C^\infty_{\text{glob}}(S)$ satisfying

$$f_x|_{U_x} = f|_{U_x},$$

then $f \in C^\infty_{\text{glob}}(S)$.

Here the vertical bar $|$ denotes the restriction. Note that we write $C^\infty_{\text{glob}}(S)$ to distinguish this set of functions, whose elements are defined on the whole of S, from sheaves of smooth functions if the space is also endowed with a smooth structure.

A differential space is a space S endowed with a differential structure $C^\infty_{\text{glob}}(S)$. Clearly, smooth manifolds are differential spaces. However, the category of differential spaces is much larger than the category of manifolds.

Let R and S be differential spaces with differential structures $C^\infty_{\text{glob}}(R)$ and $C^\infty_{\text{glob}}(S)$, respectively. A map $\phi : R \to S$ is said to be smooth if $\phi^*(f) = f \circ \phi \in C^\infty_{\text{glob}}(R)$ for all $f \in C^\infty_{\text{glob}}(S)$. A smooth map between differential spaces is a diffeomorphism if it is invertible and its inverse is smooth.

If R is a differential space with differential structure $C^\infty_{\text{glob}}(R)$ and S is a subset of R, then we can define a differential structure $C^\infty_{\text{glob}}(S)$ on S as follows. A function $f : S \to \mathbb{R}$ is in $C^\infty_{\text{glob}}(S)$ if and only if, for every $x \in S$, there is an open neighborhood U of x in R and a function $f_x \in C^\infty_{\text{glob}}(R)$ such that $f|_{S\cap U} = f_x|_{S\cap U}$. The differential structure $C^\infty_{\text{glob}}(S)$ described above is the smallest differential structure on S such that the inclusion map $\iota : S \to R$ is smooth. We shall refer to S with the differential structure $C^\infty_{\text{glob}}(S)$ described above as a differential subspace of R. If S is a closed subset of R, then the differential structure $C^\infty_{\text{glob}}(S)$ described above consists of restrictions to S of functions in $C^\infty_{\text{glob}}(R)$.

A differential space R is said to be locally diffeomorphic to a differential space S if, for every $x \in R$, there exists a neighborhood U of x diffeomorphic to an open subset V of S. More precisely, we require that the differential subspace U of R is diffeomorphic to the differential subspace V of S. A differential space R is a smooth manifold of dimension n if and only if it is locally diffeomorphic to $\mathbb{R}^n$. A Hausdorff differential space that is locally diffeomorphic to subsets of $\mathbb{R}^n$ is called a *subcartesian space*. In the following, we restrict our considerations to subcartesian spaces.

Differential spaces were introduced in [29]; see also [30] and [31]. The original definition of a subcartesian space, in terms of a singular atlas, was given in [2].

The characterization of subcartesian spaces used here can be found in [39], where the term *differential spaces of class* D_0 is used. A comprehensive bibliography of differential spaces is given in [7].

3.2. **Vector fields.** In this subsection, we review integration of vector fields and distributions on subcartesian spaces following [33].

Let S be a subcartesian space with differential structure $C^\infty_{\text{glob}}(S)$. A derivation on $C^\infty_{\text{glob}}(S)$ is an $\mathbb{R}$-linear map $X : C^\infty_{\text{glob}}(S) \to C^\infty_{\text{glob}}(S) : f \mapsto X(f)$ satisfying Leibniz' rule

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2). \tag{3.1}$$

We denote the space of derivations of $C^\infty_{\text{glob}}(S)$ by $\operatorname{Der} C^\infty_{\text{glob}}(S)$. It has the structure of a Lie algebra with the Lie bracket $[X_1, X_2]$ defined by

$$[X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f))$$

for every $X_1, X_2 \in \operatorname{Der} C^\infty_{\text{glob}}(S)$ and $f \in C^\infty_{\text{glob}}(S)$.

Let I be an interval in $\mathbb{R}$. A smooth map $c : I \to S$ is an integral curve of a derivation X if

$$\frac{d}{dt} f(c(t)) = X(f)(c(t)) \tag{3.2}$$

for all $f \in C^\infty_{\text{glob}}(S)$ and $t \in I$. If I is closed and t is an endpoint of I, then the derivative on the left hand side of equation (3.2) is one-sided. In the limiting case, when I consists of only one point, the left hand side of (3.2) is undefined. We extend the definition of an integral curve to this case by declaring that a map $c : \{t_0\} \to S$, with domain consisting of a single point in $\mathbb{R}$, is an integral curve of every derivation X. An integral curve of X through a point $x_0 \in S$ is an integral curve $c : I \to S$ of X such that $0 \in I$ and $c(0) = x_0$. An integral curve $c : I \to S$ of X through x_0 is maximal if its domain I contains the domain of every integral curve of X through x_0.

Remark 3.1. Let S be a subcartesian space. For every derivation $X \in \operatorname{Der} C^\infty_{\text{glob}}(S)$ and each $x_0 \in S$ there exists a unique maximal integral curve of X through x_0. A proof of this can be found in [32].

A *vector field on a subcartesian space* is a derivation X of $C^\infty_{\text{glob}}(S)$ such that translations along integral curves of X give rise to a one-parameter local group ϕ^X_t of local diffeomorphisms of S. In other words,

$$\frac{d}{dt} f\left(\phi^X_t(x)\right) = X(f)\left(\phi^X_t(x)\right)$$

for every $f \in C^\infty_{\text{glob}}(S)$ and each $(t, x) \in \mathbb{R} \times S$ for which $\phi^X_t(x)$ is defined. Let $\mathfrak{X}_{\text{glob}}(S)$ denote the family of all vector fields on a subcartesian space S. The orbit S_x of $\mathfrak{X}_{\text{glob}}(S)$ through x is given by

$$S_x = \{\phi^{X_n}_{t_n} \circ \ldots \circ \phi^{X_1}_{t_1} \mid n \in \mathbb{N},\ t_1, ..., t_n \in \mathbb{R},\ X_1, ..., X_n \in \mathfrak{X}(S)\}. \tag{3.3}$$

Remark 3.2. Let $\mathfrak{X}_{\text{glob}}(S)$ be the family of all vector fields on a subcartesian space S. For each $x \in S$, the orbit S_x is a manifold and the inclusion map $S_x \hookrightarrow S$ is smooth. For every family $\mathcal{F}$ of vector fields on S, orbits of $\mathcal{F}$ are contained in orbits of $\mathfrak{X}_{\text{glob}}(S)$. For a proof of this, see Theorem 4 in [33]. Smoothness of the inclusion map $S_x \hookrightarrow S$ is discussed in the proof of Theorem 3 in [33].

3.3. **Stratifications.** A *decomposition* of a differential space S is a partition of S by a locally finite family $\mathcal{D}$ of smooth manifolds S_α of S such that

(1) each manifold $S_\alpha \in \mathcal{D}$ with its manifold structure is a locally closed differential subspace of S

and

(2) for $S_\alpha, S_\beta \in \mathcal{D}$, if $S_\alpha \cap \bar{S}_\beta \neq \emptyset$, then either $S_\alpha = S_\beta$ or $S_\alpha \subset \bar{S}_\beta \backslash S_\beta$.

Manifolds $S_\alpha \in \mathcal{D}$ are called *strata* of the decomposition $\mathcal{D}$.

Decompositions of a differential space S can be partially ordered by inclusion. If $\mathcal{D}^1 = \{S^1_\alpha\}$ and $\mathcal{D}^2 = \{S^2_\beta\}$ are two decompositions of S, we say that $\mathcal{D}^1$ is a *refinement* of $\mathcal{D}^2$, and write $\mathcal{D}^1 \geq \mathcal{D}^2$, if, for every $S^1_\alpha \in \mathcal{D}^1$, there exists $S^2_\beta \in \mathcal{D}^2$ such that $S^1_\alpha \subseteq S^2_\beta$. We say that $\mathcal{D}$ is a minimal (coarsest) decomposition of P if it is not a refinement of a different decomposition of P. Note that if P is a manifold, then the minimal decomposition of M consists of a single manifold $M = P$. Similarly, we say that $\mathcal{D}$ is a maximal (finest) decomposition of P if $\mathcal{D}' \geq \mathcal{D}$ implies $\mathcal{D}' = \mathcal{D}$.

Let $\mathcal{D} = \{S_\alpha\}$ be a decomposition of S. The *stratification* corresponding to $\mathcal{D}$ is a map $\mathcal{S}$ which associates to each $x \in S$ the germ at x of the stratum S_α containing x. If all strata S_α of $\mathcal{D}$ are connected, then $\mathcal{D}$ is uniquely determined by the stratification $\mathcal{S}$ corresponding to $\mathcal{D}$ (see [21]). In the following we identify decompositions of S with connected strata with corresponding stratifications of S.

3.4. **Orbits of a proper action.** In this section we consider a smooth and proper action

$$\begin{array}{llll} \Phi & : G \times M & \to & M \\ & (g,m) & \mapsto & \Phi(g,m) \equiv \Phi_g(m) \equiv gm \equiv g \cdot m \end{array} \tag{3.4}$$

of a Lie group G on a manifold M. Our aim is to describe the differential structure of the orbit space $\bar{M} = M/G$. We denote the orbit map by $\pi : M \to \bar{M}$.

For each closed Lie subgroup H of G we define the *isotropy type* set

$$M_H = \{m \in M \mid G_m = H\},$$

where $G_m = \{g \in G \mid gm = m\}$ is the isotropy subgroup of $m \in M$. Since the action is proper, all isotropy groups are compact. The sets M_H, where H ranges over the closed Lie subgroups of G for which M_H is nonempty, form a partition of M, and therefore they are the equivalence classes of an equivalence relation in M. Define the normalizer of H in G by

$$N(H) = \{g \in G \mid gHg^{-1} = H\}.$$

$N(H)$ is a closed Lie subgroup of G. Since H is a normal subgroup of $N(H)$ the quotient $N(H)/H$ is a Lie group. If $m \in M_H$, we have $G_m = H$ and, for all $g \in G$, $G_{gm} = gHg^{-1}$. As a consequence, gm lies in M_H if and only if $g \in N(H)$. The action of G on M restricts to an action of $N(H)$ on M_H, which induces a free and proper action of $N(H)/H$ on M_H.

Define the *orbit type* set

$$M_{(H)} = \{m \in M \mid G_m \text{ is conjugated to } H\}. \tag{3.5}$$

Then,

$$M_{(H)} = \{gm \mid g \in G, m \in M_H\} = \pi^{-1}(\pi(M_H)).$$

Connected components of M_H and $M_{(H)}$ are embedded submanifolds of M; therefore M_H is called an *isotropy type manifold* and $M_{(H)}$ an *orbit type manifold*. Moreover,

$$\pi\left(M_{(H)}\right) = \{gm \mid m \in M_H\}/G = M_H/N(H) = M_H/(N(H)/H).$$

But the action of $N(H)/H$ on M_H is free and proper which implies that $M_H/(N(H)/H)$ is a quotient manifold of M_H. Hence, $\pi(M_{(H)})$ is a manifold contained in the orbit space $\bar{M} = M/G$.

Since the action of G on M is proper, the Slice Theorem of [26] ensures that for each $m \in M$ there exists a slice S_m for this action and that $\pi(S_m)$ is an open subset of $\bar{M}$ homeomorphic to S_m/G_m. It follows that

$$C^\infty_{\mathrm{glob}}(\bar{M}) = \{f \in C^0(\bar{M}) \mid \pi^*(f) \in C^\infty(M)\}$$

is a differential structure on the orbit space $\bar{M}$; see Theorem 3.4 of [11]. Moreover, for each slice S_m, its projection $\pi(S_m)$ to $\bar{M}$ is diffeomorphic to S_m/G_m in the sense of differential spaces. Since G_m is compact and the action of G_m on S_m is linear, it follows that the space $C^\infty_{\mathrm{diff}}(S_m)^{G_m}$ of G_m-invariant smooth functions on S_m is given by smooth functions of algebraic invariants (see [28]). Hilbert's theorem ensures that the ring of G_m-invariant polynomials on S_m is finitely generated ([40], page 274). Hence, $\bar{M}$ is locally diffeomorphic to a subset of a finite-dimensional space, which implies that $\bar{M}$ is subcartesian; see [32].

A partition of the orbit space $\bar{M} = M/G$ by connected components of $\pi(M_{(H)})$ is a decomposition of the differential space $\bar{M}$. The corresponding stratification of $\bar{M}$ is called the orbit type stratification of the orbit space (see [14], and [27]). It is a minimal stratification in the partial order discussed above (see [3]). This implies that the strata $\pi(M_{(H)})$ of the orbit type stratification are orbits of the family of all vector fields on $\bar{M}$ (see [21]).

Now let $C^\infty(\bar{M})$ be the sheaf of smooth functions on $\bar{M}$ defined as follows. A function $f : V \to \mathbb{R}$ is an element of $C^\infty(\bar{M})$ if $V \subseteq \bar{M}$ is an open subset and $\pi^* f \in C^\infty(M)$. This really defines a sheaf of smooth functions on $\bar{M}$; see [25] or [13]. Proposition 4.7 in [13] states that this sheaf can equivalently be constructed as follows: $f : V \to \mathbb{R}$ is an element of $C^\infty(\bar{M})$ if $V \subseteq \bar{M}$ is an open subset and for all $x \in V$ there exists $U_x \subseteq \bar{M}$ open, $x \in U_x$, and $f_x \in C^\infty_{\mathrm{glob}}(\bar{M})$ such that

$$f_x|_{U_x} = f|_{U_x}.$$

In an analogous manner, we define $C^\infty(\bar{P})$ for a stratum $\bar{P}$ of $\bar{M}$. A function $f_{\bar{P}} : V_{\bar{P}} \to \mathbb{R}$ is an element of $C^\infty(\bar{P})$ if $V_{\bar{P}} \subseteq \bar{P}$ is an open subset and for all $x \in V_{\bar{P}}$ there exists an open neighborhood $U \subseteq M$ of x such that $U \cap \bar{P} \subset V_{\bar{P}} \subseteq \bar{P}$ and $f \in C^\infty_{\mathrm{glob}}(\bar{M})$ such that

$$f_{\bar{P}}|_{U\cap\bar{P}} = f|_{U\cap\bar{P}}.$$

Note that this implies that for any $f \in C^\infty(\bar{P})$ and any point x in the domain of definition of f, there exists an open neighborhood $U \subseteq \bar{M}$ of x and $f_x \in C^\infty(\bar{M})$ such that

$$f|_{U\cap\bar{P}} = f_x|_{U\cap\bar{P}}.$$

Hence, by shrinking the domain of the definition of f, we can see the function f_x as an extension of f at x. We shall often use this property in the rest of the paper (without mentioning the "shrinking" of the domain of definition). We will see later that this smooth structure on $\bar{P}$ is exactly its smooth structure as the quotient of the stratum $P = \pi^{-1}(\bar{P})$ of M.

We end this subsection with a proposition on the uniqueness of the restriction of a vector field on $\bar{M}$ to a stratum of $\bar{M}$.

Proposition 3.3. *Let $\bar{P}$ be a stratum of $\bar{M}$. We know by the considerations above that each vector field $\bar{X}$ on $\bar{M}$ restricts to a vector field $X_{\bar{P}}$ on $\bar{P}$. We write $X_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$. If $X^1_{\bar{P}}$ and $X^2_{\bar{P}}$ are such that $X^1_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$ and $X^2_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$, then they have to be equal.*

Proof. If $\bar{X} \in \mathfrak{X}(\bar{M})$ restricts to a global vector field $X_{\bar{P}} \in \mathfrak{X}(\bar{P})$, we have for all $\bar{f} \in C^\infty_{\rm glob}(\bar{M})$:

$$X_{\bar{P}}(\iota^*_{\bar{P}}\bar{f}\,) = \bar{X}(\bar{f}) \circ \iota_{\bar{P}}.$$

Since each function $f_{\bar{P}} \in C^\infty(\bar{P})$ is locally the restriction to $\bar{P}$ of some $\bar{f} \in C^\infty_{\rm glob}(\bar{M})$ and the derivations on $\bar{P}$ correspond exactly to the vector fields on $\bar{P}$ (since $\bar{P}$ is a smooth manifold), we automatically get the uniqueness of $X_{\bar{P}}$. □

4. Generalized distributions and orthogonal spaces

We will need a few standard facts from the theory of generalized distributions on a smooth manifold M (see [35, 34, 36], [37] for the original articles and [20], [38], [27], or [25], for a quick review of this theory).

Let E be a vector bundle over M. A *generalized subdistribution* Δ of E is a subset Δ of E such that for each $m \in M$, the set $\Delta(m) := \Delta \cap E(m)$ is a vector subspace of E_m. The number $\dim \Delta(m)$ is called the *rank* of Δ at $m \in M$. A point $m \in M$ is a *regular* point of the distribution Δ if there exists a neighborhood U of m such that the rank of Δ is constant on U. Otherwise, m is a *singular* point of the distribution.

A local *differentiable section* of Δ is a smooth section $\sigma \in \Gamma(E)$ defined on some open subset $U \subset M$ such that $\sigma(u) \in \Delta(u)$ for each $u \in U$. We denote by $\Gamma(\Delta)$ the space of local sections of Δ. A generalized subdistribution is said to be *differentiable* or *smooth* if for every point $m \in M$ and every vector $v \in \Delta(m)$, there is a differentiable section $\sigma \in \Gamma(\Delta)$ defined on an open neighborhood U of m such that $\sigma(m) = v$. The generalized subdistribution Δ is said to be *locally finitely generated* if for each point $m \in M$ there exists a neighborhood U of m and smooth sections $\sigma_1, \ldots, \sigma_k \in \Gamma(E)$ defined on U such that for all m' in U we have

$$\Delta(m') = \operatorname{span}\{\sigma_1(m'), \ldots, \sigma_k(m')\}.$$

Note that a locally finitely generated distribution is necessarily smooth.

A smooth generalized subdistribution of the tangent space TM (that is, with $E = TM$) will simply be called a *smooth tangent distribution*; a smooth generalized subdistribution of the cotangent space T^*M will be called a *smooth cotangent distribution*. We will work most of the time with smooth generalized subdistributions of the Pontryagin bundle $E = TM \oplus T^*M$, which will be called *smooth generalized distributions*.

Example 4.1. A Dirac structure D on a manifold M defines two smooth tangent distributions $\mathsf{G}_0, \mathsf{G}_1 \subset TM$ and two smooth cotangent distributions $\mathsf{P}_0, \mathsf{P}_1 \subset T^*M$:

$$\mathsf{G}_0(m) := \{X(m) \in T_mM \,|X \in \mathfrak{X}(M), (X, 0) \in \Gamma(D)\},$$
$$\mathsf{G}_1(m) := \left\{ X(m) \in T_mM \;\middle|\; \begin{array}{c} X \in \mathfrak{X}(M), \text{ there exists } \alpha \in \Omega^1(M) \\ \text{such that } (X, \alpha) \in \Gamma(D) \end{array} \right\}$$

and

$$\mathsf{P}_0(m) := \{\alpha(m) \in T_m^*M \mid \alpha \in \Omega^1(M), (0, \alpha) \in \Gamma(D)\},$$
$$\mathsf{P}_1(m) := \left\{ \alpha(m) \in T_m^*M \;\middle|\; \begin{array}{c} \alpha \in \Omega^1(M), \text{ there exists } \; X \in \mathfrak{X}(M) \\ \text{such that } (X, \alpha) \in \Gamma(D) \end{array} \right\}.$$

The smoothness of $\mathsf{G}_0, \mathsf{G}_1, \mathsf{P}_0, \mathsf{P}_1$ is obvious since, by definition, they are generated by smooth local sections. In general, these are not vector subbundles of TM and T^*M, respectively. It is also clear that $\mathsf{G}_0 \subset \mathsf{G}_1$ and $\mathsf{P}_0 \subset \mathsf{P}_1$.

4.1. **Generalized foliations and integrability of tangent distributions.** To give content to the notion of integrability of a smooth tangent distribution and elaborate on it, we need to quickly review the concept and main properties of generalized foliations. A *generalized foliation* on M is a partition $\mathfrak{F} := \{\mathcal{L}_\alpha\}_{\alpha \in A}$ of M into disjoint connected sets, called *leaves*, such that each point $m \in M$ has a *generalized foliated chart* $(U, \varphi : U \to V \subseteq \mathbb{R}^{\dim M})$, $m \in U$. This means that there is some natural number $p_\alpha \leq \dim M$, called the *dimension* of the leaf $\mathcal{L}_\alpha$, and a subset $S_\alpha \subset \mathbb{R}^{\dim M - p_\alpha}$ such that $\varphi(U \cap \mathcal{L}_\alpha) = \{(x^1, \ldots, x^{\dim M}) \in V \mid (x^{p_\alpha+1}, \ldots, x^{\dim M}) \in S_\alpha\}$. The key difference with the concept of foliation is that the number p_α can change from leaf to leaf. Note that each $(x_\circ^{p_\alpha+1}, \ldots, x_\circ^{\dim M}) \in S_\alpha$ determines a connected component $(U \cap \mathcal{L}_\alpha)_\circ$ of $U \cap \mathcal{L}_\alpha$, that is, $\varphi((U \cap \mathcal{L}_\alpha)_\circ) = \{(x^1, \ldots, x^{p_\alpha}, x_\circ^{p_\alpha+1}, \ldots, x_\circ^{\dim M}) \in V\}$. The generalized foliated charts induce on each leaf a smooth manifold structure that makes them into initial submanifolds of M.

A leaf $\mathcal{L}_\alpha$ is called *regular* if it has an open neighborhood that intersects only leaves whose dimension equals $\dim \mathcal{L}_\alpha$. If such a neighborhood does not exist, then $\mathcal{L}_\alpha$ is called a *singular* leaf. A point is called *regular* (*singular*) if it is contained in a regular (singular) leaf. The set of vectors tangent to the leaves of $\mathfrak{F}$ is defined by

$$T(M, \mathfrak{F}) := \bigcup_{\alpha \in A} \bigcup_{m \in \mathcal{L}_\alpha} T_m\mathcal{L}_\alpha \subset TM.$$

Let us now turn to the relationship between distributions and generalized foliations. In all that follows, $\mathcal{T}$ is a smooth tangent distribution. An *integral manifold* of $\mathcal{T}$ is an injectively immersed connected manifold $\iota_L : L \hookrightarrow M$, where ι_L is the inclusion, satisfying the condition $T_m\iota_L(T_mL) \subset \mathcal{T}(m)$ for every $m \in L$. The integral manifold L is of *maximal dimension* at $m \in L$ if $T_m\iota_L(T_mL) = \mathcal{T}(m)$. The distribution $\mathcal{T}$ is *completely integrable* if for every $m \in M$ there is an integral manifold L of $\mathcal{T}$, $m \in L$, everywhere of maximal dimension. The distribution $\mathcal{T}$ is *involutive* if it is invariant under the (local) flows associated to differentiable sections of $\mathcal{T}$. The distribution $\mathcal{T}$ is *algebraically involutive* if for any two smooth vector fields defined on an open set of M which take values in $\mathcal{T}$, their bracket also takes values in $\mathcal{T}$. Clearly involutive distributions are algebraically involutive and the converse is true if the distribution is a subbundle.

Recall that the Frobenius theorem states that a vector subbundle of TM is (algebraically) involutive if and only if it is the tangent bundle of a foliation on M. The same is true for distributions: *A smooth distribution is involutive if and only if it coincides with the set of vectors tangent to a generalized foliation, that is, it is completely integrable.* This is known as the Stefan-Sussmann Theorem.

We will give the Stefan-Sussmann Theorem in the more general setting of a smooth tangent distribution spanned by a family of vector fields. Note that each smooth tangent distribution is spanned by the family of its smooth sections.

Let F be an everywhere defined family of local vector fields on M. By *everywhere defined* we mean that for every $m \in M$ there exists $X \in F$ such that $m \in \operatorname{Dom}(X)$. We can associate to the flows of the vector fields in F a set of local diffeomorphisms $\mathcal{A}_F := \{\phi_t \mid \phi_t \text{ flow of } X \in F\}$ of M and a pseudogroup of transformations generated by it,

$$A_F := (\mathbb{I}, M) \cup \{\phi^1_{t_1} \circ \cdots \circ \phi^n_{t_n} \mid n \in \mathbb{N} \text{ and } \phi^n_{t_n} \in \mathcal{A}_F \text{ or } (\phi^n_{t_n})^{-1} \in \mathcal{A}_F\}.$$

Analogously, we also define, for any $z \in M$, the following vector subspaces of T_zM:

$$\begin{aligned} \mathcal{D}_F(z) &:= \operatorname{span}\left\{ \left.\frac{d}{dt}\right|_{t=t_0} \phi_t(y) \,\middle|\, \phi_t \text{ flow of } X \in F,\ \phi_{t_0}(y) = z \right\} \\ &= \operatorname{span}\{X(z) \in T_zM \mid X \in F \text{ and } z \in \operatorname{Dom}(X)\}, \\ D_F(z) &:= \operatorname{span}\{T_y\mathcal{F}_T \cdot \mathcal{D}_F(y) \mid \mathcal{F}_T \in A_F, \mathcal{F}_T(y) = z\}. \end{aligned}$$

Note that, by construction, $\mathcal{D}_F$ is a smooth tangent distribution. We will say that $\mathcal{D}_F$ is the smooth tangent distribution *spanned* by F.

The A_F-orbits, also called the *accessible sets* of the family F, form a generalized foliation whose leaves have as tangent spaces the values of D_F (see, for example, [25]). An important question is determining when the smooth tangent distribution $\mathcal{D}_F$ spanned by F is integrable.

Theorem 4.2 ([35] and [37]). *Let $\mathcal{D}_F$ be a differentiable generalized distribution on the smooth manifold M spanned by an everywhere defined family of vector fields F. The following properties are equivalent:*

(1) *The distribution $\mathcal{D}_F$ is invariant under the pseudogroup of transformations generated by F; that is, for each $\mathcal{F}_T \in A_F$ and for each $z \in M$ in the domain of $\mathcal{F}_T$,*

$$T_z\mathcal{F}_T(\mathcal{D}_F(z)) = \mathcal{D}_F(\mathcal{F}_T(z)).$$

(2) $\mathcal{D}_F = D_F$.

(3) *For any $X \in F$ with flow ϕ_t and any $x \in \operatorname{Dom}(X)$, there exist:*
   (a) *A finite set $\{X_1, \ldots, X_p\} \subset F$ such that*

   $$\mathcal{D}_F(x) = \operatorname{span}\{X_1(x), \ldots, X_p(x)\}.$$

   (b) *A constant $\epsilon > 0$ and Lebesgue integrable functions $\lambda_{ij} : (-\epsilon, \epsilon) \to \mathbb{R}$ $(1 \le i, j \le p)$ such that for every $t \in (-\epsilon, \epsilon)$ and $j \in \{1, \ldots, p\}$:*

   $$[X, X_j](\phi_t(x)) = \sum_{i=1}^{p} \lambda_{ij}(t) X_i(\phi_t(x))$$

   and $\mathcal{D}_F(\phi_t(x)) = \operatorname{span}\{X_1(\phi_t(x)), \ldots, X_p(\phi_t(x))\}$.

(4) *The distribution $\mathcal{D}_F$ is integrable and its maximal integral manifolds are the A_F-orbits.*

Note that as a consequence of this theorem (consider in particular the third statement), we know that if a tangent distribution $\mathcal{T}$ is locally finitely generated, then *it is integrable if and only if it is algebraically involutive.*

As already mentioned, given an involutive (and hence a completely integrable) distribution $\mathcal{T}$, each point $m \in M$ belongs to exactly one connected integral manifold $\mathcal{L}_m$ that is maximal relative to inclusion. It turns out that $\mathcal{L}_m$ is an initial submanifold and that it is also the *accessible* set of m; that is, $\mathcal{L}_m$ equals the subset of points in M that can be reached by applying to m a composition of a finite numbers of flows of elements of $\Gamma(\mathcal{T})$. The collection of all maximal integral submanifolds of $\mathcal{T}$ forms a generalized foliation $\mathfrak{F}_{\mathcal{T}}$ such that $\mathcal{T} = T(M, \mathfrak{F}_{\mathcal{T}})$. Conversely, given a generalized foliation $\mathfrak{F}$ on M, the subset $T(M, \mathfrak{F}) \subset TM$ is a smooth completely integrable (and hence involutive) distribution whose collection of maximal integral submanifolds coincides with $\mathfrak{F}$. These two statements expand the Stefan-Sussmann Theorem cited above.

4.2. **Generalized smooth subdistributions and annihilators.** Assume in this section that E is a vector bundle on M that is endowed with a smooth nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle_E$. If $E = TM \oplus T^*M$ is the Pontryagin bundle, this pairing $\langle \cdot, \cdot \rangle_{TM \oplus T^*M}$ will always be the symmetric pairing $\langle \cdot, \cdot \rangle$ defined in (2.1). If $\Delta \subset E$ is a smooth subdistribution of E, its *smooth orthogonal* distribution is the smooth generalized subdistribution $\Delta^\perp$ of E defined by

$$\Delta^\perp(m) := \left\{ \tau(m) \;\middle|\; \begin{array}{c} \tau \in \Gamma(E) \text{ with } m \in \mathrm{Dom}(\tau) \text{ is such that for all} \\ \sigma \in \Gamma(\Delta) \text{ with } m \in \mathrm{Dom}(\sigma), \\ \text{we have } \langle \sigma, \tau \rangle_E = 0 \text{ on } \mathrm{Dom}(\tau) \cap \mathrm{Dom}(\sigma) \end{array} \right\}.$$

Here we have the, in general strict, inclusion $\Delta \subset \Delta^{\perp\perp}$. Note that the smooth orthogonal distribution of a smooth generalized subdistribution is smooth by construction. If the distribution Δ is a vector subbundle of E, then its smooth orthogonal distribution is also a vector subbundle of E. Note that the smooth orthogonal distribution of a smooth generalized subdistribution Δ of E is in general different from the *pointwise* orthogonal distribution of Δ, defined by

$$\Delta^{\perp_{\mathrm{p}}}(m) := \{ v_m \in E(m) \mid \langle v_m, w_m \rangle_E = 0 \text{ for all } w_m \in \Delta(m) \},$$

where the subscript p stands for "pointwise". The pointwise orthogonal distribution of a smooth generalized subdistribution Δ is not smooth in general. The proof of the following proposition is easy, and we omit it here.

Proposition 4.3. *Let Δ be a smooth generalized subdistribution of E. Then we have*

$$\Delta^\perp \subseteq \Delta^{\perp_{\mathrm{p}}}, \quad \Delta = \Delta^{\perp_{\mathrm{p}}\perp_{\mathrm{p}}}, \ \textit{and} \quad \Delta \subseteq \Delta^{\perp\perp}.$$

If Δ is itself a vector bundle over M, the smooth orthogonal distribution $\Delta^\perp$ of Δ is also a subbundle of E, and we have $\Delta^\perp = \Delta^{\perp_{\mathrm{p}}}$.

We use this to show the following proposition about the smooth annihilator of a sum of vector subbundles of E.

Proposition 4.4. *Let Δ_1 and Δ_2 be smooth subbundles of the vector bundle $(E, \langle , \rangle)$. Since Δ_1 and Δ_2 have constant ranks on M, their smooth orthogonal spaces $\Delta_1^\perp$ and $\Delta_2^\perp$ are also smooth subbundles of E and equal to the pointwise orthogonals of Δ_1 and Δ_2. The following are equivalent:*

(1) *The intersection $\Delta_1^\perp \cap \Delta_2^\perp$ is smooth.*

(2) $(\Delta_1+\Delta_2)^\perp = \Delta_1^\perp \cap \Delta_2^\perp$.
(3) $(\Delta_1^\perp \cap \Delta_2^\perp)^\perp = \Delta_1+\Delta_2$.
(4) $\Delta_1^\perp \cap \Delta_2^\perp$ *has constant rank on* M.

Proof. Let $\sigma \in \Gamma\left((\Delta_1+\Delta_2)^\perp\right)$. Then for all $\sigma_1 \in \Gamma(\Delta_1)$ and $\sigma_2 \in \Gamma(\Delta_2)$, we have $\langle \sigma, \sigma_1+\sigma_2\rangle = 0$ on the common domain of definition of the three sections. Applying this to $\sigma_1 \in \Gamma(\Delta_1)$ and $\sigma_2 = 0$ (respectively $\sigma_2 \in \Gamma(\Delta_2)$ and $\sigma_1 = 0$), we get $\sigma \in \Gamma(\Delta_1^\perp)$ (respectively $\sigma \in \Gamma(\Delta_2^\perp)$). Hence, we have shown that the inclusion $(\Delta_1+\Delta_2)^\perp \subseteq \Delta_1^\perp \cap \Delta_2^\perp$ is always true.

Using this, we show first that if $\Delta_1^\perp \cap \Delta_2^\perp$ is smooth, we have

$$(\Delta_1+\Delta_2)^\perp = \Delta_1^\perp \cap \Delta_2^\perp. \tag{4.1}$$

We have only to show the inclusion $(\Delta_1+\Delta_2)^\perp \supseteq \Delta_1^\perp \cap \Delta_2^\perp$. Choose $e_m \in (\Delta_1^\perp \cap \Delta_2^\perp)(m)$. Since the intersection $\Delta_1^\perp \cap \Delta_2^\perp$ is smooth, there exists a section $\sigma \in \Gamma\left(\Delta_1^\perp \cap \Delta_2^\perp\right)$ with $\sigma(m) = e_m$. Let $\sigma_1 \in \Gamma(\Delta_1)$ and $\sigma_2 \in \Gamma(\Delta_2)$. Since $\sigma \in \Gamma\left(\Delta_1^\perp \cap \Delta_2^\perp\right)$, we have $\langle\sigma,\sigma_1\rangle = \langle\sigma,\sigma_2\rangle = 0$, and hence $\langle\sigma,\sigma_1+\sigma_2\rangle = 0$. From this it follows that $\sigma \in \Gamma\left((\Delta_1+\Delta_2)^\perp\right)$ and hence $e_m \in (\Delta_1+\Delta_2)^\perp(m)$.

Conversely, if the equality in (4.1) holds, the intersection $\Delta_1^\perp \cap \Delta_2^\perp$ is the smooth annihilator of $\Delta_1+\Delta_2$ and is thus smooth by definition. Hence, we have shown "(1) $\Leftrightarrow$ (2)".

If (4.1) holds, we have

$$(\Delta_1+\Delta_2)^{\perp_{\mathrm{p}}}(m) = (\Delta_1(m)+\Delta_2(m))^\perp = \Delta_1(m)^\perp \cap \Delta_2(m)^\perp = (\Delta_1^\perp \cap \Delta_2^\perp)(m), \tag{4.2}$$

and hence, using Proposition 4.3:

$$(\Delta_1^\perp \cap \Delta_2^\perp)^\perp = \left((\Delta_1+\Delta_2)^{\perp_{\mathrm{p}}}\right)^\perp \subseteq (\Delta_1+\Delta_2)^{\perp_{\mathrm{p}}\perp_{\mathrm{p}}} = \Delta_1+\Delta_2.$$

The converse inclusion follows also from Proposition 4.3:

$$\Delta_1+\Delta_2 \subseteq (\Delta_1+\Delta_2)^{\perp\perp} = (\Delta_1^\perp \cap \Delta_2^\perp)^\perp.$$

Conversely, the equality $\Delta_1+\Delta_2 = (\Delta_1^\perp \cap \Delta_2^\perp)^\perp$ implies that $\Delta_1^\perp \cap \Delta_2^\perp \subseteq (\Delta_1^\perp \cap \Delta_2^\perp)^{\perp\perp} = (\Delta_1+\Delta_2)^\perp$ with Proposition 4.3, and we have shown the converse implication at the beginning of this proof. This shows "(2) $\Leftrightarrow$ (3)".

Assume again that (4.1) holds. Then it implies (4.2) as above. The equalities (4.2) and (4.1) then yield together:

$$(\Delta_1+\Delta_2)^{\perp_{\mathrm{p}}} \overset{(4.2)}{=} \Delta_1^\perp \cap \Delta_2^\perp \overset{(4.1)}{=} (\Delta_1+\Delta_2)^\perp.$$

But this is only possible if $\Delta_1+\Delta_2$ has constant rank on M, which yields, using (4.1), the fact that $\Delta_1^\perp \cap \Delta_2^\perp$ has constant rank on M, too. Hence, we have proved the implication "(2) $\Rightarrow$ (4)".

To finish the proof, we see that the implication "(4) $\Rightarrow$ (3)" is easy. If $\Delta_1^\perp \cap \Delta_2^\perp$ has constant rank on M, then its smooth annihilator is equal to its pointwise annihilator and we get

$$\left(\Delta_1^\perp \cap \Delta_2^\perp\right)^\perp = \Delta_1^{\perp\perp} + \Delta_2^{\perp\perp} = \Delta_1+\Delta_2$$

since Δ_1 and Δ_2 have constant rank on M. $\square$

A tangent (respectively cotangent) distribution $\mathcal{T} \subseteq TM$ (respectively $\mathcal{C} \subseteq T^*M$) can be identified with the smooth generalized distribution $\mathcal{T} \ominus \{0\}$ (respectively

$\{0\} \oplus \mathcal{C})$. The smooth orthogonal distribution of $\mathcal{T} \oplus \{0\}$ in $TM \oplus T^*M$ is easily computed to be $(\mathcal{T} \oplus \{0\})^\perp = TM \oplus \mathcal{T}^\circ$, where

$$\mathcal{T}^\circ(m) = \left\{ \alpha(m) \;\middle|\; \begin{array}{c} \alpha \in \Omega^1(M), m \in \operatorname{Dom}(\alpha) \text{ and } \alpha(X) = 0 \\ \text{on } \operatorname{Dom}(\alpha) \cap \operatorname{Dom}(X) \text{ for all } X \in \Gamma(\mathcal{T}) \end{array} \right\}$$

for all $m \in M$. This smooth cotangent distribution will be called the *smooth annihilator* of $\mathcal{T}$. Analogously, we define the smooth annihilator $\mathcal{C}^\circ$ of a cotangent distribution $\mathcal{C}$. Then $\mathcal{C}^\circ$ is a smooth tangent distribution and we have $(\{0\} \oplus \mathcal{C})^\perp = \mathcal{C}^\circ \oplus T^*M$. The pointwise annihilator of a smooth tangent distribution $\mathcal{T}$ (respectively of a smooth cotangent distribution $\mathcal{C}$), will be written $\mathcal{T}^{\text{ann}}$ (respectively $\mathcal{C}^{\text{ann}}$), and is such that $(\mathcal{T} \oplus \{0\})^{\perp_{\text{P}}} = TM \oplus \mathcal{T}^{\text{ann}}$ (respectively $(\{0\} \oplus \mathcal{C})^{\perp_{\text{P}}} = \mathcal{C}^{\text{ann}} \oplus T^*M$). We get as in Proposition 4.3:

$$\mathcal{T}^\circ \subseteq \mathcal{T}^{\text{ann}}, \quad \mathcal{T} = \mathcal{T}^{\text{ann}\,\text{ann}}, \quad \text{and } \mathcal{T} \subseteq \mathcal{T}^{\circ\circ},$$

and analogously for $\mathcal{C}$. If $\mathcal{T}$ is a smooth subbundle of TM, then $\mathcal{T}^\circ = \mathcal{T}^{\text{ann}}$ is also a smooth subbundle of T^*M.

The tangent distribution $\mathcal{V}$ spanned by the fundamental vector fields of the action of a Lie group G on a manifold M will be of great importance later on. At every point $m \in M$ it is defined by

$$\mathcal{V}(m) = \{\xi_M(m) \mid \xi \in \mathfrak{g}\}.$$

If the action is not free, the rank of the fibers of $\mathcal{V}$ can vary on M. The smooth annihilator $\mathcal{V}^\circ$ of $\mathcal{V}$ is given by

$$\mathcal{V}^\circ(m) = \left\{ \alpha(m) \;\middle|\; \begin{array}{c} \alpha \in \Omega^1(M),\, m \in \operatorname{Dom}(\alpha), \\ \text{such that } \alpha(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g} \end{array} \right\}.$$

We will also use the smooth generalized distribution $\mathcal{K} := \mathcal{V} \oplus \{0\}$ and its smooth orthogonal space $\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ$.

5. Proper actions and orbit type manifolds

5.1. **Tube Theorem and G-invariant average.** If the action of the Lie group G on M is proper, we can find for each point $m \in M$ a G-invariant neighborhood of m such that the action can be described easily on this neighborhood. The proof of the following theorem can be found, for example, in [25].

Theorem 5.1 (Tube Theorem)**.** *Let M be a manifold and G a Lie group acting properly on M. For a given point $m \in M$ denote $H := G_m$. Then there exists a G-invariant open neighborhood U of the orbit $G \cdot m$, called* the tube at *m, and a G-equivariant diffeomorphism $G \times_H B \xrightarrow{\sim} U$. The set B is an open H-invariant neighborhood of 0 in an H-representation space H-equivariantly isomorphic to $T_mM/T_m(G \cdot m)$. The H-representation on $T_mM/T_m(G \cdot m)$ is given by $h \cdot (v + T_m(G \cdot m)) := T_m\Phi_h \cdot v + T_m(G \cdot m)$, $h \in H$, $v \in T_mM$. The smooth manifold $G \times_H B$ is the quotient of the smooth free and proper (twisted) action Ψ of H on $G \times B$ given by $\Psi(h, (g, b)) := (gh^{-1}, h \cdot b)$, $g \in G$, $h \in H$, $b \in B$. The G-action on $G \times_H B$ is given by $k \cdot [g, b] := [kg, b]_H$, where $k, g \in G$, $b \in B$, and $[g, b]_H \in G \times_H B$ is the equivalence class (i.e., H-orbit) of (g, b).*

G-invariant average. Let $m \in M$ and $H := G_m$. If the action of G on M is proper, then the isotropy subgroup H of m is a compact Lie subgroup of G. Hence, there exists a Haar measure dh on H, that is, a G-invariant measure on H satisfying $\int_H dh = 1$ (see, for example, [14]). Here the left G-invariance of dh is equivalent to the right G-invariance of dh, and we have $R_{h'}^* dh = dh = L_{h'}^* dh$ for all $h' \in H$, where $L_h : H \to H$ (respectively $R_h : H \to H$) denotes left (respectively right) translation by h on H.

Let $X \in \mathfrak{X}(M)$ be defined on the tube U at $m \in M$ of the proper action of the Lie group G on M. Using the Tube Theorem, we write the points of U as equivalence classes $[g,b]_H$ with $g \in G$ and $b \in B$. Note that for all $h \in H$, we have $[g,b]_H = [gh^{-1}, hb]_H$. Furthermore, the action of G on U is given by $\Phi_{g'}([g,b]_H) = [g'g, b]_H$. Define the vector field X_G by the following:

$$X_G([g,b]_H) = \left(\Phi_{g^{-1}}^* \left(\int_H \Phi_h^* X dh\right)\right)([g,b]_H);$$

that is, for each point $m' = [g,b]_H \in U$ we have

$$X_G([g,b]_H) = T_{[e,b]_H}\Phi_g \left(\int_H \left(T_{[h,b]_H}\Phi_{h^{-1}} X([h,b]_H)\right) dh\right).$$

We have to show that this definition doesn't depend on the choice of the representative $[g,b]_H$ for the point m'. Write $m' = [gh^{-1}, hb]_H$ with some $h \in H$, and compute

$$\begin{aligned}
&X_G([gh^{-1}, hb]_H)\\
&\quad = T_{[e,hb]_H}\Phi_{gh^{-1}} \left(\int_H \left(T_{[\tilde{h},hb]_H}\Phi_{\tilde{h}^{-1}} X([\tilde{h}, hb]_H)\right) d\tilde{h}\right)\\
&\quad = T_{[h^{-1},hb]_H}\Phi_g \circ T_{[e,hb]_H}\Phi_{h^{-1}} \left(\int_H \left(T_{[e,\tilde{h}hb]_H}\Phi_{\tilde{h}^{-1}} X([e, \tilde{h}hb]_H)\right) d\tilde{h}\right)\\
&\quad = T_{[e,b]_H}\Phi_g \left(\int_H \left(T_{[e,\tilde{h}hb]_H}\Phi_{h^{-1}\tilde{h}^{-1}} X([e, \tilde{h}hb]_H)\right) d\tilde{h}\right)\\
&\quad = T_{[e,b]_H}\Phi_g \left(\int_H \left(T_{[e,\tilde{h}hb]_H}\Phi_{(\tilde{h}h)^{-1}} X([e, \tilde{h}hb]_H)\right) R_h^* d\tilde{h}\right)\\
&\quad \overset{h' := \tilde{h}h}{=} T_{[e,b]_H}\Phi_g \left(\int_H \left(T_{[e,h'b]_H}\Phi_{h'^{-1}} X([e, h'b]_H)\right) dh'\right)\\
&\quad = X_G([g,b]_H),
\end{aligned}$$

where we have used the equality $d\tilde{h} = R_h^* d\tilde{h}$. The vector field X_G is an element of $\mathfrak{X}(M)^G$: letting $[g,b]_H \in U$ and $g' \in G$, we have

$$\begin{aligned}
(\Phi_{g'}^* X_G)([g,b]_H) &= T_{[g'g,b]_H}\Phi_{g'^{-1}} X_G([g'g, b]_H)\\
&= T_{[g'g,b]_H}\Phi_{g'^{-1}} \circ T_{[e,b]_H}\Phi_{g'g} \left(\int_H \left(T_{[h,b]_H}\Phi_{h^{-1}} X([h,b]_H)\right) dh\right)\\
&= T_{[e,b]_H}\Phi_g \left(\int_H \left(T_{[h,b]_H}\Phi_{h^{-1}} X([h,b]_H)\right) dh\right)\\
&= X_G([g,b]_H).
\end{aligned}$$

At last, we should show that X_G is smooth. Let $X^H := \int_H \Phi_h^* X dh$ be the averaged vector field which is clearly smooth on $U \simeq G \times_H B$. Let $\Psi : H \times (G \times B) \to G \times B$

be the (smooth free and proper) twisted action of H on $G\times B$, that is, $\Psi(h(g,b)) = \Psi_h(g,b) = (gh^{-1}, hb)$ for all $g \in G$, $b \in B$, $h \in H$, and let $\pi_H : G\times B \to G\times_H B \simeq U$ be the projection. We write $\Phi : G \times (G \times B) \to G \times B$ for the left action of G on $G\times B$, given by $g\cdot(g', b) = (gg', b)$. Note that π_H is G-equivariant. Let $\widetilde{X^H}$ be an H-invariant vector field on $G\times B$ such that $\widetilde{X^H} \sim_{\pi_H} X^H$. Since $\widetilde{X^H} \in \mathfrak{X}(G\times B)$, it can be written as a sum $\widetilde{X^H} = X^G + X^B$ with $X^G \in \Gamma(TG\times 0_B)$ and $X^B \in \Gamma(0_G\times TB)$. Since X^G is smooth, $X^G|_{\{e\}\times B}$ is also smooth, and there exists a smooth function $\xi : B \to \mathfrak{g}$ such that $X^G(e,b) = (\xi(b), 0) \in \mathfrak{g} \times 0_b$ for all $b \in B$. Let ϕ^B_t be the flow of X^B. The points $\phi^B_t(e,b)$ are elements of $\{e\} \times B$ for each t where $\phi^B_t(e,b)$ is defined. Define $Y \in \mathfrak{X}(G \times B)$ by

$$\begin{aligned} Y(g,b) &:= T_{(e,b)}\Phi_g \widetilde{X^H}(e,b) = T_{(e,b)}\Phi_g X^G(e,b) + T_{(e,b)}\Phi_g X^B(e,b) \\ &=: Y^G(g,b) + Y^B(g,b). \end{aligned}$$

The vector fields Y^G and Y^B have $\phi^G_t(g,b) = \Phi_{g\exp(t\xi(b))}(e,b)$ and $\phi^B_t(g,b) = \Phi_g \circ \phi^B_t(e,b)$ as flows, which are obviously smooth. Hence the two vector fields Y^G and Y^B are smooth and so is Y. It is easy to see, using the fact that $\Psi_h \circ \Phi_g = \Phi_g \circ \Psi_h$ for all $g \in G$ and $h \in H$, that the vector field Y remains H-invariant and hence descends to $G \times_H B$. The construction of Y and the G-equivariance of π_H, yield that $Y \sim_{\pi_H} X_G$. This automatically implies that X_G is smooth. We call X_G the *G-invariant average* of the vector field X. Note that X_G is, in general, not equal to X (at any point); it can even vanish. Indeed, we will see in the following that G-invariant vector fields are tangent to the orbit type manifolds (in reality, they are even tangent to the isotropy type manifolds; see [25]). Hence, if we choose a G-invariant Riemannian metric on M and a section X of the (G-invariant) orthogonal $TP^\perp \subseteq TM|_P$ of TP relative to this metric, where P is a stratum of M, its G-invariant average will remain a section of $TP^\perp$, but will also be tangent to P. Hence, it will be the zero section. For an analogous statement, see [11], Lemma 2.4.

In the same manner, define for $\alpha \in \Omega^1(M)$ the G-invariant average $\alpha_G \in \Omega^1(M)^G$ of α as follows:

$$\alpha_G([g,b]_H) = \left(\Phi^*_{g^{-1}}\left(\int_H \Phi^*_h \alpha dh\right)\right)([g,b]_H);$$

that is, for each point $m' = [g,b]_H \in U$ we have

$$\begin{aligned} \alpha_G([g,b]_H) &= \left(\int_H \Phi^*_h \alpha dh\right)_{[e,b]_H} \circ T_{[g,b]_H}\Phi_{g^{-1}} \\ &= \left(\int_H (\alpha([h,b]_H) \circ T_{[e,b]_H}\Phi_h) dh\right) \circ T_{[g,b]_H}\Phi_{g^{-1}}. \end{aligned} \tag{5.1}$$

In an analogous manner as above, we can show that α_G is well-defined, smooth, and G-invariant. In the following, the one-form $\int_H \Phi^*_h \alpha dh$ will be called α^H.

If (X, α) is a section of a G-invariant generalized distribution $\mathcal{D}$, the section (X_G, α_G) is a G-invariant section of $\mathcal{D}$.

Note that, in the same manner, we can define the G-invariant average f_G of a smooth function f defined on the tube U of the action of G at m. The function f_G is defined by

$$f_G([g,b]_H) = \int_{h\in H} f([h,b]_H) dh.$$

Again, it is easy to check that f_G is well-defined. The smoothness of f_G can be shown with similar arguments as for the smoothness of X_G.

Let P be a connected component of an orbit type manifold (recall (3.5)) and $\bar{P} := \pi(P)$, where $\pi : M \to M/G =: \bar{M}$ is the orbit space projection. Since G is connected, the subgroup G^P of G such that $\Phi_g(P) \subseteq P$ for all $g \in G^P$ is equal to G. Hence the proper action of G on M restricts to a proper action Φ^P of G on P satisfying $\iota_P \circ \Phi^P_g = \Phi_g \circ \iota_P$ for all $g \in G$. Moreover, the action of G on P has conjugated isotropy subgroups and thus the quotient P/G is a smooth manifold. Let π_P be the quotient map. Using the previous discussion, we can relate the differential structures on $\bar{P}$, seen as the quotient manifold of P by the smooth and proper G-action, and as a stratum of the stratified space $\bar{M}$.

Proposition 5.2. *Let P be a connected component of an orbit type manifold $M_{(H)}$. The quotient P/G is diffeomorphic to the stratum $\pi(P) = \bar{P}$ of $\bar{M}$.*

Proof. The bijectivity of the well-defined map $\Lambda : P/G \to \bar{P}$, $\pi_P(p) \mapsto \pi(\iota_P(p))$ is easy. Note that we have $\pi \circ \iota_P = \iota_{\bar{P}} \circ \Lambda \circ \pi_P$.

Let $f_{\bar{P}} \in C^\infty(\bar{P})$ and $\bar{p} \in \bar{P}$ in the domain of definition of $f_{\bar{P}}$. We have to find a neighborhood $U_{\bar{P}} \subseteq \bar{P}$ of $\bar{p}$ such that $\Lambda^*(f_{\bar{P}}|_{U_{\bar{P}}}) \in C^\infty(P/G)$. Since $f_{\bar{P}} \in C^\infty(\bar{P})$, there exists a neighborhood $U \subset M$ of $\bar{p}$ and $\bar{f} \in C^\infty(\bar{M})$ such that $f_{\bar{P}}|_{U_{\bar{P}}} = \bar{f} \circ \iota_{\bar{P}}|_{U_{\bar{P}}}$. Assume without loss of generality that $U_{\bar{P}} = U \cap \bar{P}$. Since $\bar{f}$ is a smooth function on $\bar{M}$, there exists $f \in C^\infty(M)^G$ such that $f = \pi^*(\bar{f})$. But then we have

$$\pi_P^*(\Lambda^*(f_{\bar{P}}|_{U_{\bar{P}}})) = (\pi_P^* \circ \Lambda^* \circ \iota_{\bar{P}}^*)(\bar{f}) = (\iota_P^* \circ \pi^*)(\bar{f}) = \iota_P^*(f) \in C^\infty(P),$$

and hence $\Lambda^*(f_{\bar{P}}|_{U_{\bar{P}}}) \in C^\infty(P/G)$.

Let $f_{P/G} \in C^\infty(P/G)$. We have to show that $(\Lambda^{-1})^*(f_{P/G})$ is an element of $C^\infty(\bar{P})$. Define $f_P := \pi_P^*(f_{P/G}) \in C^\infty(P)^G$ and extend it to a function $f \in C^\infty(M)$, that is, $\iota_P^*(f) = f_P$. Without loss of generality we can assume that f is G-invariant (otherwise, the G-invariant average of f will also pull back to f_P), and thus pushes forward to $\bar{f} \in C^\infty(\bar{M})$. Then we have

$$(\pi_P^* \circ \Lambda^* \circ \iota_{\bar{P}}^*)(\bar{f}) = (\iota_P^* \circ \pi^*)(\bar{f}) = f_P = \pi_P^*(f_{P/G});$$

hence

$$(\Lambda^* \circ \iota_{\bar{P}}^*)(\bar{f}) = f_{P/G}$$

since π_P is a smooth surjective submersion. From this follows

$$(\Lambda^{-1})^*(f_{P/G}) = \iota_{\bar{P}}^*(\bar{f}),$$

which is an element of $C^\infty(\bar{P})$. □

Thus, in the following, we will identify $\bar{P}$ and P/G without further mentioning it.

5.2. **Push-forward of vector fields and one-forms.** Consider a G-invariant local vector field X on M. Since X is G-invariant, the push-forward $\bar{X} := \pi_* X$, defined by $\pi^*((\pi_* X)(\bar{f})) = X(\pi^*(\bar{f}))$ for every $\bar{f} \in C^\infty(\bar{M})$, is a well-defined (local) derivation of $C^\infty(\bar{M})$. Moreover, X generates a local one-parameter group ϕ_t^X of local diffeomorphisms of M. Since X is G-invariant, ϕ_t^X commutes with the action of G on M, and it induces a local one-parameter group of local diffeomorphisms of $\bar{M}$ generated by $\pi_* X$. Hence, $\pi_* X$ is a (local) vector field on $\bar{M}$.

We write $\mathfrak{X}(\bar{M})$ for the sheaf of (local) vector fields on $\bar{M}$. Then we have

$$\pi_*\left(\mathfrak{X}(M)^G\right) = \mathfrak{X}(\bar{M}) \tag{5.2}$$

(see [13], Theorem 6.10). In particular, for each stratum of $\bar{M}$ the tangent bundle space of the stratum is spanned by push forwards by π of G-invariant vector fields on M. It is easy to see that the sheaf of local vector fields on $\bar{P}$ is the set of local restrictions to $\bar{P}$ of elements of $\bar{M}$. Also, Proposition 3.3 is also true for local vector fields.

Yet, the class of vector fields on M that push forward to vector fields on $\bar{M}$ is bigger than the class of G-invariant vector fields, as the next lemma shows.

Lemma 5.3. *If $X \in \mathfrak{X}(M)$ is such that $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$, then it defines a derivation of the ring $C^\infty(M)^G$ of G-invariant functions. Therefore, it pushes down to a derivation $\bar{X}$ of $C^\infty(\bar{M})$. The derivation $\bar{X}$ is a vector field on the subcartesian space $\bar{M}$.*

Proof. Let $f \in C^\infty(M)^G$ and $g \in G$. Since $[X, V] \in \Gamma(\mathcal{V})$ for all sections $V \in \Gamma(\mathcal{V})$, we have, in particular, $[X, \xi_M] = V_\xi \in \Gamma(\mathcal{V})$ for each $\xi \in \mathfrak{g}$ and thus:

$$\xi_M(X(f)) = X(\xi_M(f)) - V_\xi(f) = 0,$$

since $V(f) = 0$ for all $V \in \Gamma(\mathcal{V})$. We get for all $m \in M$:

$$\left.\frac{d}{dt}\right|_{t=0} X(f) \circ \Phi_{\exp(t\xi)}(m) = 0$$

and hence, for all $t \in \mathbb{R}$:

$$\begin{aligned} \frac{d}{dt} X(f) \circ \Phi_{\exp(t\xi)}(m) &= \left.\frac{d}{ds}\right|_{s=0} X(f) \circ \Phi_{\exp((t+s)\xi)}(m) \\ &= \left.\frac{d}{ds}\right|_{s=0} X(f) \circ \Phi_{\exp(s\xi)}\left(\Phi_{\exp(t\xi)}(m)\right) \\ &= 0. \end{aligned}$$

Since the Lie group G is connected, it is spanned as a group by every neighborhood of its neutral element, hence from the image of the exponential map. With this it follows that the function $X(f)$ is G-invariant. Hence X defines a derivation of $C^\infty(M)^G$, and hence it induces a derivation $\bar{X}$ of the ring $C^\infty(\bar{M})$ of smooth functions on $\bar{M}$ as follows:

$$\pi^*(\bar{X}(\bar{f})) := X(\pi^*(\bar{f}))$$

for all $\bar{f} \in C^\infty(\bar{M})$.

We have to show that $\bar{X}$ is a vector field on the quotient space $\bar{M}$. Let $\bar{\phi}_t$ be the flow of $\bar{X}$ and let $\bar{f} \in C^\infty(\bar{M})$, i.e., $\pi^*(\bar{f}) \in C^\infty(M)$. We have to show that $\bar{\phi}_t^*(\bar{f}) \in C^\infty(\bar{M})$. From the definition of $\bar{X}$ follows the equality

$$\bar{\phi}_t \circ \pi = \pi \circ \phi_t,$$

where ϕ_t is the flow of X. Thus we have

$$\pi^*\left(\bar{\phi}_t^*(\bar{f})\right) = \phi_t^*\left(\pi^*(\bar{f})\right).$$

Since X is a vector field on M, the function $\phi_t^*\left(\pi^*(\bar{f})\right)$ is an element of $C^\infty(M)$, and hence $\bar{\phi}_t^*(\bar{f})$ an element of $C^\infty(\bar{M})$. □

Let X be as in the last lemma, and let $\bar{X}$ be the vector field on $\bar{M}$ with $X \sim_\pi \bar{X}$. Since $\bar{X}$ is a vector field, there exists a G-invariant vector field $X^G \in \mathfrak{X}(M)^G$ with $X^G \sim_\pi \bar{X}$ (see (5.2)). Thus, the vector field X can be written as a sum $X = X^G + X^{\mathcal{V}}$, with $X^{\mathcal{V}}$ a section of $\mathcal{V}$ (note that X^G is in general not equal to the G-invariant average X_G of X).

Let α be a (local) G-invariant one-form on M annihilating vectors tangent to orbits of the action of G on M. For each G-invariant vector field X on M, the evaluation $\alpha(X)$ is G-invariant. Hence, there exists a smooth function $\pi_*(\alpha(X))$ defined on $\bar{M}$ by $\pi^*(\pi_*(\alpha(X))) = \alpha(X)$. Since α annihilates vectors tangent to orbits of G, it follows that $\pi_*(\alpha(X))$ depends on X through its push-forwards $\pi_* X$. In other words, there is a linear form $\pi_*\alpha$ on the space of push-forwards by π of G-invariant vector fields on M such that

$$(\pi_*\alpha)(\pi_* X) = \pi_*(\alpha(X)) \quad \text{ for all } X \in \mathfrak{X}(M)^G.$$

Moreover, for every $\bar{f} \in C^\infty(\bar{M})$,

$$\begin{aligned}(\pi_*\alpha)(\bar{f}\pi_* X) &= (\pi_*\alpha)(\pi_*(\pi^*(\bar{f})X)) = \pi_*(\alpha(\pi^*(\bar{f})X)) \\ &= \pi_*(\pi^*(\bar{f}))\,\pi_*(\alpha(X)) = \bar{f}(\pi_*\alpha)(\pi_* X),\end{aligned}$$

that is, $\pi_*\alpha$ is $C^\infty(\bar{M})$-linear. This implies that, for every stratum of $\bar{M}$, the restriction of $\pi_*\alpha$ to the stratum gives rise to a well-defined one-form on the stratum.

Definition 5.4. Let G be a Lie group acting properly on the manifold M. Let $\mathcal{V}$ be the vertical space of the action. A section (X, α) in $\Gamma(TM \oplus \mathcal{V}^\circ)$ satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ and $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ will be called a *descending section*.

We will also need a few more facts about *one-forms* of $\bar{M}$. Indeed, we have a notion of vector fields on $\bar{M}$, and we know that these are exactly the push-forwards of descending vector fields on M. We also want to introduce objects which will play the role of *one-forms* on $\bar{M}$. The definition of a one-form on $\bar{M}$ should be such that each element $\alpha_{\bar{P}} \in \Omega^1(\bar{P})$, where $\bar{P}$ is a stratum of $\bar{M}$, is the restriction to $\bar{P}$ of a one-form on $\bar{M}$. Thus, we could define a one-form as a $C^\infty(\bar{M})$ linear map $\mathfrak{X}(\bar{M}) \to C^\infty(\bar{M})$, but since we want a one-to-one correspondence between sections $\Gamma(\mathcal{V}^\circ)^G$ and one-forms on $\bar{M}$, we need to define these more carefully.

By the space of *Kähler differentials of* $C^\infty(\bar{M})$ *over* $\mathbb{R}$ one understands a $C^\infty(\bar{M})$-module $\Omega_{C^\infty(\bar{M})/\mathbb{R}}$ together with a derivation $\mathbf{d} : C^\infty(\bar{M}) \to \Omega_{C^\infty(\bar{M})/\mathbb{R}}$ called *the Kähler derivative* such that the following universal property is satisfied (see [27]):

For every $C^\infty(\bar{M})$-module $\mathcal{M}$ and every derivation $\delta : C^\infty(\bar{M}) \to \mathcal{M}$ there exists a unique $\mathbb{R}$-linear mapping $\mathbf{i}_\delta : \Omega_{C^\infty(\bar{M})/\mathbb{R}} \to \mathcal{M}$ such that the diagram

$$\begin{array}{ccc} C^\infty(\bar{M}) & \xrightarrow{\ \delta\ } & \mathcal{M} \\ {\scriptstyle \mathbf{d}}\big\downarrow & \nearrow_{\mathbf{i}_\delta} & \\ \Omega_{C^\infty(\bar{M})/\mathbb{R}} & & \end{array}$$

commutes.

In particular, if $\mathcal{M} = C^\infty(\bar{M})$ and δ is a vector field $\bar{X}$ on $\bar{M}$, we get $\bar{X}(\bar{f}) = \mathbf{i}_{\bar{X}}\mathbf{d}\bar{f}$ for each function $\bar{f} \in C^\infty(\bar{M})$ and $\mathbf{i}_{\bar{X}}$ is the inner product with $\bar{X}$.

Proposition 5.5 ([24]). *The space $\Omega_{C^\infty(\bar{M})/\mathbb{R}}$ exists and can be represented as follows. Let Ω be the free $C^\infty(\bar{M})$-module over the symbols $\mathbf{d}\bar{f}$ with $\bar{f} \in C^\infty(\bar{M})$, and $\mathfrak{J}$ the $C^\infty(\bar{M})$-submodule generated by the relations*

$$\mathbf{d}(\lambda\bar{f} + \mu\bar{g}) - \lambda\mathbf{d}\bar{f} - \mu\mathbf{d}\bar{g} = 0 \quad \text{for all} \quad \lambda, \mu \in \mathbb{R}, \bar{f}, \bar{g} \in C^\infty(\bar{M}),$$
$$\mathbf{d}(\bar{f}\bar{g}) - \bar{f}\mathbf{d}\bar{g} - \bar{g}\mathbf{d}\bar{f} = 0 \quad \text{for all} \quad \bar{f}, \bar{g} \in C^\infty(\bar{M}).$$

Then $\Omega_{C^\infty(\bar{M})/\mathbb{R}} = \Omega/\mathfrak{J}$ and $\mathbf{d} : C^\infty(\bar{M}) \to \Omega_{C^\infty(\bar{M})/\mathbb{R}}$ is defined by $\bar{f} \mapsto \mathbf{d}\bar{f} + \mathfrak{J}$.

From this it follows immediately that each element of $\Omega_{C^\infty(\bar{M})/\mathbb{R}}$ can be written as a sum $\sum_j \bar{g}_j \mathbf{d}\bar{f}_j$ with finitely many $\bar{g}_j, \bar{f}_j \in C^\infty(\bar{M})$.

Hence, let $\bar{\alpha} = \sum_{j=1}^k \bar{g}_j \mathbf{d}\bar{f}_j \in \Omega_{C^\infty(\bar{M})/\mathbb{R}}$ and set $\alpha = \sum_{j=1}^n \pi^*\bar{g}_j \mathbf{d}(\pi^*\bar{f}_j) \in \Gamma(\mathcal{V}^\circ)$. We then have for each G-invariant vector field X on M:

$$\pi^*((\pi_*\alpha)(\pi_*X)) = \alpha(X) = \sum_{j=1}^n \pi^*\bar{g}_j \mathbf{d}(\pi^*\bar{f}_j)(X) = \sum_{j=1}^n \pi^*\bar{g}_j X(\pi^*\bar{f}_j)$$
$$= \pi^*\left(\sum_{j=1}^n \bar{g}_j\,(\pi_*X)(\bar{f}_j)\right) = \pi^*\left(\sum_{j=1}^n \bar{g}_j \mathbf{i}_{\pi_*X}\mathbf{d}\bar{f}_j\right).$$

Hence, the $C^\infty(\bar{M})$-linear map $\pi_*\alpha : \mathfrak{X}(\bar{M}) \to C^\infty(\bar{M})$ corresponds exactly to the $C^\infty(\bar{M})$-linear map $\mathfrak{X}(\bar{M}) \to C^\infty(\bar{M})$ defined by $\bar{\alpha}$ as follows:

$$\bar{\alpha}(\bar{X}) := \sum_{j=1}^k \bar{g}_j\, \mathbf{i}_{\bar{X}}\mathbf{d}\bar{f}_j \text{ for all } \bar{X} \in \mathfrak{X}(\bar{M}).$$

We set $\alpha =: \pi^*\bar{\alpha}$. Thus, each Kähler differential on $C^\infty(\bar{M})$ can be realized as the push-forward of an element of $\Gamma(\mathcal{V}^\circ)^G$. Conversely, we will see later that each element $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ can be written as a sum $\alpha = \sum_{j=1}^k g_j \mathbf{d}f_j$ with $g_j, f_j \in C^\infty(M)^G$ (see Lemma 5.9) and thus pushes forward to the Kähler differential $\sum_{j=1}^k (\pi_*g_j)\mathbf{d}(\pi_*f_j)$. An element $\bar{\alpha} \in \Omega_{C^\infty(\bar{M})/\mathbb{R}}$ will be called a *one-form* on $\bar{M}$ and the set of one-forms on $\bar{M}$ will be denoted by $\Omega^1(\bar{M})$. We have shown the following proposition.

Proposition 5.6. *The one-forms on $\bar{M}$ correspond exactly to the push-forwards of elements of $\Gamma(\mathcal{V}^\circ)^G$.*

Note that not every smooth section of the stratified cotangent space on $\bar{M}$, i.e., a smooth $C^\infty(\bar{M})$-linear map $\mathfrak{X}(\bar{M}) \to C^\infty(\bar{M})$, can be realized as a one-form on $\bar{M}$ (see [27] for the definition and discussion). There is a nontrivial condition for this to hold; see Proposition 2.3.7 in [27]. Hence, since each element of $\Gamma(\mathcal{V}^\circ)^G$ pushes forward to a one-form on $\bar{M}$, there should be smooth $C^\infty(\bar{M})$-linear maps $\mathfrak{X}(\bar{M}) \to C^\infty(\bar{M})$ which cannot be realized as push-forwards of elements of $\Gamma(\mathcal{V}^\circ)^G$.

Note that for vector fields, we have the analogous fact that each vector field on $\bar{M}$ is the push-forward of a G-invariant vector field on M (see (5.2)), but that not *all* derivations on $\bar{M}$ are vector fields on $\bar{M}$.

5.3. **Connected components of the orbit types.** Let F^G be the everywhere defined family of local vector fields

$$F^G = \{X \in \mathfrak{X}(M)^G \mid X = X^G + X^{\mathcal{V}} \text{ with } X^G \in \mathfrak{X}(M)^G \text{ and } X^{\mathcal{V}} \in \Gamma(\mathcal{V})\},$$

$\mathcal{A}^G := \{\phi_t \mid X \in F^G,\ \phi_t$ flow of $X\}$, and denote by A^G the pseudogroup of local diffeomorphisms associated to the flows of the family F^G, i.e.,

$$A^G = \{\mathbb{I}\} \cup \{\phi^1_{t_1} \circ \cdots \circ \phi^n_{t_n} \mid n \in \mathbb{N} \text{ and } \phi^n_{t_n} \text{ or } (\phi^n_{t_n})^{-1} \text{ flow of } X^n \in F^G\}.$$

Let $\mathcal{T}$ be the smooth generalized distribution spanned by F^G, that is,

$$\mathcal{T}(m) = \operatorname{span}\{X(m) \mid X \in F^G,\ m \in \operatorname{Dom}(X)\}.$$

Note that with Lemma 5.3 and the considerations following its proof, F^G is equal to

$$\{X \in \mathfrak{X}(M) \mid [X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})\}.$$

We will show that the distribution $\mathcal{T}$ is integrable in the sense of Stefan-Sussman and compare its leaves with the connected components of the orbit type manifolds.

Lemma 5.7. *For each $\mathcal{F} \in A^G$ and for each $m \in \operatorname{Dom}(\mathcal{F}) \subseteq M$, we have*

$$T_m\mathcal{F}(\mathcal{T}(m)) = \mathcal{T}(\mathcal{F}(m)).$$

As a consequence, the distribution $\mathcal{T}$ is integrable in the sense of Stefan-Sussman and its leaves are the A^G-orbits.

Proof. Assume first that $\mathcal{F} = \phi^X_t \in A^G$ for one vector field $X \in \mathfrak{X}(M)$ satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ (the general statement will follow inductively, since each element of A^G is a composition of finitely many such diffeomorphisms). Write X as a sum $X^G + X^\mathcal{V}$ with $X^G \in \mathfrak{X}(M)^G$ and $X^\mathcal{V} \in \Gamma(\mathcal{V})$. Let $v \in \mathcal{T}(m)$. Then $v = Y(m) = Y^G(m) + Y^\mathcal{V}(m)$ for sections $Y^G \in \mathfrak{X}(M)^G$ and $Y^\mathcal{V} \in \Gamma(\mathcal{V})$. By the Trotter Product Formula (see, for example, [25]), the flows ϕ^X and ϕ^Y of the vector fields X and Y are given by

$$\phi^X_t = \lim_{n\to\infty}\left(\phi^{X^G}_{t/n} \circ \phi^{X^\mathcal{V}}_{t/n}\right)^n \quad \text{and} \quad \phi^Y_t = \lim_{n\to\infty}\left(\phi^{Y^G}_{t/n} \circ \phi^{Y^\mathcal{V}}_{t/n}\right)^n,$$

where ϕ^{X^G}, $\phi^{X^\mathcal{V}}$, ϕ^{Y^G}, and $\phi^{Y^\mathcal{V}}$ are the flows of the vector fields X^G, $X^\mathcal{V}$, Y^G, and $Y^\mathcal{V}$. But since X^G and Y^G are G-invariant and $X^\mathcal{V}$ and $Y^\mathcal{V}$ are sections of $\mathcal{V}$, the flows of the vector fields X^G and Y^G commute with the flows of $X^\mathcal{V}$ and $Y^\mathcal{V}$. Hence, we get

$$\phi^X_t = \phi^{X^G}_t \circ \phi^{X^\mathcal{V}}_t = \phi^{X^\mathcal{V}}_t \circ \phi^{X^G}_t \quad \text{and} \quad \phi^Y_t = \phi^{Y^G}_t \circ \phi^{Y^\mathcal{V}}_t = \phi^{Y^\mathcal{V}}_t \circ \phi^{Y^G}_t.$$

The compositions

$$\phi_s := \phi^X_t \circ \phi^Y_s \circ \phi^X_{-t}, \qquad \phi^G_s := \phi^{X^G}_t \circ \phi^{Y^G}_s \circ \phi^{X^G}_{-t}$$

and

$$\phi^\mathcal{V}_s := \phi^{X^\mathcal{V}}_t \circ \phi^{Y^\mathcal{V}}_s \circ \phi^{X^\mathcal{V}}_{-t}$$

define flows on M. Let Z, Z^G and $Z^\mathcal{V}$ be the vector fields associated to those flows. We then have $Z^G \in \mathfrak{X}(M)^G$, $Z^\mathcal{V} \in \Gamma(\mathcal{V})$ and

$$\begin{aligned}\phi_s = \phi^X_t \circ \phi^Y_s \circ \phi^X_{-t} &= \phi^{X^G}_t \circ \phi^{X^\mathcal{V}}_t \circ \phi^{Y^G}_s \circ \phi^{Y^\mathcal{V}}_s \circ \phi^{X^G}_{-t} \circ \phi^{X^\mathcal{V}}_{-t}\\ &= \left(\phi^{X^G}_t \circ \phi^{Y^G}_s \circ \phi^{X^G}_{-t}\right) \circ \left(\phi^{X^\mathcal{V}}_t \circ \phi^{Y^\mathcal{V}}_s \circ \phi^{X^\mathcal{V}}_{-t}\right)\\ &= \left(\phi^{X^\mathcal{V}}_t \circ \phi^{Y^\mathcal{V}}_s \circ \phi^{X^\mathcal{V}}_{-t}\right) \circ \left(\phi^{X^G}_t \circ \phi^{Y^G}_s \circ \phi^{X^G}_{-t}\right).\end{aligned}$$

The vector field Z is then equal to the sum $Z^G+Z^{\mathcal{V}}$ and it satisfies $[Z,\Gamma(\mathcal{V})]\subseteq\Gamma(\mathcal{V})$. The equality

$$\begin{aligned} T_m\phi_t^X(Y(m)) &= \left.\frac{d}{ds}\right|_{s=0}\phi_t^X\circ\phi_s^Y(m) = \left.\frac{d}{ds}\right|_{s=0}\phi_t^X\circ\phi_s^Y\circ\phi_{-t}^X(\phi_t^X(m)) \\ &= \left.\frac{d}{ds}\right|_{s=0}\phi_s(\phi_t^X(m)) = Z(\phi_t^X(m)) \end{aligned}$$

then yields the first inclusion $T_m\phi_t^X(\mathcal{T}(m))\in\mathcal{T}(\phi_t^X(m))$.

For the other inclusion, we use a similar method: let $Y=Y^G+Y^{\mathcal{V}}$ be a vector field satisfying $[Y,\Gamma(\mathcal{V})]\subseteq\Gamma(\mathcal{V})$ and defined on a neighborhood of $\phi_t^X(m)$. As above, the vector field Z corresponding to the flow $\phi_s:=\phi_{-t}^X\circ\phi_s^Y\circ\phi_t^X$ can be written as a sum $Z=Z^G+Z^{\mathcal{V}}$ and is hence a section of $\mathcal{T}$. We get

$$\begin{aligned} Y(\phi_t^X(m)) &= \left.\frac{d}{ds}\right|_{s=0}\phi_s^Y(\phi_t^X(m)) = \left.\frac{d}{ds}\right|_{s=0}\left(\phi_t^X\circ\phi_{-t}^X\circ\phi_s^Y\circ\phi_t^X\right)(m) \\ &= \left.\frac{d}{ds}\right|_{s=0}\left(\phi_t^X\circ\phi_s\right)(m) = T_m\phi_t^X(Z(m))\in T_m\phi_t(\mathcal{T}(m)). \end{aligned}$$

□

Theorem 5.8. *The integrable leaves of the distribution* $\mathcal{T}$ *are exactly the connected components of the orbit type manifolds.*

Proof. Let N be the A^G-orbit through the point $m\in M$. We have to show that $N=P$, where P is the connected component through m of the orbit type manifold $M_{(G_m)}$. Let $m'\in N$. Then there exist vector fields $X_1,\ldots,X_l\in\mathfrak{X}(M)^G$ and $V_1,\ldots,V_l\in\Gamma(\mathcal{V})$ such that

$$m'=\left(\phi_{t_1}^{V_1}\circ\ldots\circ\phi_{t_k}^{V_k}\circ\phi_{s_1}^{X_1}\circ\ldots\circ\phi_{s_l}^{X_l}\right)(m)$$

(recall that the flows of the G-invariant vector fields commute with the flows of the sections of $\mathcal{V}$). Hence, we can assume without loss of generality that $m'=\phi_t^X(m)$ with a vector field $X\in\mathfrak{X}(M)^G$ or $m'=\phi_t^V(m)$ with V a section of $\mathcal{V}$. In the first case, the vector field X pushes down to a vector field $\bar{X}$ on $\bar{M}$. Let $\phi_t^{\bar{X}}$ be the flow of the vector field $\bar{X}$. Since the strata of $\bar{M}$ are the accessible sets of the vector fields on $\bar{M}$, the points $\pi(m')=\left(\pi\circ\phi_t^X\right)(m)=\phi_t^{\bar{X}}(\pi(m))$ and $\pi(m)$ lie in the same stratum of $\bar{M}$, hence in $\pi(P)$. Since X is G-invariant, its flow is also G-equivariant. Thus, we have $\phi_t^X(gm)=g\cdot\phi_t^X(gm)$ for all t where it is defined, and hence $\phi_t^X(m)\in M_{(G_m)}$ for all t. This yields that m and $\phi_t^X(m)$ can be joined by a smooth path in $M_{(G_m)}$, and consequently that they are in the same connected component of $M_{(G_m)}$, that is, the connected component P. But since $\pi(m')=\pi(\phi_t^X(m))$, there exists $g\in G$ such that $m'=\Phi_g(\phi_t^X(m))$ and since the action of G on M restricts to P, the point m' is also an element of P. In the second case we have $m'=\phi_t^V(m)$ with $V\in\Gamma(\mathcal{V})$. Since the vector field V is tangent to the G-orbits, its integral curve through m lies entirely in the connected component of the orbit type manifold through m and we are finished.

For the other inclusion, let m' be a point of P. We then have $\pi(m)$ and $\pi(m')\in\pi(P)$, a stratum of $\bar{M}$. Thus, we can assume without loss of generality that $\pi(m')=\phi_t^{\bar{X}}(\pi(m))$ for some vector field $\bar{X}\in\mathfrak{X}(\bar{M})$ (in reality, $\pi(m)$ and $\pi(m')$ can be joined by finitely many such curves). Let $X\in\mathfrak{X}(M)^G$ be such that $X\sim_\pi\bar{X}$ and let ϕ_t^X be its flow. Then we have $\left(\pi\circ\phi_t^X\right)(m)=\phi_t^{\bar{X}}(\pi(m))=\pi(m')$. Thus, there exists

$g \in G$ such that $\Phi_g(\phi_t^X(m)) = m'$. But since G is connected, we find finitely many elements $\xi^1, \ldots, \xi^l \in \mathfrak{g}$ such that $g = \exp(\xi^1) \cdot \ldots \cdot \exp(\xi^l)$, and hence we have

$$m' = \left(\Phi_{\exp(\xi^1)} \circ \ldots \circ \Phi_{\exp(\xi^l)} \circ \phi_t^X\right)(m).$$

The curves $\Phi_{\exp(t\xi^i)} : [0,1] \to M$, $i = 1, \ldots, l$, are segments of integral curves of the sections ξ^i_M of $\mathcal{V}$, and $\Phi_{\exp(\xi^1)} \circ \ldots \circ \Phi_{\exp(\xi^l)} \circ \phi_t^X$ is consequently an element of A^G. From this it follows that $m' \in N$. □

An alternative proof of this theorem is based on Theorem 3.5.1 (stating that the distribution $\mathcal{T}_G$ spanned by G-invariant vector fields is integrable with leaves the connected components of the isotropy type manifolds), Proposition 3.4.6 in [25], and the fact that $G \cdot M_H^m = M_{(H)}^m$, where M_H^m (respectively $M_{(H)}^m$) is the connected component of M_H (respectively $M_{(H)}$) containing m.

Let P be a stratum of M, that is, a leaf of the distribution $\mathcal{T}$. Since M is paracompact, there exists a G-invariant Riemannian metric ρ on M (see, for example, [14]). Consider the vector bundle $TP = \mathcal{T}|_P \subseteq TM|_P$ over P, and let $TP^\perp$ be a G-invariant orthogonal complement of TP viewed as a subbundle of $TM|_P$. We can describe the codistribution TP° in the following way:

$$TP^\circ(p) = \{\mathbf{i}_X\rho(p) \mid X \in \Gamma(TP^\perp)\}$$

for all $p \in P$. Note that the Riemannian metric ρ allows an identification of the tangent bundle of M with the cotangent bundle via

$$X \in \mathfrak{X}(M) \leftrightarrow \mathbf{i}_X\rho \in \Omega^1(M).$$

The section $\mathbf{i}_X\rho$ is G-invariant if and only if X is G-invariant. We will use this in the proof of many of the following propositions and lemmas.

In the following, we will make use of the codistribution $\mathcal{V}_G^\circ$ defined as the span of the G-invariant sections of $\mathcal{V}^\circ$:

$$\mathcal{V}_G^\circ(x) = \{\alpha_x \mid \alpha \in \Gamma(\mathcal{V}^\circ)^G\}$$

for all $x \in M$.

This codistribution is in fact spanned by the exterior derivative of all G-invariant functions on M, as stated in the following lemma.

Lemma 5.9. *The codistribution $\mathcal{V}_G^\circ$ can be described as follows: for each $m \in M$ we have*

$$\mathcal{V}_G^\circ(m) = \operatorname{span}\{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\}.$$

Proof. We use the identity $((T_mGm)^{\mathrm{ann}})^{G_m} = \operatorname{span}\{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\}$ (see [25], Theorem 2.5.10), where

$$(T_mGm)^{\mathrm{ann}} := \{\alpha_m \in T_m^*M \mid \alpha_m(\xi_M(m)) = 0 \text{ for all } \xi \in \mathfrak{g}\}$$

is the pointwise annihilator of the tangent space T_mGm to the orbit Gm. We show

$$\operatorname{span}\{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\} \subseteq \mathcal{V}_G^\circ(m) \subseteq ((T_mGm)^{\mathrm{ann}})^{G_m},$$

and our claim follows from the equality above. The first inclusion is easy since for each function $f \in C^\infty(M)^G$, we have $\mathbf{d}f \in \Gamma(\mathcal{V}^\circ)^G$. For the second inclusion, choose $\alpha(m) \in \mathcal{V}_G^\circ(m)$, with α a G-invariant section of $\mathcal{V}^\circ$. Then we have $\alpha(\xi_M) = 0$ for all $\xi \in \mathfrak{g}$ and hence $\alpha(m)(\xi_M(m)) = 0$ for all $\xi \in \mathfrak{g}$, that is, $\alpha(m) \in (T_mGm)^{\mathrm{ann}}$.

Since α is G-invariant, we have $\Phi_h^*\alpha = \alpha$ for all $h \in G_m \subseteq G$ and hence, for all $v \in T_mM$ we get

$$\alpha_m(T_m\Phi_h v) = \alpha_{\Phi_h(m)}(T_m\Phi_h v) = (\Phi_h^*\alpha)_m(v) = \alpha_m(v),$$

where we have used that $h \cdot m = m$ since $h \in G_m$. Hence we have $(T_m\Phi_h)^*(\alpha(m)) = \alpha(m)$ for all $h \in G_m$ and hence $\alpha(m) \in ((T_mGm)^{\mathrm{ann}})^{G_m}$. □

Using this, we can show the following lemma.

Lemma 5.10. *Let P be a stratum of M, and let $\mathcal{V}_P$ be the vertical space of the induced action of G on P. We have the equality*

$$\iota_P^*(\mathcal{V}_G^\circ) = (\mathcal{V}_P)^\circ \subseteq T^*P.$$

Hence, the map $\iota_P^* : \mathcal{V}_G^\circ|_P \to (\mathcal{V}_P)^\circ$ is an isomorphism. Thus, $\mathcal{V}_G^\circ|_P$ is a vector bundle over P and

$$(\mathcal{V}_G^\circ|_P)^\circ = \mathcal{V}|_P \oplus TP^\perp.$$

For the proof of this, we will need the following lemma.

Lemma 5.11. *If the action of a Lie group G on a manifold M is with conjugated isotropy subgroups, then the (smooth) annihilator $\mathcal{V}^\circ$ of the vertical bundle $\mathcal{V}$ is spanned by its G-invariant sections.*

Proof. Since the action of G on M is with conjugated isotropy subgroups, the vertical space $\mathcal{V}$ is a smooth integrable subbundle of TM. Thus, for each $p \in M$, we find a coordinate neigborhood U of p with coordinates $(x_1, \ldots, x_n)$ such that $\mathcal{V}$ is spanned by $\partial_{x_1}, \ldots, \partial_{x_k}$, where $k = \dim G - \dim G_p = \dim \mathcal{V}$. The annihilator $\mathcal{V}^\circ$ of $\mathcal{V}$ is then spanned by $\mathbf{d}x_{k+1}, \ldots, \mathbf{d}x_n$ on U. Since $\mathbf{d}x_{k+1}, \ldots, \mathbf{d}x_n$ vanish on $\mathcal{V}$, and G is connected, the functions $x_{k+1}, \ldots, x_n$ are then G-invariant and we get $\mathbf{d}x_{k+1}, \ldots, \mathbf{d}x_n \in \Gamma(\mathcal{V}^\circ)^G$. □

Proof of Lemma 5.10. The inclusion $\iota_P^*(\mathcal{V}_G^\circ) \subseteq (\mathcal{V}_P)^\circ$ is easy. For the other inclusion, note that since all isotropy type manifolds of the action of G on P are conjugated, the codistribution $(\mathcal{V}_P)^\circ$ is spanned by its G-invariant sections by Lemma 5.11. Therefore, by Lemma 5.9, each G-invariant section of $(\mathcal{V}_P)^\circ$ is in the $C^\infty(P)^G$-span of $\{\mathbf{d}f \mid f \in C^\infty(P)^G\}$. Hence, each element $\tilde{\alpha}(p) \in (\mathcal{V}_P)^\circ(p)$ can be written as $\tilde{\alpha}(p) = \sum_{j=1}^k \tilde{f}_j(p)\mathbf{d}\tilde{g}_j|_p$ with $\tilde{f}_j, \tilde{g}_j \in C^\infty(P)^G$. Choose $f_j, g_j \in C^\infty(M)$ such that $\tilde{f}_j = \iota_P^* f_j$ and $\tilde{g}_j = \iota_P^* g_j$ for $j = 1, \ldots, k$. Without loss of generality, the functions $f_1, \ldots, f_k, g_1, \ldots, g_k$ are G-invariant (otherwise, their G-invariant averages will also restrict to $\tilde{f}_1, \ldots, \tilde{f}_k, \tilde{g}_1, \ldots, \tilde{g}_k$). Let $\alpha = \sum_{j=1}^k f_j \mathbf{d}g_j \in \Gamma(\mathcal{V}^\circ)^G$. Then we have

$$\begin{aligned}(\iota_P^*\alpha)(p) &= \left(\iota_P^*\left(\sum_{j=1}^k f_j \mathbf{d}g_j\right)\right)(p) = \sum_{j=1}^k (\iota_P^* f_j)(p)\mathbf{d}(\iota_P^* g_j)|_p \\ &= \sum_{j=1}^k \tilde{f}_j(p)\mathbf{d}\tilde{g}_j|_p = \tilde{\alpha}(p),\end{aligned}$$

and the proof of the first assertion is complete, since we have shown that $\tilde{\alpha}(p) = (\iota_P^*\alpha)(p) \in (\iota_P^*(\mathcal{V}_G^\circ))(p)$.

From this it follows that the map $\iota_P^* : \mathcal{V}_G^\circ|_P \to (\mathcal{V}_P)^\circ$ is surjective. For the injectivity, let $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ be defined on a neighborhood of $p \in P$ and such that

$\iota_P^*\alpha = 0$. The vector field $X \in \mathfrak{X}(M)$ satisfying $\mathbf{i}_X\rho = \alpha$ is G-invariant and hence tangent to P on P. Therefore, there exists $\tilde{X} \in \mathfrak{X}(P)$ with $\tilde{X} \sim_{\iota_P} X$ and we have $\iota_P^*\alpha = \iota_P^*\mathbf{i}_X\rho = \mathbf{i}_{\tilde{X}}\iota_P^*\rho$. But since $\iota_P^*\alpha = 0$ we get $\tilde{X} = 0$ using the fact that $\iota_P^*\rho$ is a Riemannian metric on P. Hence, we have shown that $\alpha|_P = 0$.

It remains to show the equality

$$(\mathcal{V}_G^\circ|_P)^\circ = \mathcal{V}|_P \oplus TP^\perp.$$

Since $\mathcal{V}|_P \subset TP \subset TM|_P$, we have $\mathcal{V}|_P \cap TP^\perp = 0_P$. First let $X \in \Gamma(TP^\perp)$, $V \in \Gamma(\mathcal{V})$ and $\alpha \in \Gamma(\mathcal{V}^\circ)^G$. Then we have $\alpha = \mathbf{i}_Y\rho$ with $Y \in \mathfrak{X}(M)^G$ and hence

$$\begin{aligned}\alpha|_P(X + V|_P) &= \rho|_P(Y|_P, X + V|_P) = \rho|_P(Y|_P, X) + \rho|_P(Y|_P, V|_P)\\ &= \rho|_P(Y|_P, X) + \alpha(V) \circ \iota_P = 0,\end{aligned}$$

since Y is tangent to P on P, that is, $Y|_P \in \Gamma(TP)$.

Now choose $X \in \Gamma((\mathcal{V}_G^\circ|_P)^\circ) \subseteq \Gamma(TM|_P)$ and write $X = X^\top + X^\perp$ with $X^\top \in \Gamma(TP)$ and $X^\perp \in \Gamma(TP^\perp)$. Choose an arbitrary $\alpha \in \Gamma(\mathcal{V}^\circ)^G$. Then $\alpha = \mathbf{i}_Y\rho$ with $Y \in \mathfrak{X}(M)^G$. Again, Y is tangent to P on P and we compute

$$\begin{aligned}0 = \alpha|_P(X) &= \alpha|_P(X^\top + X^\perp) = \rho|_P(Y|_P, X^\top + X^\perp)\\ &= \rho|_P(Y|_P, X^\top) + \rho|_P(Y|_P, X^\perp) = \rho|_P(Y|_P, X^\top).\end{aligned}$$

Thus, we have $X^\top \in \Gamma((\mathcal{V}_G^\circ|_P)^\circ)$. Since $X^\top \in \Gamma(TP)$, there exists $X \in \mathfrak{X}(M)$ with $X|_P = X^\top$ and $\tilde{X} \in \mathfrak{X}(P)$ with $\tilde{X} \sim_{\iota_P} X$. For each section $\tilde{\alpha} \in \Gamma(\mathcal{V}_P^\circ) = \iota_P^*(\Gamma(\mathcal{V}_G^\circ))$, we have $\tilde{\alpha} = \iota_P^*\alpha$ with $\alpha \in \Gamma(\mathcal{V}_G^\circ)$ and

$$\tilde{\alpha}(\tilde{X}) = \alpha(X) \circ \iota_P = 0.$$

But then $\tilde{X} \in \Gamma(\mathcal{V}_P)$, which leads to $X^\top \in \Gamma(\mathcal{V}|_P)$. □

With analogous methods as in the proof of the first part of the last lemma, we can show the following proposition.

Proposition 5.12. *Each local one-form on $\bar{P}$ is the restriction to $\bar{P}$ of a local one-form on $\bar{M}$.*

Proof. Let $\alpha_{\bar{P}} \in \Omega^1(\bar{P})$ and consider $\pi_P^*\alpha_{\bar{P}} \in \Omega^1(P)$. Hence, we have $\pi_P^*\alpha_{\bar{P}} \in \Gamma(\mathcal{V}_P^\circ)^G$, and we can find, as in the proof of Lemma 5.10, an element α of $\Gamma(\mathcal{V}^\circ)^G$ satisfying $\iota_P^*\alpha = \pi_P^*\alpha_{\bar{P}}$. The one-form α pushes forward to $\bar{\alpha} \in \Omega^1(\bar{M})$ and, with

$$\pi_P^*\alpha_{\bar{P}} = \iota_P^*\alpha = \iota_P^*\pi^*\bar{\alpha} = \pi_P^*\iota_{\bar{P}}^*\bar{\alpha}$$

and the fact that π_P is a smooth surjective submersion, we get the equality of $\alpha_{\bar{P}}$ and $\iota_{\bar{P}}^*\bar{\alpha}$. □

Our last two lemmas are rather technical. Let E_P be the vector bundle $E_P = TM|_P \oplus T^*M|_P$ over P, endowed with $\langle\cdot\,,\cdot\rangle_{E_P} = \langle\cdot\,,\cdot\rangle|_{E_P\times E_P}$. Note that this pairing is automatically symmetric and nondegenerate, since these properties are satisfied pointwise.

Lemma 5.13. *If the intersection $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is smooth, then we have*

$$(D|_P \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)|_P)^\perp = D|_P + \mathcal{K}|_P + (TP^\perp \oplus TP^\circ)$$

as smooth generalized subdistributions of E_P endowed with $\langle\cdot\,,\cdot\rangle_{E_P}$.

Proof. By Lemma 5.10, we know that $(\mathcal{V}_G^\circ|_P)^\circ = \mathcal{V}|_P \oplus TP^\perp$. From this follows immediately the equality:

$$(\mathcal{T} \ominus \mathcal{V}_G^\circ)|_P{}^\perp = (\mathcal{V}|_P \oplus TP^\perp) \ominus TP^\circ = \mathcal{K}|_P \oplus (TP^\perp \ominus TP^\circ),$$

and hence also

$$\left(\mathcal{K}|_P \oplus (TP^\perp \ominus TP^\circ)\right)^\perp = (\mathcal{T} \ominus \mathcal{V}_G^\circ)|_P$$

since $(\mathcal{T} \ominus \mathcal{V}_G^\circ)|_P$ and $\mathcal{K}|_P \oplus (TP^\perp \ominus TP^\circ)$ are vector bundles over P.

Now, since the intersection $D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)$ is spanned by the descending sections of D, it is in particular smooth. Its restriction to P is then also smooth and Proposition 4.4 yields

$$(D|_P \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)|_P)^\perp = D|_P + \mathcal{K}|_P + (TP^\perp \ominus TP^\circ).$$

(Note that the sum is not necessarily direct anymore.) □

Corollary 5.14. *If the intersection $D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)$ is smooth, we have*

$$(D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ))^\perp = D + \mathcal{K}$$

as smooth generalized distributions.

Proof. The inclusion $D + \mathcal{K} \subseteq (D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ))^\perp$ is easy.

Let $m \in M$. If $m \in M^{\text{reg}}$, the previous lemma shows that

$$(D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ))^\perp(m) = (D + \mathcal{K})(m)$$

since M^{reg} is open and dense in M.

Let $m \in P \subseteq M \backslash M^{\text{reg}}$, where P is a connected component of the orbit type manifold of m. Let (X, α) be a section of $(D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ))^\perp$ defined on a neighborhood U of m. Since $U \cap M^{\text{reg}}$ is open and dense in U, we find a sequence $(x_n)_{n \in \mathbb{N}}$ in $U \cap M^{\text{reg}}$ converging to m. Since (X, α) is smooth, we have $\lim_{n\to\infty}(X(x_n), \alpha(x_n)) = (X(m), \alpha(m))$. But from the above we know that $(X(x_n), \alpha(x_n)) \in (D+\mathcal{K})(x_n)$ for all $n \in \mathbb{N}$. Since the sum $D+\mathcal{K}$ is closed, we have $(X(m), \alpha(m)) \in (D+\mathcal{K})(m)$. □

Here we present an example inspired by [3] to illustrate the theory. In the following, we denote by $\mathcal{T}_G$ the distribution on M spanned by the family of G-invariant vector fields on M.

Example 5.15. We consider the diagonal action Φ of $G := \text{SO}(3)$ on $M := \mathbb{R}^3 \times \mathbb{R}^3$, that is, $\Phi : \text{SO}(3) \times (\mathbb{R}^3 \times \mathbb{R}^3) \to \mathbb{R}^3 \times \mathbb{R}^3$, $\Phi(A, v, w) := A \cdot (v, w) := (Av, Aw)$. This action is proper since SO(3) is a compact Lie group.

We have $\Phi_A(v, w) = (v, w)$ if and only if $Av = v$ and $Aw = w$, i.e., the rotation A fixes v and w. Hence, we have the following three cases:

(1) $(v, w) = (0, 0)$: in this case the isotropy subgroup is $G_{(0,0)} = \text{SO}(3)$,
(2) v and w are linearly independent: $G_{(v,w)} = \{\text{Id}_3\}$,
(3) v and w are linearly dependent and not both equal to zero; without loss of generality assume that $v \neq 0$: $G_{(v,w)} = \{A \in \text{SO}(3) \mid A \text{ is a rotation with axis } v\}$.

Thus there are infinitely many isotropy type manifolds (one for each fixed direction $v \in \mathbb{R}^3$ in the third case) and three orbit type manifolds $M_0 := M_{(\text{SO}(3))}, M_2 :=$

$M_{(\{\mathrm{Id}_3\})}$, and $M_1 := M_{(\mathrm{SO}(2))}$, where

$$\mathrm{SO}(2) \simeq \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \alpha \in \mathbb{R} \right\} \subset \mathrm{SO}(3)$$

is the isotropy subgroup of (e_3, e_3), corresponding to the isotropy type manifold

$$M_{\mathrm{SO}(2)} = \left\{ (ae_3, be_3) \mid (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \right\}.$$

Define $f_1, f_2, f_3 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $f_1(v,w) = \|v\|^2$, $f_2(v,w) = \|w\|^2$ and $f_3(v,w) = \langle v, w\rangle$. The pairs (v,w) and (v',w') are in the same G-orbit if and only if the three functions are equal on (v,w) and (v',w'). Indeed, if $\|v\| = \|v'\|$, $\|w\| = \|w'\|$, and $\langle v,w\rangle = \langle v', w'\rangle$, then there exists a rotation $A \in \mathrm{SO}(3)$ such that $Av = v'$ and $Aw = w'$.

The orbit space is thus diffeomorphic to the subset $\bar{M}$ of $\mathbb{R}^3$ defined by

$$\bar{M} := \{(f_1, f_2, f_3)(v,w) \mid (v,w) \in \mathbb{R}^3 \times \mathbb{R}^3\};$$

hence $\bar{M} := \{(x,y,z) \in \mathbb{R}^3 \mid x, y \geq 0 \text{ and } z^2 \leq xy\}$ by the Cauchy-Schwarz inequality. This is a stratified space with three strata

$$\begin{aligned}
&\bar{P}_0 := \{(0,0,0)\} = M_0/\,\mathrm{SO}(3),\\
&\bar{P}_1 := \{(x,y,z) \in \mathbb{R}^3 \mid x,y \geq 0,\ (x,y) \neq (0,0) \text{ and } z^2 = xy\} = M_1/\,\mathrm{SO}(3),\\
&\bar{P}_2 := \{(x,y,z) \in \mathbb{R}^3 \mid x,y > 0 \text{ and } z^2 < xy\} = M_2/\,\mathrm{SO}(3)
\end{aligned}$$

(compare with Proposition 5.2). In Figure 1 we have represented the strata $\bar{P}_0$ (the point $(0,0,0)$) and $\bar{P}_1$ (the surface without the singular point $(0,0,0)$) of the reduced space $\bar{M} = (\mathbb{R}^3 \times \mathbb{R}^3)/\,\mathrm{SO}(3)$. The manifold $\bar{P}_2$ is the open set "inside" the surface.

We use the coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$p := (v,w) = (x_1, y_1, z_1, x_2, y_2, z_2).$$

The invariant functions f_1, f_2, f_3 are given in these coordinates by $f_1(p) = x_1^2 + y_1^2 + z_1^2$, $f_2(p) = x_2^2 + y_2^2 + z_2^2$, and $f_3(p) = x_1x_2 + y_1y_2 + z_1z_2$. Using Lemma 5.9 and the fact that the three invariant polynomials f_1, f_2, f_3 form a Hilbert basis for the set of $\mathbb{S}^1$-invariant polynomials on $\mathbb{R}6$ (see [3] and the Theorem of Schwarz-Mather as presented in e.g., [27]; [25] has a quick summary), we get:

$$\begin{aligned}
\mathcal{V}_G^\circ(p) &= \operatorname{span}\{\mathbf{d}f_1, \mathbf{d}f_2, \mathbf{d}f_3\}(p)\\
&= \operatorname{span}\left\{ \begin{array}{c} x_1\mathbf{d}x_1 + y_1\mathbf{d}y_1 + z_1\mathbf{d}z_1, x_2\mathbf{d}x_2 + y_2\mathbf{d}y_2 + z_2\mathbf{d}z_2, \\ x_1\mathbf{d}x_2 + x_2\mathbf{d}x_1 + y_1\mathbf{d}y_2 + y_2\mathbf{d}y_1 + z_1\mathbf{d}z_2 + z_2\mathbf{d}z_1 \end{array} \right\}(p).
\end{aligned}$$

The vertical distribution is easily computed to be

$$\mathcal{V}(p) = \operatorname{span}\left\{ \begin{array}{c} x_1\partial_{y_1} - y_1\partial_{x_1} + x_2\partial_{y_2} - y_2\partial_{x_2}, \\ x_1\partial_{z_1} - z_1\partial_{x_1} + x_2\partial_{z_2} - z_2\partial_{x_2}, \\ z_1\partial_{y_1} - y_1\partial_{z_1} + z_2\partial_{y_2} - y_2\partial_{z_2} \end{array} \right\}(p).$$

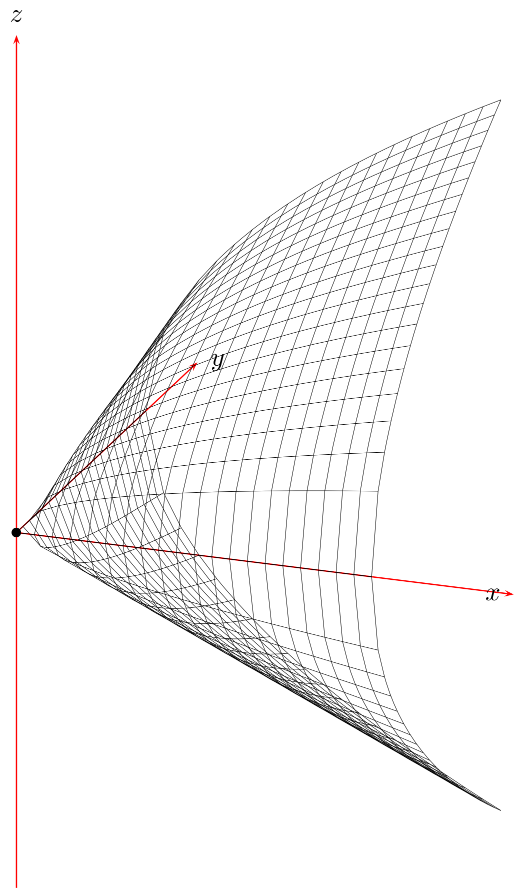

FIGURE 1

We also get (see the appendix of [18])

$$\mathcal{T}(p) = \text{span}\left\{\begin{array}{c} X_1 := x_1\partial_{x_1} + y_1\partial_{y_1} + z_1\partial_{z_1}, \\ X_2 := x_2\partial_{x_2} + y_2\partial_{y_2} + z_2\partial_{z_2}, \\ X_3 := x_1\partial_{x_2} + y_1\partial_{y_2} + z_1\partial_{z_2}, \\ X_4 := x_2\partial_{x_1} + y_2\partial_{y_1} + z_2\partial_{z_1}, \\ X_5 := x_1\partial_{y_1} - y_1\partial_{x_1} + x_2\partial_{y_2} - y_2\partial_{x_2}, \\ X_6 := x_1\partial_{z_1} - z_1\partial_{x_1} + x_2\partial_{z_2} - z_2\partial_{x_2}, \\ X_7 := z_1\partial_{y_1} - y_1\partial_{z_1} + z_2\partial_{y_2} - y_2\partial_{z_2}, \\ X_8 := (y_2z_1 - z_2y_1)\partial_{x_1} + (z_2x_1 - z_1x_2)\partial_{y_1} \\ +(x_2y_1 - y_2x_1)\partial_{z_1}, \\ X_9 := (v\times w)_x\partial_{x_2} + (v\times w)_y\partial_{y_2} + (v\times w)_z\partial_{z_2}, \\ X_{10} := ((v\times w)\times v)_x\partial_{x_1} + ((v\times w)\times v)_y\partial_{y_1} \\ +((v\times w)\times v)_z\partial_{z_1} + ((v\times w)\times w)_x\partial_{x_2} \\ +((v\times w)\times w)_y\partial_{y_2} + ((v\times w)\times w)_z\partial_{z_2}, \end{array}\right\}(p),$$

where $(v\times w)_x$, $(v\times w)_y$ and $(v\times w)_z$ are the x-, y- and z-components of the vector product $v\times w$, and

$$\mathcal{T}_G(p)=\operatorname{span}_{\mathbb{R}}\{X_1,X_2,X_3,X_4,X_8,X_9,X_{10}\}(p)$$

for all $p\in\mathbb{R}^3\times\mathbb{R}^3$.

We verify the statement of Lemma 5.10 for this particular example. We denote by $\iota_{M_i}:M_i\hookrightarrow M$ the inclusions for $i=1,2,3$. We have to show that $\iota_{M_i}^*(\mathcal{V}_G^\circ)=(\mathcal{V}_{M_i})^\circ$ for $i=1,2,3$, where $\mathcal{V}_{M_i}$ is the vertical space of the induced action of G on the stratum M_i. The statement is obvious for the two strata M_0 and M_2 since the first is a point and the second is an open set in M. Hence, we study the manifold M_1. We have a G-equivariant diffeomorphism

$$\begin{array}{rccc}\psi: & \mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\}) & \to & M_1,\\ & (u,(a,b)) & \mapsto & (au,bu),\end{array}$$

where the G-action on $\mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\})$ is given by

$$\begin{array}{rccc}\Phi: & \mathrm{SO}(3)\times\left(\mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\})\right) & \to & \mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\}),\\ & (A,(u,(a,b))) & \mapsto & (Au,(a,b)).\end{array}$$

The vertical space of the SO(3)-action on M_1 thus corresponds to the tangent space of the sphere $T\mathbb{S}^2\oplus\{0\}$ via the identification ψ. Hence $\psi^*\left((\mathcal{V}_{M_1})^\circ\right)$ is spanned by the two one-forms $\mathbf{d}a$ and $\mathbf{d}b$, where a,b are the coordinates on the $\mathbb{R}^2\setminus\{(0,0)\}$-factor.

The functions $(\psi^*\circ\iota_{M_1}^*)f_i$, $i=1,2,3$, are given by

$$\left((\psi^*\circ\iota_{M_1}^*)f_1\right)(u,a,b)=a^2,\quad \left((\psi^*\circ\iota_{M_1}^*)f_2\right)(u,a,b)=b^2,$$

and

$$\left((\psi^*\circ\iota_{M_1}^*)f_3\right)(u,a,b)=ab.$$

Hence we get

$$\left((\psi^*\circ\iota_{M_1}^*)\mathbf{d}f_1\right)(u,a,b)=2a\mathbf{d}a,\quad \left((\psi^*\circ\iota_{M_1}^*)\mathbf{d}f_2\right)(u,a,b)=2b\mathbf{d}b$$

and

$$\left((\psi^*\circ\iota_{M_1}^*)\mathbf{d}f_3\right)(u,a,b)=a\mathbf{d}b+b\mathbf{d}a.$$

Since the coordinates a and b are never simultaneously zero on $\mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\})$, we conclude that $(\psi^*\circ\iota_{M_1}^*)(\mathcal{V}_G^\circ)$ is spanned at each point of $\mathbb{S}^2\times(\mathbb{R}^2\setminus\{(0,0)\})$ by the values at this point of $\mathbf{d}a$ and $\mathbf{d}b$. This proves the desired equality $(\mathcal{V}_{M_1})^\circ=\iota_{M_1}^*(\mathcal{V}_G^\circ)$.

Another interesting fact to be checked directly is the equality between the accessible sets of the distribution $\mathcal{T}$ (respectively $\mathcal{T}_G$) and the orbit type manifolds (respectively the isotropy type manifolds). The flows $\phi^1,\ldots,\phi^{10}$ of the vector fields $X_1,\ldots,X_{10}$ are given by

$$\begin{aligned}
&\phi_t^1(v,w)=(e^tv,w),\qquad \phi_t^2(v,w)=(v,e^tw),\\
&\phi_t^3(v,w)=(v,tv+w),\qquad \phi_t^4(v,w)=(tw+v,w),\\
&\phi_t^5(v,w)=R_{e_3,t}\cdot(v,w),\qquad \phi_t^6(v,w)=R_{e_2,-t}\cdot(v,w),\\
&\phi_t^7(v,w)=R_{e_1,t}\cdot(v,w),\\
&\phi_t^8(v,w)=\exp(tB_w)\cdot(v,w),\qquad \phi_t^9(v,w)=\exp(tB_v)\cdot(v,w),\\
&\phi_t^{10}(v,w)=\exp(tB_{v\times w})\cdot(v,w),
\end{aligned}$$

where $R_{e_i,t} \in \mathrm{SO}(3)$ is the rotation about the e_i-axis by the angle $t \in \mathbb{R}$ and

$$B_w := \begin{pmatrix} 0 & z_2 & -y_2 \\ -z_2 & 0 & x_2 \\ y_2 & -x_2 & 0 \end{pmatrix} \in \mathfrak{so}(3), \quad \text{for} \quad w = (x_2, y_2, z_2).$$

A straightforward computation for the rotations about the axes shows that $A \exp(tB_w) A^{-1} = \exp(tB_{Aw})$ for all $A \in \mathrm{SO}(3)$. Since $B_w w = 0$ it follows that $\exp(tB_w)w = w$ and $\exp(tB_w) \in \mathrm{SO}(3)$ is a rotation with axis w if $w \neq 0$, and the identity if $w = 0$.

We have $\phi_t^i(0,0) = (0,0)$ for $i = 1, \ldots, 10$ and all $t \in \mathbb{R}$, which shows that the accessible set of $\mathcal{T}$ and of $\mathcal{T}_G$ through the origin $(0,0)$ is the origin and we recover the orbit and isotropy type manifold $\{(0,0)\} = M_{(\mathrm{SO}(3))} = M_{\mathrm{SO}(3)}$.

If v and w are linearly independent (respectively dependent), it is easy to verify that the two components of $\phi_t^i(v,w)$ are linearly independent (respectively dependent) for $i = 1, \ldots, 10$ and all $t \in \mathbb{R}$. This shows that the flow of each of the vector fields $X_1, \ldots, X_{10}$ leaves the orbit type manifolds invariant and that the flows of each of the spanning vector fields of $\mathcal{T}_G$ leave the isotropy type manifolds invariant (note that $\phi_t^8(v,w) = \phi_t^9(v,w) = \phi_t^{10}(v,w) = (v,w)$ if v and w are linearly dependent). Hence, we have to verify that each two pairs of vectors in the same isotropy (respectively orbit) type manifold can be joined by a concatenation of paths formed by pieces of integral curves of the vector fields spanning $\mathcal{T}_G$ (respectively $\mathcal{T}$).

We start with linearly dependent pairs. Choose $(v,w) \neq (0,0)$ and $(v',w') \neq (0,0)$ in the same isotropy type $\{(au, bu) \mid a, b \in \mathbb{R},\, (a,b) \neq (0,0)\}$ for some $u \neq 0$ in $\mathbb{R}^3$. Write $(v,w) = (au, bu)$ and $(v', w') = (a'u, b'u)$. There are several different cases to be considered.

(1) If $aa' > 0$ and $bb' > 0$, then (v,w) can be joined to (v',w') by flow lines of ϕ^1 and ϕ^2. Indeed, if we set $t_1 = \ln(\frac{a'}{a})$ and $t_2 = \ln(\frac{b'}{b})$, we get $(a'u, b'u) = (\phi_{t_2}^2 \circ \phi_{t_1}^1)(au, bu)$.
(2) If $b \neq 0$ and $a' \neq 0$, set $t_1 = \frac{a'-a}{b}$ and $t_2 = \frac{b'-b}{a'}$ and get $(\phi_{t_2}^3 \circ \phi_{t_1}^4)(au, bu) = \phi_{t_2}^3(a'u, bu) = (a'u, b'u)$.
(3) If $a \neq 0$ and $b' \neq 0$, set $t_1 = \frac{b'-b}{a}$ and $t_2 = \frac{a'-a}{b'}$ and get $(\phi_{t_2}^4 \circ \phi_{t_1}^3)(au, bu) = \phi_{t_2}^3(a'u, bu) = (a'u, b'u)$.
(4) If $aa' > 0$ and $b = b' = 0$, set $t = \ln(\frac{a'}{a})$. Then $(a'u, 0) = \phi_t^1(au, 0)$. Use the same method with ϕ^2 for the case $bb' > 0$ and $a = a' = 0$.
(5) If $aa' < 0$ and $b = b' = 0$, then $(au, -au) = \phi_{-1}^3(au, 0)$ and we can continue as in case 2. Use the same method with ϕ^4 and case 3 for the case $bb' < 0$ and $a = a' = 0$.

To join two pairs (au, bu) and $(a'v, b'v)$ in the *orbit type* manifold of linearly dependent pairs (choose u and v of unit length), we use a combination of integral curves of X_5, X_6, X_7 to send (au, bu) to (av, bv) by a rotation and then we proceed as above using integral curves of X_1, X_2, X_3, and X_4.

The isotropy type manifold through a linearly independent pair is equal to the orbit type manifold through this pair:

$$M_{\mathrm{Id}_3} = M_{(\mathrm{Id}_3)} = \{(v,w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid (v,w) \text{ linearly independent}\}.$$

It is then possible to join two pairs of this type by integral curves of the vector fields $X_1, \ldots, X_4$ and X_8, X_9, X_{10}. First, we show that if v', w' lie in the span of

v, w, we can join (v,w) to (v',w') by pieces of the integral curves of $X_1,\dots,X_4$. Indeed, there exist $a,b,c,d\in\mathbb{R}$ such that $v'=av+bw$ and $w'=cv+dw$.

(1) If $b=0$ (that is, v and v' are linearly dependent), then a and d have to be nonzero because $av+bw=av$ and $cv+dw$ are linearly independent. We then have several subcases:

(a) If $a>0$ and $d>0$, then $(av, cv+dw)=(\phi^1_{\ln a}\circ\phi^3_c\circ\phi^2_{\ln d})(v,w)$.

(b) If $a>0$ and $d<0$, then we have

$$\begin{aligned}&\left(\phi^1_{\ln a}\circ\phi^3_{c+d\frac{\langle v,w\rangle}{\langle v,v\rangle}}\circ\phi^2_{\ln(-d)}\circ\phi^9_\pi\circ\phi^3_{-\frac{\langle v,w\rangle}{\langle v,v\rangle}}\right)(v,w)\\ &=\left(\phi^1_{\ln a}\circ\phi^3_{c+d\frac{\langle v,w\rangle}{\langle v,v\rangle}}\circ\phi^2_{\ln(-d)}\circ\phi^9_\pi\right)\left(v, w-\frac{\langle v,w\rangle}{\langle v,v\rangle}v\right)\\ &=\left(\phi^1_{\ln a}\circ\phi^3_{c+d\frac{\langle v,w\rangle}{\langle v,v\rangle}}\circ\phi^2_{\ln(-d)}\right)\left(v, -w+\frac{\langle v,w\rangle}{\langle v,v\rangle}v\right)=(av, cv+dw).\end{aligned}$$

We have used the fact that since v and $w-\frac{\langle v,w\rangle}{\langle v,v\rangle}v$ are orthogonal, the rotation $\exp(\pi B_v)$ of angle π around the axis spanned by v sends $w-\frac{\langle v,w\rangle}{\langle v,v\rangle}v$ to $-w+\frac{\langle v,w\rangle}{\langle v,v\rangle}v$.

(c) If $a<0$ and $d>0$, then we have in an analogous manner

$$\left(\phi^1_{\ln(-a)}\circ\phi^3_{-\left(c+d\frac{\langle v,w\rangle}{\langle v,v\rangle}\right)}\circ\phi^2_{\ln d}\circ\phi^8_\pi\circ\phi^3_{-\frac{\langle v,w\rangle}{\langle v,v\rangle}}\right)(v,w)=(av,cv+dw).$$

(d) Finally, if $a<0$ and $d<0$, we have

$$\left(\phi^1_{\ln(-a)}\circ\phi^3_{-c}\circ\phi^2_{\ln(-d)}\circ\phi^{10}_\pi\right)(v,w)=(av,cv+dw).$$

(2) If $b\neq 0$, choose t_1 such that $t_1b-a\neq 0$ and $t_2=\frac{-b}{t_1b-a}$. Then we have $(\phi^4_{t_2}\circ\phi^3_{t_1})(v,w)=((1+t_1t_2)v+t_2w, w+t_1v)$. Since $(1+t_1t_2)b-t_2a=\left(1+t_1\frac{-b}{t_1b-a}\right)b+\frac{ab}{t_1b-a}=b\left(1+\frac{a-t_1b}{t_1b-a}\right)=0$, the vectors $av+bw$ and $(1+t_1t_2)v+t_2w$ are linearly dependent and we continue as in case 1. (Using an integral curve of ϕ^{10}, we can also first rotate v and w around the axis $v\times w$ so that the images of v and v' are linearly dependent, and then continue as in case 1.)

Then, to simplify the problem, we assume, without loss of generality, that the plane spanned by (v,w) is the (x,y)-plane α_{xy} (spanned by e_1 and e_2). By the considerations above, we can bring (v,w) to (e_1,e_2) along pieces of integral curves of $X_1,\dots,X_4$. Hence, to finish the proof it suffices to show that we can use pieces of integral curves of X_8, X_9, X_{10} to bring the plane spanned by e_1, e_2 to the plane α spanned by an arbitrary linearly independent pair (v',w'). If (v',w') spans the (x,y)-plane α_{xy}, we are done by the considerations above. Otherwise, there are again two cases (Figure 2 illustrates this second case).

(1) If α is equal to the plane α_{xz} spanned by e_1 and e_3, then we have $(e_1,e_3)=\phi^9_{\pi/2}(e_1,e_2)$ and we are done.

(2) If not, there exists a unit vector $u\in\mathbb{R}^3$ spanning the intersection $\alpha_{xz}\cap\alpha$. Then there exists $t_1\in\mathbb{R}$ such that $(u,e_2)=\exp(t_1B_{e_2})\cdot(e_1,e_2)=\phi^8_{t_1}(e_1,e_2)$. If e_2 lies in α, the vectors u and e_2 are linearly independent by construction and we are done. Otherwise, let u' be a unit vector spanning the intersection of α with the plane spanned by e_2 and $\exp(tB_{e_2})e_3$ (this

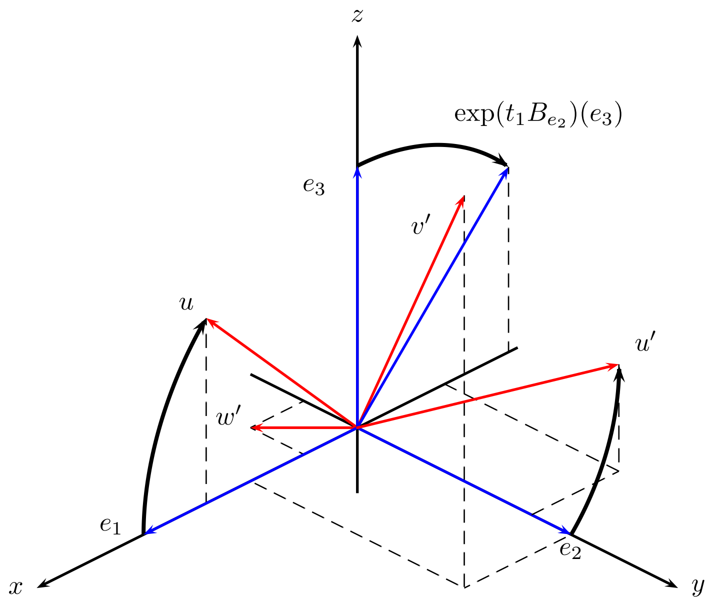

FIGURE 2

is the plane orthogonal to u). Then there exists t_2 such that $(u, u') = \exp(t_2 B_u) \cdot (u, e_2) = \exp(t_2 B_u) \exp(t_1 B_{e_2}) \cdot (e_1, e_2) = (\phi^9_{t_2} \circ \phi^8_{t_1})(e_1, e_2)$.

These considerations illustrate Theorem 5.8 stating that the integral leaves of $\mathcal{T}$ are the connected components of the orbit type manifolds and Theorem 3.5.1 in [25] stating that the integral leaves of $\mathcal{T}_G$ are the connected components of the isotropy type manifolds.

We now study properties of the stratified space $\bar{M}$. We want to show that the restrictions of the *vector fields* on $\bar{M}$ to the strata of $\bar{M}$ span the tangent space of each stratum. The flows $\bar{\phi}^1, \ldots, \bar{\phi}^{10}$ associated to the vector fields $\bar{X}_1, \ldots, \bar{X}_{10}$ defined by $X_i \sim_\pi \bar{X}_i$ for $i = 1, \ldots, 10$ are given by

$$
\begin{aligned}
&\bar{\phi}^1_t(x, y, z) = \left(\bar{\phi}^1_t \circ \pi\right)(v, w) = \left(\pi \circ \phi^1_t\right)(v, w) = \pi(e^t v, w) = (e^{2t} x, y, e^t z),\\
&\bar{\phi}^2_t(x, y, z) = (x, e^{2t} y, e^t z),\\
&\bar{\phi}^3_t(x, y, z) = \pi(v, tv + w) = (x, t^2 x + 2tz + y, xt + z),\\
&\bar{\phi}^4_t(x, y, z) = (t^2 y + 2tz + x, y, yt + z),\\
&\bar{\phi}^5_t(x, y, z) = \bar{\phi}^6_t(x, y, z) = \bar{\phi}^7_t(x, y, z) = \bar{\phi}^8_t(x, y, z)\\
&\qquad\qquad\quad = \bar{\phi}^9_t(x, y, z) = \bar{\phi}^{10}_t(x, y, z) = (x, y, z).
\end{aligned}
$$

This leads to

$$\begin{aligned}
\bar{X}_1(x,y,z) &= 2x\partial_x + z\partial_z, \qquad \bar{X}_2(x,y,z) = 2y\partial_y + z\partial_z,\\
\bar{X}_3(x,y,z) &= 2z\partial_y + x\partial_z, \qquad \bar{X}_4(x,y,z) = 2z\partial_x + y\partial_z,\\
\bar{X}_5(x,y,z) &= \bar{X}_6(x,y,z) = \bar{X}_7(x,y,z) = \bar{X}_8(x,y,z)\\
&= \bar{X}_9(x,y,z) = \bar{X}_{10}(x,y,z) = 0.
\end{aligned}$$

The last equalities are consistent with the fact that $X_5, \ldots, X_{10}$ are sections of the vertical space $\mathcal{V}$. At the point $(0,0,0)$, we have hence $\bar{X}_1(0,0,0) = \ldots = \bar{X}_4(0,0,0) = 0$, and we conclude that the (trivial) tangent space of $\bar{P}_0$ is spanned by the values at $(0,0,0)$ of the vector fields on $\bar{M}$. The stratum $\bar{P}_1$ can be seen as the manifold given by the equation $xy = z^2$ in $(\mathbb{R}^2 \setminus \{(0,0)\}) \times \mathbb{R}$. Thus, we know that the tangent space of $\bar{P}_1$ is equal to the kernel of $x\mathbf{d}y + y\mathbf{d}x - 2z\mathbf{d}z$ at points of $\bar{P}_1$. We thus find that the values of $\bar{X}_1, \ldots, \bar{X}_4$ at points of $\bar{P}_1$ span the tangent space of $\bar{P}_1$ (recall that x and y never vanish simultaneously on $\bar{P}_1$). The points (x,y,z) of the last stratum $\bar{P}_2$ satisfy $x, y > 0$ and $z^2 < xy$. Hence, for $p = (x,y,z) \in \bar{P}_2$, we have

$$\begin{aligned}
&\operatorname{span}_{\mathbb{R}}\{2x\partial_x + z\partial_z, 2y\partial_y + z\partial_z, 2z\partial_y + x\partial_z, 2z\partial_x + y\partial_z\}(p)\\
&\quad = \operatorname{span}_{\mathbb{R}}\left\{\partial_x + \frac{z}{2x}\partial_z, \partial_y + \frac{z}{2y}\partial_z, \frac{2z}{x}\partial_y + \partial_z, \frac{2z}{y}\partial_x + \partial_z\right\}(p)\\
&\quad = \operatorname{span}_{\mathbb{R}}\left\{\partial_x, \partial_y, \partial_z\right\}(p),
\end{aligned}$$

where we have used $\frac{z^2}{xy} < 1$ and the identities

$$\begin{aligned}
\left(\partial_x + \frac{z}{2x}\partial_z\right) - \frac{z}{2x}\left(\frac{2z}{y}\partial_x + \partial_z\right) &= \left(1 - \frac{z^2}{xy}\right)\partial_x,\\
\left(\partial_y + \frac{z}{2y}\partial_z\right) - \frac{z}{2y}\left(\frac{2z}{x}\partial_y + \partial_z\right) &= \left(1 - \frac{z^2}{xy}\right)\partial_y.
\end{aligned}$$

Finally, we study in the same manner the push-forwards of the three one-forms $\mathbf{d}f_1$, $\mathbf{d}f_2$, $\mathbf{d}f_3$ spanning $\mathcal{V}_G^\circ$. Denote by $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\alpha}_3$ these three "one-forms" on $\bar{M}$ (see Subsection 5.2). Since $\mathbf{d}f_1$, $\mathbf{d}f_2$, $\mathbf{d}f_3$ vanish at 0, we have $\bar{\alpha}_1(0) = \bar{\alpha}_2(0) = \bar{\alpha}_3(0) = 0$, by definition, and we conclude $\operatorname{span}_{\mathbb{R}}\{\bar{\alpha}_1(0), \bar{\alpha}_2(0), \bar{\alpha}_3(0)\} = T_0^*\bar{P}_0$. At points of $\bar{P}_2$, we have $\bar{\alpha}_1 = \mathbf{d}x$, $\bar{\alpha}_2 = \mathbf{d}y$, and $\bar{\alpha}_3 = \mathbf{d}z$. Finally, at points of M_1, we have the equality $2f_3\mathbf{d}f_3 = f_1\mathbf{d}f_2 + f_2\mathbf{d}f_1$ and we obtain, as desired,

$$\operatorname{span}_{\mathbb{R}}\{\bar{\alpha}_1(x,y,z), \bar{\alpha}_2(x,y,z), \bar{\alpha}_3(x,y,z)\} = \operatorname{span}_{\mathbb{R}}\{\mathbf{d}x, \mathbf{d}y, \mathbf{d}z\}_{(x,y,z)}/\sim,$$

where $\sim$ is the equivalence relation on $\operatorname{span}_{\mathbb{R}}\{\mathbf{d}x, \mathbf{d}y, \mathbf{d}z\}_{(x,y,z)}$ defined by $x\mathbf{d}y + y\mathbf{d}x - 2z\mathbf{d}z = 0$.

This shows that the restrictions of the "one-forms" on $\bar{M}$ to each of its strata span the cotangent space of each stratum, as stated in the considerations at the beginning of Subsection 5.2 together with Propositions 5.2 and 5.12.

6. Singular reduction of Dirac structures

6.1. The special case of conjugated isotropy subgroups. In the special case of a proper action with conjugated orbit subgroups, the reduction theorem is shown in [19]. We recall its formulation here because the understanding of the construction of the reduced Dirac structure in this case can be helpful for the understanding of the general case.

Theorem 6.1. *Let G be a connected Lie group acting in a proper way on the manifold M, such that all isotropy subgroups are conjugated. Assume that $D \cap \mathcal{K}^{\perp}$ has constant rank on M. Then the Dirac structure D on M induces a Dirac structure $\bar{D}$ on the quotient $\bar{M} = M/G$ given by*

(6.1)
$$\bar{D}(\bar{m}) = \left\{ \big(\bar{X}(\bar{m}), \bar{\alpha}(\bar{m})\big) \in T_{\bar{m}}\bar{M} \times T^*_{\bar{m}}\bar{M} \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(M) \text{ such that} \\ X \sim_\pi \bar{X} \text{ and } (X, \pi^*\bar{\alpha}) \in \Gamma(D) \end{array} \right\}$$

for all $\bar{m}$ in $\bar{M}$. If D is integrable, then $\bar{D}$ is also integrable.

Remark 6.2. Note that the method we use for singular reduction yields this *regular reduction theorem* as a corollary of our general singular reduction theorems for Dirac structures (see Theorems 6.6 and 6.5 in the next subsection).

As in the Poisson case (compare with [15]), it is also possible to prove singular reduction by using regular reduction. Indeed, if $Q \subseteq M_H \subseteq M$ is a connected component of an isotropy type, it is possible to show that the Dirac structure D on M restricts naturally to a Dirac structure D_Q on Q, which would be $N(H)/H$-invariant if D were G-invariant, and integrable if D were integrable. To prove these statements (see [17]) one needs Proposition 4.4 and G-invariant averaging (see Subsection 5.1). Since the action of $N(H)/H$ on Q is free and proper, we can use regular Dirac reduction on the Dirac manifold (Q, D_Q) and get a smooth quotient Dirac manifold $(\bar{Q}, D_{\bar{Q}})$. The manifold $\bar{Q}$ is diffeomorphic to the quotient P/G if $P = G \cdot Q$ is the connected component of $M_{(H)}$ containing Q. In fact, we have $\bar{P} = \pi(P) = \pi(Q)$ if $\pi : M \to \bar{M}$ is the orbit map, and $\bar{Q}$ is then a stratum of the reduced space $\bar{M}$. By construction, it is then easy to see that the reduced Dirac manifold $(\bar{Q}, D_{\bar{Q}})$ is diffeomorphic to the reduced space $(\bar{P}, D_{\bar{P}})$ that we will get in the next subsection. We want to thank R. Loja Fernandes for a discussion that resulted in this remark.

6.2. **The general setting of a proper action.** Consider the subset $\mathcal{D}^G$ of $\Gamma(D)$ defined by

$$\mathcal{D}^G = \{(X, \alpha) \in \Gamma(D) \mid \alpha \in \Gamma(\mathcal{V}^\circ)^G \text{ and } [X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})\},$$

that is, the set of the descending sections of D.

We have seen in Lemma 5.3 that each vector field X satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ pushes forward to a vector field $\bar{X}$ on $\bar{M}$. By the considerations in Subsection 3.4 (see also Proposition 3.3), we know that for each stratum $\bar{P}$ of $\bar{M}$, the restriction of $\bar{X}$ to points of $\bar{P}$ is a vector field $X_{\bar{P}}$ on $\bar{P}$. On the other hand, if $(X, \alpha) \in \mathcal{D}^G$, then we have $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ and it pushes forward to the one-form $\bar{\alpha} := \pi_*\alpha$ such that, for every $\bar{Y} \in \mathfrak{X}(\bar{M})$ and every section Y of TM satisfying $Y \sim_\pi \bar{Y}$, we have

$$\pi^*(\bar{\alpha}(\bar{Y})) = \alpha(Y).$$

Moreover, for each stratum $\bar{P}$ of $\bar{M}$, the restriction of $\bar{\alpha}$ to points of $\bar{P}$ defines a 1-form $\alpha_{\bar{P}}$ on $\bar{P}$. Let

$$\bar{\mathcal{D}} = \{(\bar{X}, \bar{\alpha}) \mid (X, \alpha) \in \mathcal{D}^G\}$$

and for each stratum $\bar{P}$ of $\bar{M}$, set

$$\mathcal{D}_{\bar{P}} = \{(X_{\bar{P}}, \alpha_{\bar{P}}) \mid (\bar{X}, \bar{\alpha}) \in \bar{\mathcal{D}}\}.$$

Define the smooth distribution $D_{\bar{P}}$ on $\bar{P}$ by

$$D_{\bar{P}}(s) = \{(X_{\bar{P}}(s), \alpha_{\bar{P}}(s)) \in T_s\bar{P} \times T_s^*\bar{P} \mid (X_{\bar{P}}, \alpha_{\bar{P}}) \in \mathcal{D}_{\bar{P}}\}. \tag{6.2}$$

Remark 6.3. Note that $\Gamma(D_{\bar{P}}) = \mathcal{D}_{\bar{P}}$. Indeed, any $(X_{\bar{P}}, \alpha_{\bar{P}}) \in \Gamma(D_{\bar{P}})$ can be written as

$$(X_{\bar{P}}, \alpha_{\bar{P}}) = \sum_{i=1}^{k} f_{\bar{P}}^i (X_{\bar{P}}^i, \alpha_{\bar{P}}^i), \quad (X_{\bar{P}}^i, \alpha_{\bar{P}}^i) \in \mathcal{D}_{\bar{P}}, \quad f_{\bar{P}}^i \in C^\infty(\bar{P}).$$

Each $(X_{\bar{P}}^i, \alpha_{\bar{P}}^i)$ has a smooth extension $(\bar{X}^i, \bar{\alpha}^i) \in \bar{\mathcal{D}}$ which is a push-forward of some element $(X^i, \alpha^i) \in \mathcal{D}^G$. Each function $f_{\bar{P}}^i$ smoothly extends to a function $\bar{f}^i \in C^\infty(\bar{M})$, by the smooth structure of $\bar{P}$ as a stratum of $\bar{M}$, which is a push-forward of a function $f^i \in C^\infty(M)^G$. Therefore $\sum_{i=1}^k f^i(X^i, \alpha^i)$ is a descending section of D and the restriction to $\bar{P}$ of its push-forward to $\bar{M}$ coincides with $(X_{\bar{P}}, \alpha_{\bar{P}})$.

Theorem 6.4. *Let (M, D) be a Dirac manifold with a proper Dirac action of a connected Lie group G on it. Assume that the intersection $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections. Then each element $(\bar{X}, \bar{\alpha}) \in \mathfrak{X}(\bar{M}) \times \Omega^1(\bar{M})$ orthogonal to all the sections in $\bar{\mathcal{D}}$ is already an element of $\bar{\mathcal{D}}$.*

Proof. Let $(\bar{X}, \bar{\alpha}) \in \mathfrak{X}(\bar{M}) \times \Omega^1(\bar{M})$ be such that $\bar{\alpha}(\bar{Y}) + \bar{\beta}(\bar{X}) = 0$ for all elements $(\bar{Y}, \bar{\beta}) \in \bar{\mathcal{D}}$. Let $(Y, \beta) \in \Gamma(D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ))$ be such that $Y \sim_\pi \bar{Y}$ and $\beta = \pi^*\bar{\beta}$. Choose also $X \in \mathfrak{X}(M)$ such that $X \sim_\pi \bar{X}$ and set $\alpha = \pi^*\bar{\alpha} \in \Omega^1(M)$ (see Proposition 5.6 and the considerations after Lemma 5.5). Then we get

$$\langle (X, \alpha), (Y, \beta) \rangle = \left(\bar{\alpha}(\bar{Y}) + \bar{\beta}(\bar{X})\right) \circ \pi = 0.$$

Since $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections, we get $(X, \alpha) \in \Gamma((D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ))^\perp)$. Hence, with Lemma 5.14, we get $(X, \alpha) \in \Gamma(D + \mathcal{K})$ and there exist $X' \in \mathfrak{X}(M)$ and $V \in \Gamma(\mathcal{V})$ such that (X', α) is a section of D and $(X, \alpha) = (X', \alpha) + (V, 0)$. Because of the definition of α, X, and V, we immediately get that (X', α) is a descending section of $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$. It is easy to see that $X' \sim_\pi \bar{X}$, and we have $\alpha = \pi^*\bar{\alpha}$. Thus $(\bar{X}, \bar{\alpha})$ is an element of $\bar{\mathcal{D}}$. □

As a consequence of this theorem, we get that the set of pairs $(\bar{X}, \bar{\alpha}) \in \mathfrak{X}(\bar{M}) \times \Omega^1(\bar{M})$ orthogonal to all the elements of $\bar{\mathcal{D}}$ is $\bar{\mathcal{D}}$ itself. Hence, it is natural to ask if $D_{\bar{P}}$ defines a Dirac structure on $\bar{P}$ for each stratum $\bar{P}$ of $\bar{M}$. For the stratum $\bar{M}^{\text{reg}} = \pi(M^{\text{reg}})$, this is automatically true since M^{reg} is open and dense in M.

Theorem 6.5. *Let (M, D) be a Dirac manifold with a proper Dirac action of a connected Lie group G on it. Let $\bar{P}$ be a stratum of the quotient space $\bar{M}$. If $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections, then $D_{\bar{P}}$ defined in* (6.2) *is a Dirac structure on $\bar{P}$.*

Proof. Let the stratum $\bar{P}$ be a connected component of $\pi(M_{(H)})$ for a compact Lie subgroup H of G, and let P be the connected component of $M_{(H)}$ such that $\pi^{-1}(\bar{P}) = P$.

The inclusion $D_{\bar{P}} \subseteq D_{\bar{P}}^\perp$ is easy. For the other inclusion, choose $\bar{p} \in \bar{P}$ and $p \in P \subseteq M_{(H)}$ such that $\pi(p) = \bar{p}$. Recall from Proposition 5.2 that the action Φ of G on M restricts to the proper action Φ^P on P, and the quotient map π restricts to $\pi_P := \pi|_P : P \to \bar{P}$.

Let $(X_{\bar P},\alpha_{\bar P})\in\mathfrak{X}(\bar P)\times\Omega^1(\bar P)$ be a section of $D_{\bar P}^{\perp}$ defined on a neighborhood $U_{\bar P}$ of $\bar p$. Then there exists $(\bar X,\bar\alpha)\in\mathfrak{X}(\bar M)\times\Omega^1(\bar M)$, with $\operatorname{Dom}(\bar X,\bar\alpha)=:\bar U$, such that $X_{\bar P}\sim_{\iota_{\bar P}}\bar X$ and $\alpha_{\bar P}=\iota_{\bar P}^*\bar\alpha$, and $(X,\alpha)\in\mathfrak{X}(M)^G\times\Gamma(\mathcal{V}^\circ)^G$ defined on $U:=\pi^{-1}\bar U$, such that $X\sim_\pi\bar X$ and $\alpha=\pi^*\bar\alpha$. By Remark 6.3, we find for each $(Y_{\bar P},\beta_{\bar P})\in\Gamma(D_{\bar P})$ sections $(\bar Y,\bar\alpha)\in\bar{\mathcal{D}}$ and $(Y,\alpha)\in\mathcal{D}^G$ such that $Y\sim_\pi\bar Y$, $Y_{\bar P}\sim_{\iota_{\bar P}}\bar Y$, and $\alpha=\pi^*\bar\alpha$, $\alpha_{\bar P}=\iota_{\bar P}^*\bar\alpha$. We get the equalities

$$\begin{aligned}\langle(X,\alpha),(Y,\beta)\rangle\circ\iota_P&=\langle(\bar X,\bar\alpha),(\bar Y,\bar\beta)\rangle\circ\pi\circ\iota_P\\&=\langle(\bar X,\bar\alpha),(\bar Y,\bar\beta)\rangle\circ\iota_{\bar P}\circ\pi_P\\&=\langle(X_{\bar P},\alpha_{\bar P}),(Y_{\bar P},\beta_{\bar P})\rangle\circ\pi_P=0.\end{aligned}$$

Since $D\cap(\mathcal{T}\oplus\mathcal{V}_G^\circ)$ is spanned by its descending sections, we get that $(X,\alpha)|_P$ is a section of

$$\big((D\cap(\mathcal{T}\oplus\mathcal{V}_G^\circ))|_P\big)^{\perp}.$$

But since $D\cap(\mathcal{T}\oplus\mathcal{V}_G^\circ)$ is spanned by its descending sections it is in particular smooth. By Proposition 5.13, we get that $(X,\alpha)|_P$ is a section of

$$D|_P+\mathcal{K}|_P+(TP^\perp\oplus TP^\circ).$$

Thus, there exist for all x in the G-invariant set $U\cap P$ an open neighborhood $U_x\subseteq M$ of x, and sections (Z^x,ζ^x) of D and V^x of $\mathcal{V}$ defined on the whole of M (otherwise multiply them with an appropriate bump-function), and sections $W^x\in\Gamma(TP^\perp)$ and $\gamma^x\in\Gamma(TP^\circ)$ defined on P such that

$$(X,\alpha)|_{P\cap U_x}=(Z^x,\zeta^x)|_{P\cap U_x}+(V^x,0)|_{P\cap U_x}+(W^x,\gamma^x)|_{P\cap U_x}.$$

Since M is paracompact, its open submanifold $U':=\bigcup_{x\in P\cap U}U_x$ is also paracompact and there exists a locally finite refinement $\mathcal{U}_\Lambda$ of its open covering $\{U_x\mid x\in P\cap U\}$, where Λ is a subset of $P\cap U$, and a partition of unity $\{\rho_\lambda\}_{\lambda\in\Lambda}$ subordinate to $\mathcal{U}_\Lambda$. Set $\rho_\lambda|_{M\setminus U'}=0$ for all $\lambda\in\Lambda$. Then all the functions ρ_λ are defined on the whole of M.

Define the global sections

$$(Z,\zeta)=\sum_{\lambda\in\Lambda}\rho_\lambda(Z^\lambda,\zeta^\lambda),\qquad (V,0)=\sum_{\lambda\in\Lambda}\rho_\lambda(V^\lambda,0),$$

and

$$(W,\gamma)=\sum_{\lambda\in\Lambda}\rho_\lambda(W^\lambda,\gamma^\lambda).$$

Then (Z,ζ) is a section of D, V a section of $\mathcal{V}$, $W\in\Gamma(TP^\perp)$ and $\gamma\in\Gamma(TP^\circ)$, and we have for all $p'\in P\cap U$:

$$\begin{aligned}&\big((Z,\zeta)|_P+(V,0)|_P+(W,\gamma)\big)(p')\\&=\sum_{\lambda\in\Lambda}\rho_\lambda(p')\big((Z^\lambda,\zeta^\lambda)(p')+(V^\lambda,0)(p')+(W^\lambda,\gamma^\lambda)(p')\big)\\&=\sum_{\lambda\in\Lambda}\rho_\lambda(p')(X,\alpha)(p')=(X,\alpha)(p').\end{aligned}$$

Consider the G-invariant averages Z_G, ζ_G, V_G, γ_G, W_G of Z, ζ, V, γ, and W. We get for all $p'=[g,b]_H\in P\cap U$,

$$\begin{aligned}\big(Z_G+V_G+W_G\big)(p')&=T_{[e,b]_H}\Phi_g\left(\int_H\left(\Phi_h^*(Z+V+W)\right)([e,b]_H)dh\right)\\&=T_{[e,b]_H}\Phi_g\left(\int_H T_{[h,b]_H}\Phi_{h^{-1}}(Z+V+W)([h,b]_H)dh\right)\\&=T_{[e,b]_H}\Phi_g\left(\int_H T_{[h,b]_H}\Phi_{h^{-1}}X([h,b]_H)dh\right)\\&=T_{[e,b]_H}\Phi_g\left(\int_H X([e,b]_H)dh\right)=T_{[e,b]_H}\Phi_gX([e,b]_H)\\&=(\Phi_g^*X)(p')=X(p'),\end{aligned}$$

where we have used the fact that X is G-invariant and that $[g,b]_H$ in $P\cap U$ implies $[e,b]_H$ and $[h,b]_H\in P\cap U$ for all $h\in H$. In the same manner, we show that

$$\alpha(p')=(\zeta_G+\gamma_G)(p').$$

Thus we have

$$(X,\alpha)=(Z_G,\zeta_G)+(V_G,0)+(W_G,\gamma_G)$$

on $P\cap U$. Since all involved distributions $\mathcal{V}$, $TP^{\perp}$, TP°, and D are G-invariant, we still have $V_G\in\Gamma(\mathcal{V})$, $W_G\in\Gamma(TP^{\perp})$, $\gamma_G\in\Gamma(TP^{\circ})$, and $(Z_G,\zeta_G)\in\Gamma(D)$. But now we have $X-Z_G-V_G\in\Gamma(\mathcal{T})$ and hence $(X-Z_G-V_G)|_P\in\Gamma(TP)$. Thus, the equality $(X-Z_G-V_G)|_P=W_G\in\Gamma(TP^{\perp})$ leads to $W_G=(X-Z_G-V_G)|_P=0$. The section $X_{\gamma_G}\in\Gamma(TP^{\perp})$ satisfying $\mathbf{i}_{X_{\gamma_G}}\rho|_P=\gamma_G$ is G-invariant, since γ_G is. Hence it is tangent to P and has to be the zero section. This shows that $\gamma_G=0$.

Thus, we have

$$(X,\alpha)|_P=(Z_G,\zeta_G)|_P+(V_G,0)|_P,$$

and simultaneously $(Z_G,\zeta_G)|_P\in\Gamma\left((D\cap(\mathcal{T}\oplus\mathcal{V}_G^{\circ}))|_P\right)$. Hence, there exists a section (X',α') of $D\cap(\mathcal{T}\oplus\mathcal{V}_G^{\circ})$ defined on a neighborhood $U'\subseteq U$ of p such that $(X',\alpha')|_{P\cap U'}=(Z_G,\zeta_G)|_{P\cap U'}$, and hence $(X,\alpha)|_{P\cap U'}=(X',\alpha')|_{P\cap U'}+(V_G,0)|_{P\cap U'}$. By G-invariant averaging (in the same manner as above, with a partition of unity if needed), we can assume that (X',α') is G-invariant and that U' is G-invariant. Hence, (X',α') is a descending section of D, i.e., an element of $\mathcal{D}^G$. Thus, there exists $(\bar{X}',\bar{\alpha}')\in\bar{\mathcal{D}}$ such that $X'\sim_\pi\bar{X}'$ and $\alpha'=\pi^*\bar{\alpha}'$, and $(X'_{\bar{P}},\alpha'_{\bar{P}})\in\mathcal{D}_{\bar{P}}$ such that $X'_{\bar{P}}\sim_{\iota_{\bar{P}}}\bar{X}'$ and $\alpha'_{\bar{P}}=\iota_{\bar{P}}^*\bar{\alpha}'$. At last, we compute

$$\begin{aligned}\pi_P^*\alpha'_{\bar{P}}&=\pi_P^*\iota_{\bar{P}}^*\bar{\alpha}'=\iota_P^*\pi^*\bar{\alpha}'=\iota_P^*\alpha'\\&=\iota_P^*\alpha=\iota_P^*\pi^*\bar{\alpha}=\pi_P^*\iota_{\bar{P}}^*\bar{\alpha}=\pi_P^*\alpha_{\bar{P}},\end{aligned}$$

which yields $\alpha'_{\bar{P}}=\alpha_{\bar{P}}$ on the open set $\pi(U'\cap P)\subseteq U_{\bar{P}}$ with $\bar{p}\in\pi(U'\cap P)$. In the same manner, we compute

$$\begin{aligned}(T\iota_{\bar{P}}\circ X'_{\bar{P}})\circ\pi_P&=\bar{X}'\circ\iota_{\bar{P}}\circ\pi_P=\bar{X}'\circ\pi\circ\iota_P\\&=(T\pi\circ X')\circ\iota_P=(T\pi\circ(X'+V_G))\circ\iota_P=(T\pi\circ X)\circ\iota_P\\&=\bar{X}\circ\pi\circ\iota_P=\bar{X}\circ\iota_{\bar{P}}\circ\pi_P=(T\iota_{\bar{P}}\circ X_{\bar{P}})\circ\pi_P.\end{aligned}$$

Thus, we have shown that $(X_{\bar{P}},\alpha_{\bar{P}})$ is an element of $\mathcal{D}_{\bar{P}}$, that is, $(X_{\bar{P}},\alpha_{\bar{P}})$ is a section of $D_{\bar{P}}$ and $(X_{\bar{P}},\alpha_{\bar{P}})(\bar{p})\in D_{\bar{P}}(\bar{p})$. $\square$

Analogously to the regular case, we also have:

Theorem 6.6. *Let (M, D) be a Dirac manifold with a proper Dirac action of a connected Lie group G on it. Let $\bar{P}$ be a stratum of the quotient space $\bar{M}$. Assume that the Dirac structure D is integrable and that $D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)$ is spanned by its descending sections. Then the Dirac structure $D_{\bar{P}}$ on $\bar{P}$ introduced in Theorem* 6.5 *is integrable.*

Proof. Let $(X_{\bar{P}}, \alpha_{\bar{P}})$ and $(Y_{\bar{P}}, \beta_{\bar{P}})$ be sections of $D_{\bar{P}}$. We want to show that

$$[(X_{\bar{P}}, \alpha_{\bar{P}}), (Y_{\bar{P}}, \beta_{\bar{P}})] = ([X_{\bar{P}}, Y_{\bar{P}}], \pounds_{X_{\bar{P}}}\beta_{\bar{P}} - \mathbf{i}_{Y_{\bar{P}}}\mathbf{d}\alpha_{\bar{P}})$$

is also a section of $D_{\bar{P}}$. From Remark 6.3, $(X_{\bar{P}}, \alpha_{\bar{P}})$ and $(Y_{\bar{P}}, \beta_{\bar{P}})$ are elements of $\mathcal{D}_{\bar{P}}$ and thus we find $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \bar{\mathcal{D}}$ such that $X_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$, $Y_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{Y}$, $\alpha_{\bar{P}} = \iota_{\bar{P}}^*\bar{\alpha}$, and $\beta_{\bar{P}} = \iota_{\bar{P}}^*\bar{\beta}$. Furthermore, let (X, α) and (Y, β) be elements of $\mathcal{D}^G$, i.e., descending sections of D such that (X, α) descends to $(\bar{X}, \bar{\alpha})$ and (Y, β) descends to $(\bar{Y}, \bar{\beta})$. By the proof of Theorem 6.5, we can assume that (X, α) and (Y, β) are G-invariant. The section $[(X, \alpha), (Y, \beta)] = ([X, Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)$ is then also a G-invariant section of D, since D is integrable. We have

$$\begin{aligned}
(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha)(\xi_M) &= \xi_M(\beta(X)) + \mathbf{d}\beta(X, \xi_M) - \mathbf{d}\alpha(Y, \xi_M) \\
&= 0 + X(\beta(\xi_M)) - \xi_M(\beta(X)) - \beta([X, \xi_M]) \\
&\quad - Y(\alpha(\xi_M)) + \xi_M(\alpha(Y)) + \alpha([Y, \xi_M]) \\
&= X(0) - 0 - \beta(0) - Y(0) + 0 + \alpha(0)
\end{aligned}$$

for all $\xi \in \mathfrak{g}$, where we have used that $\beta(X)$ and $\alpha(Y) \in C^\infty(M)^G$. Hence $\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha \in \Gamma(\mathcal{V}^\circ)^G$, $[(X, \alpha), (Y, \beta)] \in \mathcal{D}^G$ and there exists $(\bar{Z}, \bar{\gamma}) \in \bar{\mathcal{D}}$ such that $[X, Y] \sim_\pi \bar{Z}$ and $\pi^*\bar{\gamma} = \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha$. Let $(Z_{\bar{P}}, \gamma_{\bar{P}})$ be the corresponding pair in $\mathcal{D}_{\bar{P}}$. We want to show that $Z_{\bar{P}} = [X_{\bar{P}}, Y_{\bar{P}}]$ and $\pounds_{X_{\bar{P}}}\beta_{\bar{P}} - \mathbf{i}_{Y_{\bar{P}}}\mathbf{d}\alpha_{\bar{P}} = \gamma_{\bar{P}}$.

Since we have $(X, \alpha), (Y, \beta) \in \Gamma(D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ))$, there exist $\tilde{X}$ and $\tilde{Y}$ in $\mathfrak{X}(P)$ such that $\tilde{X} \sim_{\iota_P} X$ and $\tilde{Y} \sim_{\iota_P} Y$. Set $\tilde{\alpha} = \iota_P^*\alpha$ and $\tilde{\beta} = \iota_P^*\beta$. We have $[\tilde{X}, \tilde{Y}] \sim_{\iota_P} [X, Y]$ and $[\tilde{X}, \tilde{Y}] \in \mathfrak{X}(P)^G$. We have the equality $\iota_{\bar{P}} \circ \pi_P = \pi \circ \iota_P$ and consequently, since $\tilde{X} \sim_{\iota_P \circ \pi} \bar{X}$ and $\tilde{Y} \sim_{\iota_P \circ \pi} \bar{Y}$, we have $\tilde{X} \sim_{\iota_{\bar{P}} \circ \pi_P} \bar{X}$ and $\tilde{Y} \sim_{\iota_{\bar{P}} \circ \pi_P} \bar{Y}$. Hence, because $X_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$, $Y_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{Y}$, and by Proposition 3.3, we get $\tilde{X} \sim_{\pi_P} X_{\bar{P}}$ and $\tilde{Y} \sim_{\pi_P} Y_{\bar{P}}$, and also $[\tilde{X}, \tilde{Y}] \sim_{\pi_P} [X_{\bar{P}}, Y_{\bar{P}}]$. But in the same manner, we have $[\tilde{X}, \tilde{Y}] \sim_{\iota_P \circ \pi} \bar{Z}$; thus $[\tilde{X}, \tilde{Y}] \sim_{\iota_{\bar{P}} \circ \pi_P} \bar{Z}$ and $[X_{\bar{P}}, Y_{\bar{P}}] \sim_{\iota_{\bar{P}}} \bar{Z}$. Because of the uniqueness of $Z_{\bar{P}}$ (Proposition 3.3), we get $Z_{\bar{P}} = [X_{\bar{P}}, Y_{\bar{P}}]$.

In the same manner, we have

$$\begin{aligned}
\pi_P^*(\gamma_{\bar{P}}) = \pi_P^*(\iota_{\bar{P}}^*\bar{\gamma}) = (\iota_{\bar{P}} \circ \pi_P)^*(\bar{\gamma}) &= (\pi \circ \iota_P)^*(\bar{\gamma}) = \iota_P^*(\pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha) \\
&= (\pounds_{\tilde{X}}\tilde{\beta} - \mathbf{i}_{\tilde{Y}}\mathbf{d}\tilde{\alpha}) \\
&= \pi_P^*(\pounds_{X_{\bar{P}}}\beta_{\bar{P}} - \mathbf{i}_{Y_{\bar{P}}}\mathbf{d}\alpha_{\bar{P}}).
\end{aligned}$$

Thus, using the fact that π_P is a smooth surjective submersion, we get the equality of $\pounds_{X_{\bar{P}}}\beta_{\bar{P}} - \mathbf{i}_{Y_{\bar{P}}}\mathbf{d}\alpha_{\bar{P}}$ and $\gamma_{\bar{P}}$. □

We end this subsection with examples.

Example 6.7. Let $(M, \{\cdot\,, \cdot\})$ be a smooth Poisson manifold with a canonical and proper action of a Lie group G on it (recall that the action of G on $(M, \{\cdot\,, \cdot\})$ is canonical if $\{\Phi_g^*f_1, \Phi_g^*f_2\} = \Phi_g^*\{f_1, f_2\}$ for all $f_1, f_2 \in C^\infty(M)$ and $g \in G$). Let

$D_{\{\cdot\,,\cdot\}}$ be the Dirac structure associated to the Poisson structure; that is, $D_{\{\cdot\,,\cdot\}}(m)$ is defined by

$$D_{\{\cdot\,,\cdot\}}(m) = \{(X_f(m), \mathbf{d}f(m)) \mid f \in C^\infty(M) \text{ and } X_f = \sharp(\mathbf{d}f) \in \mathfrak{X}(M)\}$$

for all $m \in M$, where $\sharp : T^*M \to TM$, $\mathbf{d}f \mapsto X_f = \{\cdot, f\}$, is the homomorphism of vector bundles associated to $\{\cdot\,,\cdot\}$. Since the action of G on $(M, \{\cdot\,,\cdot\})$ is canonical, it is a Dirac action on $(M, D_{\{\cdot\,,\cdot\}})$.

By Lemma 5.9, we know that $\mathcal{V}_G^\circ$ is generated by the exterior differentials of the G-invariant functions on M. Using the fact that the action of G on M is canonical, it is easy to check that the vector field X_f associated to a G-invariant function f is G-invariant. Hence, we get

$$(D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ))\,(m) = \operatorname{span}\{(X_f(m), \mathbf{d}f(m)) \mid f \in C^\infty(M)^G\}.$$

This yields, automatically, that $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections. Hence we can apply Theorems 6.4 and 6.5 to the Dirac G-manifold $\left(M, D_{\{\cdot\,,\cdot\}}\right)$. Thus, each stratum of $\bar{M}$ inherits a Dirac structure $D_{\bar{P}}$ induced by $D_{\{\cdot\,,\cdot\}}$. Since $D_{\{\cdot\,,\cdot\}}$ is integrable, the Dirac structure $D_{\bar{P}}$ is also integrable by Theorem 6.6.

We want to show that the codistribution $\mathsf{P}_1^{\bar{P}}$ induced by $D_{\bar{P}}$ (see Example 4.1) on $\bar{P}$ is equal to $T^*\bar{P}$. To see this, we show that $\mathbf{d}f_{\bar{P}} \in \Gamma\left(\mathsf{P}_1^{\bar{P}}\right)$ for all $f_{\bar{P}} \in C^\infty\left(\bar{P}\right)$. Let $f_{\bar{P}} \in C^\infty\left(\bar{P}\right)$ and choose $\bar{f} \in C^\infty\left(\bar{M}\right)$ with $\iota_{\bar{P}}\left(\bar{f}\right) = f_{\bar{P}}$. Set $f := \pi^*\bar{f}$. Then, as above, we have $(X_f, \mathbf{d}f) \in \mathcal{D}^G$, and hence there exists $\bar{X} \in \mathfrak{X}\left(\bar{M}\right)$ such that $(X_f, \mathbf{d}f)$ descends to $\left(\bar{X}, \mathbf{d}\bar{f}\right) \in \bar{\mathcal{D}}$. Since the restriction of $\mathbf{d}\bar{f}$ to $\bar{P}$ is equal to $\mathbf{d}f_{\bar{P}}$, we get the existence of $X_{\bar{P}} \in \mathfrak{X}\left(\bar{P}\right)$ such that $(X_{\bar{P}}, \mathbf{d}f_{\bar{P}})$ in $\mathcal{D}_{\bar{P}}$. Hence, $\mathbf{d}f_{\bar{P}}$ is a section of $\mathsf{P}_1^{\bar{P}}$.

Since $D_{\bar{P}}$ is integrable and $\mathsf{P}_1^{\bar{P}}$ is constant dimensional and equal to $T^*\bar{P}$, the Dirac structure $D_{\bar{P}}$ defines a Poisson bracket $\{\cdot\,,\cdot\}_{\bar{P}}$ on $C^\infty(\bar{P})$ by

$$\{f_{\bar{P}}, g_{\bar{P}}\}_{\bar{P}} = -X_{f_{\bar{P}}}(g_{\bar{P}}) = X_{g_{\bar{P}}}(f_{\bar{P}}),$$

where $X_{f_{\bar{P}}}$ and $X_{g_{\bar{P}}}$ are such that $(X_{f_{\bar{P}}}, \mathbf{d}f_{\bar{P}}), (X_{g_{\bar{P}}}, \mathbf{d}g_{\bar{P}}) \in \mathcal{D}_{\bar{P}} = \Gamma(D_{\bar{P}})$. For a proof of this, see, for example, [4].

For $f_{\bar{P}}, g_{\bar{P}} \in C^\infty\left(\bar{P}\right)$ choose, as above, extensions $\bar{f}, \bar{g} \in C^\infty\left(\bar{M}\right)$ and set $f = \pi^*\bar{f}$, $g = \pi^*\bar{g}$. Then we have $\pi_P^* g_{\bar{P}} = \iota_P^* g$ and there exists $\tilde{X} \in \mathfrak{X}(P)$ such that $\tilde{X} \sim_{\iota_P} X_f$ and $\tilde{X} \sim_{\pi_P} X_{f_{\bar{P}}}$. Thus,

$$\begin{aligned}\iota_P^*\{f, g\} &= -\iota_P^*(X_f(g)) = -\tilde{X}(\iota_P^* g) = -\tilde{X}(\pi_P^* g_{\bar{P}}) \\ &= -\pi_P^*(X_{f_{\bar{P}}}(g_{\bar{P}})) = \pi_P^*\{f_{\bar{P}}, g_{\bar{P}}\}_{\bar{P}}.\end{aligned}$$

This shows that, in the terminology of [25], $(M, \{\cdot\,,\cdot\}, P, G)$ is always reducible if G acts properly and canonically on the Poisson manifold M and P is a connected component of an orbit type manifold of the action.

Example 6.8. We consider the example of the proper action Φ of $G := \mathbb{S}^1$ on $M := \mathbb{R}^3$ given by

$$\alpha \cdot (x, y, z) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha, z).$$

The orbit type manifolds of this action are $P_1 = \{0\} \times \{0\} \times \mathbb{R}$, that is, $P_1 = M_{H_1}$ with $H_1 = \mathbb{S}^1$, and $P_2 = \mathbb{R}^3 \setminus P_1$, so $P_2 = M_{H_2}$ with $H_2 = \{1\}$. The orbit of a point

$(x,y,z) \in \mathbb{R}^3$ is $\{(x',y',z') \mid x'^2+y'^2 = x^2+y^2 \text{ and } z'=z\}$. Thus the reduced space $\bar{M}$ can be identified with $[0,+\infty)\times\mathbb{R}$ with the projection π given by

$$\pi(x,y,z) = (x^2+y^2, z).$$

It is easy to compute, for each $\alpha \in \mathbb{S}^1$:

$$\begin{aligned}\Phi_\alpha^*(\partial_x) &= \cos\alpha\partial_x - \sin\alpha\partial_y,\\ \Phi_\alpha^*(\partial_y) &= \sin\alpha\partial_x + \cos\alpha\partial_y,\\ \Phi_\alpha^*(\partial_z) &= \partial_z\end{aligned}$$

and

$$\begin{aligned}\Phi_\alpha^*(\mathbf{d}x) &= \cos\alpha\mathbf{d}x - \sin\alpha\mathbf{d}y,\\ \Phi_\alpha^*(\mathbf{d}y) &= \sin\alpha\mathbf{d}x + \cos\alpha\mathbf{d}y,\\ \Phi_\alpha^*(\mathbf{d}z) &= \mathbf{d}z.\end{aligned}$$

Hence, the Dirac structure D given as the span of the sections

$$(\partial_x, \mathbf{d}y), (\partial_y, -\mathbf{d}x), (\partial_z, 0)$$

is $\mathbb{S}^1$-invariant; that is, the Lie group $\mathbb{S}^1$ acts on (M,D) by Dirac actions.

The set $\mathcal{D}^{\mathbb{S}^1}$ is spanned as a $C^\infty(M)$-module by the sections

$$(y\partial_x - x\partial_y, x\mathbf{d}x + y\mathbf{d}y) \quad \text{and} \quad (\partial_z, 0).$$

Note that the section $(x\partial_x + y\partial_y, y\mathbf{d}x - x\mathbf{d}y)$ is also $\mathbb{S}^1$-invariant but its cotangent part doesn't annihilate the vertical space. Also, since $\mathbb{S}^1$ is Abelian, the vertical space is spanned by the $\mathbb{S}^1$-invariant vector field $y\partial_x - x\partial_y$ and we only have to consider $\mathbb{S}^1$-invariant vector fields and not descending vector fields, in general. Thus $\bar{\mathcal{D}}$ is the $C^\infty\left(\bar{M}\right)$-module generated by the pairs $(\partial_{\bar{z}}, 0)$ and $(0, \bar{x}\mathbf{d}\bar{x})$, with the coordinates $\bar{x}$ and $\bar{z}$ on $\bar{M} \simeq [0,\infty)\times\mathbb{R}$.

Then we have $\mathcal{D}_{\bar{P}_1} = \operatorname{span}_{C^\infty(\bar{P}_1)}\{(\partial_{\bar{z}},0)\}$ since $\bar{x}=0$ for all $p=(\bar{x},\bar{z})\in\bar{P}_1$. Hence, the Dirac structure $D_{\bar{P}_1}$ on $\bar{P}_1 = P_1/G = P_1$ is given as the span of the section $(\partial_{\bar{z}},0)$.

We have $\bar{x}\neq 0$ for all $p=(\bar{x},\bar{z})\in\bar{P}_2$. Hence, since

$$\mathcal{D}_{\bar{P}_2} = \operatorname{span}_{C^\infty(\bar{P}_2)}\{(\partial_{\bar{z}},0),(0,\bar{x}\mathbf{d}\bar{x})\},$$

the Dirac structure $D_{\bar{P}_2}$ on $\bar{P}_2 = P_2/G = (0,\infty)\times\mathbb{R}$ is given as the span of the sections $(\partial_{\bar{z}},0)$ and $(0,\mathbf{d}\bar{x})$.

Example 6.9. Consider here again Example 5.15 with the Dirac structure given by $D = (T\mathbb{R}^3\oplus\{0\})\ominus(\{0\}\oplus T^*\mathbb{R}^3)$, i.e.,

$$D = \operatorname{span}\left\{(\partial_{x_1},0),(\partial_{y_1},0),(\partial_{z_1},0),(0,\mathbf{d}x_2),(0,\mathbf{d}y_2),(0,\mathbf{d}z_2)\right\}.$$

This Dirac bundle on $\mathbb{R}^3\times\mathbb{R}^3$ is obviously invariant under the diagonal action of SO(3), but the intersection $D\cap(\mathcal{T}\ominus\mathcal{V}_G^\circ)$ is not smooth. We have the descending sections

$$(x_1\partial_{x_1}+y_1\partial_{y_1}+z_1\partial_{z_1},0),(0,x_2\mathbf{d}x_2+y_2\mathbf{d}y_2+z_2\mathbf{d}z_2)$$

of D. Let (v,w) be a point where the function f_3 or one of the coordinates x_2,y_2,z_2 vanishes. Then linear algebra arguments show that there exists a linear combination of $(x_1\partial_{x_2}+y_1\partial_{y_2}+z_1\partial_{z_2},0)$, $(x_2\partial_{x_1}+y_2\partial_{y_1}+z_2\partial_{z_1},0)$, $(y_1\partial_{x_1}-x_1\partial_{y_1}+y_2\partial_{x_2}-x_2\partial_{y_2},0)$, $(z_1\partial_{y_1}-y_1\partial_{z_1}+z_2\partial_{y_2}-y_2\partial_{z_2},0)$, and $(x_1\partial_{z_1}-z_1\partial_{x_1}+x_2\partial_{z_2}-z_2\partial_{x_2},0)$ (sections of $\mathcal{T}\ominus\{0\}$) which is an element of $((T\mathbb{R}^3\oplus\{0\})\ominus\{0\})(v,w)$ and hence of

$D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)(v,w)$, but there exists no open neighborhood of this point such that this vector is the value at (v,w) of a vector field defined on this whole neighborhood and having all its values in $D \cap (\mathcal{T} \ominus \mathcal{V}_G^\circ)$.

Hence the reduction theorems of this paper (Theorems 6.4, 6.5 and 6.6) do not apply to this example: the action is canonical, but the hypothesis on smoothness of the intersection of D with $\mathcal{T} \ominus \mathcal{V}_G^\circ$ is not satisfied.

Example 6.10. Let us illustrate Theorems 6.5 and 6.6. Consider again the manifold $M = \mathbb{R}^3 \times \mathbb{R}^3$ this time with the (automatically proper) diagonal action of $G = \mathbb{S}^1$ on it, i.e.,

$$\begin{array}{rccc} \Phi: & \mathbb{S}^1 \times (\mathbb{R}^3 \times \mathbb{R}^3) & \to & \mathbb{R}^3 \times \mathbb{R}^3 \\ & \left(\alpha, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) & \mapsto & \left(\begin{pmatrix} x_1\cos\alpha - y_1\sin\alpha \\ x_1\sin\alpha + y_1\cos\alpha \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2\cos\alpha - y_2\sin\alpha \\ x_2\sin\alpha + y_2\cos\alpha \\ z_2 \end{pmatrix}\right). \end{array}$$

The functions

$$\begin{aligned} R_1(v,w) &= r_1^2(v,w) = x_1^2 + y_1^2 = \left\| \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\|^2, \\ R_2(v,w) &= r_2^2(v,w) = x_2^2 + y_2^2 = \left\| \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\|^2, \\ d(v,w) &= x_1y_2 - y_1x_2 = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \\ s(v,w) &= x_1x_2 + y_1y_2 = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle, \\ z_1(v,w) &= z_1, \quad z_2(v,w) = z_2 \end{aligned}$$

are $\mathbb{S}^1$-invariant. They also characterize the $\mathbb{S}^1$-orbits of the action since d and s determine in a unique way the angle between the vectors (x_1, y_1) and (x_2, y_2). Hence, the reduced manifold is the stratified space $\bar{M} = \pi(\mathbb{R}^3 \times \mathbb{R}^3) \subseteq \mathbb{R}^6$, where $\pi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^6$ is given by

$$\pi(v,w) = (R_1, R_2, d, s, z_1, z_2)(v,w).$$

We conclude that $\bar{M}$ is the semi-algebraic set

$$\bar{M} = \{(f_1, f_2, \delta, \sigma, z_1, z_2) \in \mathbb{R}^6 \mid f_1, f_2 \geq 0 \text{ and } \sigma^2 + \delta^2 = f_1 f_2\}.$$

The two strata of $\bar{M}$ are $\bar{M}_0 = \{(0,0,0,0,z_1,z_2) \mid z_1, z_2 \in \mathbb{R}\} \subseteq \mathbb{R}^6$, corresponding to the orbit (isotropy) type manifold

$$M_{\mathbb{S}^1} = M_{(\mathbb{S}^1)} = \{(0,0,0,0,z_1,z_2) \mid z_1, z_2 \in \mathbb{R}\} \subseteq \mathbb{R}^6$$

with trivial $\mathbb{S}^1$-action on it, and $\bar{M}_1 = \{(f_1, f_2, \delta, \sigma, z_1, z_2) \in \mathbb{R}^6 \mid (f_1, f_2) \neq (0,0) \text{ and } \delta^2 + \sigma^2 = f_1 f_2\}$, corresponding to the orbit (isotropy) type manifold

$$\begin{aligned} M_{\{0\}} &= M_{(\{0\})} \\ &= \{(x_1, y_1, z_1, x_2, y_2, z_2) \in \mathbb{R}^6 \mid (x_1, y_1) \neq (0,0) \text{ or } (x_2, y_2) \neq (0,0)\}. \end{aligned}$$

Let U be the open set $U := \mathbb{R}_{>0} \times \mathbb{R}^4 \subset \mathbb{R}^5$. Since the points $(f_1, f_2, \delta, \sigma, z_1, z_2)$ in $\bar{M}_1$ satisfy $f_1 > 0$ or $f_2 > 0$, we have two charts for $\bar{M}_1$, namely $(\psi_1(U), \psi_1^{-1})$

and $(\psi_2(U), \psi_2^{-1})$, where

$$\begin{array}{rccc} \psi_1: & \mathbb{R}_{>0}\times\mathbb{R}^4 & \to & \bar{M}_1 \\ & (f_1,\delta,\sigma,z_1,z_2,) & \mapsto & \left(f_1, \frac{\delta^2+\sigma^2}{f_1}, \delta,\sigma,z_1,z_2\right), \\ \psi_1^{-1}: & \psi_1(U)\subseteq \bar{M}_1 & \to & \mathbb{R}_{>0}\times\mathbb{R}^4 \\ & (f_1,f_2,\delta,\sigma,z_1,z_2) & \mapsto & (f_1,\delta,\sigma,z_1,z_2) \end{array}$$

and

$$\begin{array}{rccc} \psi_2: & \mathbb{R}_{>0}\times\mathbb{R}^4 & \to & \bar{M}_1 \\ & (f_2,\delta,\sigma,z_1,z_2) & \mapsto & \left(\frac{\delta^2+\sigma^2}{f_2}, f_2, \delta,\sigma,z_1,z_2\right), \\ \psi_2^{-1}: & \psi_2(U)\subseteq \bar{M}_1 & \to & \mathbb{R}_{>0}\times\mathbb{R}^4 \\ & (f_1,f_2,\delta,\sigma,z_1,z_2) & \mapsto & (f_2,\delta,\sigma,z_1,z_2). \end{array}$$

We compute the distributions $\mathcal{T}=\mathcal{T}_G$ (note that $\mathcal{V}$ is spanned by $\mathbb{S}^1$-invariant sections because the Lie group is Abelian) and the codistribution $\mathcal{V}_G^\circ$. We have

$$\mathcal{V}_G^\circ = \operatorname{span}_{C^\infty(M)}\left\{\begin{array}{c} \mathbf{d}z_1, \quad \mathbf{d}z_2, \quad x_1\mathbf{d}x_1+y_1\mathbf{d}y_1, \\ x_2\mathbf{d}x_2+y_2\mathbf{d}y_2, \quad x_1\mathbf{d}y_2+y_2\mathbf{d}x_1-x_2\mathbf{d}y_1-y_1\mathbf{d}x_2, \\ x_1\mathbf{d}x_2+x_2\mathbf{d}x_1+y_1\mathbf{d}y_2+y_2\mathbf{d}y_1 \end{array}\right\}$$

and (see the appendix of [18])

$$\mathcal{T} = \operatorname{span}_{C^\infty(M)}\left\{\begin{array}{l} X_1 := \partial_{z_1}, \quad X_2 := \partial_{z_2}, \\ X_3 := x_1\partial_{x_1}+y_1\partial_{y_1}, \quad X_4 := x_2\partial_{x_2}+y_2\partial_{y_2}, \\ X_5 := y_1\partial_{x_2}-x_1\partial_{y_2}, \quad X_6 := y_2\partial_{x_1}-x_2\partial_{y_1}, \\ X_7 := x_1\partial_{x_2}+y_1\partial_{y_2}, \quad X_8 := x_2\partial_{x_1}+y_2\partial_{y_1}, \\ X_9 := x_1\partial_{y_1}-y_1\partial_{x_1}, \quad X_{10} := x_2\partial_{y_2}-y_2\partial_{x_2} \end{array}\right\}.$$

Note that $\mathcal{V}$ is spanned on M by $X_9+X_{10} = x_1\partial_{y_1}-y_1\partial_{x_1}+x_2\partial_{y_2}-y_2\partial_{x_2}$.

We compute the flows associated to the spanning vector fields of $\mathcal{T}$ and find (still using the coordinates $(v,w)=(x_1,y_1,z_1,x_2,y_2,z_2)$):

$$\phi_t^1(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1+t\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}\right), \quad \phi_t^2(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2+t\end{pmatrix}\right),$$

$$\phi_t^3(v,w) = \left(\begin{pmatrix} e^tx_1\\ e^ty_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}\right), \quad \phi_t^4(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}, \begin{pmatrix} e^tx_2\\ e^ty_2\\ z_2\end{pmatrix}\right),$$

$$\phi_t^5(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}, \begin{pmatrix} y_1t+x_2\\ -x_1t+y_2\\ z_2\end{pmatrix}\right), \phi_t^6(v,w) = \left(\begin{pmatrix} y_2t+x_1\\ -x_2t+y_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}\right),$$

$$\phi_t^7(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_1t+x_2\\ y_1t+y_2\\ z_2\end{pmatrix}\right), \phi_t^8(v,w) = \left(\begin{pmatrix} x_2t+x_1\\ y_2t+y_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}\right),$$

$$\phi_t^9(v,w) = \left(\begin{pmatrix} x_1\cos t - y_1\sin t\\ x_1\sin t + y_1\cos t\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}\right),$$

$$\phi_t^{10}(v,w) = \left(\begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}, \begin{pmatrix} x_2\cos t - y_2\sin t\\ x_2\sin t + y_2\cos t\\ z_2\end{pmatrix}\right),$$

which are all easily verified to be $\mathbb{S}^1$-invariant. It is easy to check that the two orbit (isotropy) type manifolds are the accessible sets of the distribution $\mathcal{T} = \mathcal{T}_G$.

We compute with this the flow $\bar{\phi}^i$ of the vector field $\bar{X}_i$ satisfying $X_i \sim_\pi \bar{X}_i$ for each $i = 1, \ldots, 10$. We have

$$\begin{aligned}
\bar{\phi}^1_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= \bar{\phi}^1_t(\pi(v, w)) = \pi \circ \phi^1_t(v, w) = (f_1, f_2, \delta, \sigma, z_1 + t, z_2),\\
\bar{\phi}^2_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= \bar{\phi}^2_t(\pi(v, w)) = \pi \circ \phi^2_t(v, w) = (f_1, f_2, \delta, \sigma, z_1, z_2 + t),\\
\bar{\phi}^3_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (e^{2t} f_1, f_2, e^t\delta, e^t\sigma, z_1, z_2),\\
\bar{\phi}^4_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1, e^{2t} f_2, e^t\delta, e^t\sigma, z_1, z_2),\\
\bar{\phi}^5_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1, t^2 f_1 + f_2 - 2t\delta, -f_1 t + \delta, \sigma, z_1, z_2),\\
\bar{\phi}^6_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1 + t^2 f_2 + 2t\delta, f_2, f_2 t + \delta, \sigma, z_1, z_2),\\
\bar{\phi}^7_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1, t^2 f_1 + f_2 + 2t\sigma, \delta, f_1 t + \sigma, z_1, z_2),\\
\bar{\phi}^8_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1 + t^2 f_2 + 2t\sigma, f_2, \delta, f_1 t + \sigma, z_1, z_2),\\
\bar{\phi}^9_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1, f_2, \delta\cos t - \sigma \sin t, \delta \sin t + \sigma \cos t, z_1, z_2),\\
\bar{\phi}^{10}_t(f_1, f_2, \delta, \sigma, z_1, z_2) &= (f_1, f_2, \sigma \sin t + \delta \cos t, \sigma \cos t - \delta \sin t, z_1, z_2).
\end{aligned}$$

This leads to

$$\begin{aligned}
\bar{X}_1(f_1, f_2, \delta, \sigma, z_1, z_2) &= \partial_{z_1}, \qquad \bar{X}_2(f_1, f_2, \delta, \sigma, z_1, z_2) = \partial_{z_2},\\
\bar{X}_3(f_1, f_2, \delta, \sigma, z_1, z_2) &= 2f_1\partial_{f_1} + \delta\partial_\delta + \sigma\partial_\sigma,\\
\bar{X}_4(f_1, f_2, \delta, \sigma, z_1, z_2) &= 2f_2\partial_{f_2} + \delta\partial_\delta + \sigma\partial_\sigma,\\
\bar{X}_5(f_1, f_2, \delta, \sigma, z_1, z_2) &= -2\delta\partial_{f_2} - f_1\partial_\delta,\\
\bar{X}_6(f_1, f_2, \delta, \sigma, z_1, z_2) &= 2\delta\partial_{f_1} + f_2\partial_\delta,\\
\bar{X}_7(f_1, f_2, \delta, \sigma, z_1, z_2) &= 2\sigma\partial_{f_2} + f_1\partial_\sigma,\\
\bar{X}_8(f_1, f_2, \delta, \sigma, z_1, z_2) &= 2\sigma\partial_{f_1} + f_2\partial_\sigma,\\
\bar{X}_9(f_1, f_2, \delta, \sigma, z_1, z_2) &= -\sigma\partial_\delta + \delta\partial_\sigma,\\
\bar{X}_{10}(f_1, f_2, \delta, \sigma, z_1, z_2) &= \sigma\partial_\delta - \delta\partial_\sigma = -\bar{X}_9.
\end{aligned}$$

Recalling that the tangent bundle to the manifold $\bar{M}_1$ is the kernel of the one-form $\mathbf{d}(f_1 f_2 - \delta^2 - \sigma^2) = f_1\mathbf{d}f_2 + f_2\mathbf{d}f_1 - 2\sigma\mathbf{d}\sigma - 2\delta\mathbf{d}\delta$, we see that the two strata of $\bar{M}$ are indeed the accessible sets of the distribution spanned by $\bar{X}_1, \ldots, \bar{X}_{10}$.

Consider the Dirac structure $D \subseteq TM \oplus T^*M$ spanned by the pairs

$$(\partial_{x_1}, \mathbf{d}y_1), (\partial_{y_1}, -\mathbf{d}x_1), (\partial_{z_1}, 0), (\partial_{x_2}, -\mathbf{d}y_2), (\partial_{y_2}, \mathbf{d}x_2), (0, \mathbf{d}z_2).$$

Comparing this with the sections of $\mathcal{T}$ and $\mathcal{V}_G^\circ$ given above, we find

$$\begin{aligned}
\mathcal{D}^{\mathbb{S}^1} &= \operatorname{span}_{C^\infty(M)^{\mathbb{S}^1}} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2),\\ (-x_1\partial_{y_1} + y_1\partial_{x_1}, x_1\mathbf{d}x_1 + y_1\mathbf{d}y_1),\\ (x_2\partial_{y_2} - y_2\partial_{x_2}, x_2\mathbf{d}x_2 + y_2\mathbf{d}y_2),\\ (-x_1\partial_{x_2} - y_2\partial_{y_1} - x_2\partial_{x_1} - y_1\partial_{y_2},\\ x_1\mathbf{d}y_2 + y_2\mathbf{d}x_1 - x_2\mathbf{d}y_1 - y_1\mathbf{d}x_2),\\ (x_1\partial_{y_2} - x_2\partial_{y_1} - y_1\partial_{x_2} + y_2\partial_{x_1},\\ x_1\mathbf{d}x_2 + x_2\mathbf{d}x_1 + y_1\mathbf{d}y_2 + y_2\mathbf{d}y_1) \end{array} \right\}\\
&= \operatorname{span}_{C^\infty(M)^{\mathbb{S}^1}} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2), \left(-X_9, \tfrac{1}{2}\mathbf{d}R_1\right),\\ \left(X_{10}, \tfrac{1}{2}\mathbf{d}R_2\right), (-X_7 - X_8, \mathbf{d}d), (X_6 - X_5, \mathbf{d}s) \end{array} \right\}.
\end{aligned}$$

Hence, we get

$$\bar{\mathcal{D}} = \operatorname{span}_{C^\infty(\bar{M})} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2), \left(-\bar{X}_9, \frac{1}{2}\mathbf{d}f_1\right), \\ \left(\bar{X}_{10}, \frac{1}{2}\mathbf{d}f_2\right), (-\bar{X}_7 - \bar{X}_8, \mathbf{d}\delta), (\bar{X}_6 - \bar{X}_5, \mathbf{d}\sigma) \end{array} \right\}$$
$$= \operatorname{span}_{C^\infty(\bar{M})} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2), \\ \left(\sigma\partial_\delta - \delta\partial_\sigma, \frac{1}{2}\mathbf{d}f_1\right), \left(\sigma\partial_\delta - \delta\partial_\sigma, \frac{1}{2}\mathbf{d}f_2\right), \\ (-2\sigma(\partial_{f_1} + \partial_{f_2}) - (f_1 + f_2)\partial_\sigma, \mathbf{d}\delta), \\ (2\delta(\partial_{f_1} + \partial_{f_2}) + (f_1 + f_2)\partial_\delta, \mathbf{d}\sigma) \end{array} \right\}.$$

Recall the definition of one-forms on the stratified space $\bar{M}$ in Subsection 5.2. The "one-forms" $\mathbf{d}f_1$ and $\mathbf{d}f_2$ are *not* derivatives of smooth coordinates; they vanish at the points where f_1 and, respectively, f_2 vanish, by definition.

Now we compute the induced Dirac structures on the two strata $\bar{M}_0$ and $\bar{M}_1$. The pairs $\left(-\bar{X}_9, \frac{1}{2}\mathbf{d}f_1\right), \left(\bar{X}_{10}, \frac{1}{2}\mathbf{d}f_2\right), (-\bar{X}_7 - \bar{X}_8, \mathbf{d}\delta)$ and $(\bar{X}_6 - \bar{X}_5, \mathbf{d}\sigma)$ are all zero on $\bar{M}_0$. So we get $\mathcal{D}_{\bar{M}_0} = \operatorname{span}_{C^\infty(\bar{M}_0)}\{(\partial_{z_1}, 0), (0, \mathbf{d}z_2)\}$ and hence $D_{\bar{M}_0}(\bar{m}) = \operatorname{span}_{\mathbb{R}}\{(\partial_{z_1}|_{\bar{m}}, 0), (0, \mathbf{d}z_2(\bar{m}))\}$ for all $\bar{m} \in \bar{M}_0$.

For the stratum $\bar{M}_1$, we give the Dirac structure in the two charts $(\psi_1(U), \psi_1^{-1})$ and $(\psi_2(U), \psi_2^{-1})$. We start with $(\psi_1(U), \psi_1^{-1})$, that is, the points of $\bar{M}_1$ where f_1 does not vanish. We have

$$\psi_1^*\mathbf{d}f_2 = \frac{1}{f_1}\left(-\left(\frac{\sigma^2 + \delta^2}{f_1}\right)\mathbf{d}f_1 + 2\sigma\mathbf{d}\sigma + 2\delta\mathbf{d}\delta\right) \qquad \text{and} \qquad \partial_{f_2} \sim_{\psi_1^{-1}} 0.$$

Hence, the pairs in $\bar{\mathcal{D}}$ restrict to sections on $\psi_1(U) \subseteq \bar{M}_1$ that span the Dirac structure defined by

$$D_{\bar{M}_1}(f_1, \sigma, \delta, z_1, z_2)$$
$$= \operatorname{span}_{\mathbb{R}} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2), (2\sigma\partial_\delta - 2\delta\partial_\sigma, \mathbf{d}f_1), \\ \left(0, \mathbf{d}f_1 - \frac{1}{f_1}\left(-\left(\frac{\sigma^2+\delta^2}{f_1}\right)\mathbf{d}f_1 + 2\sigma\mathbf{d}\sigma + 2\delta\mathbf{d}\delta\right)\right), \\ \left(-2\sigma\partial_{f_1} - \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma, \mathbf{d}\delta\right)), \\ \left(2\delta\partial_{f_1} + \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta, \mathbf{d}\sigma\right)) \end{array} \right\} (f_1, \sigma, \delta, z_1, z_2)$$
$$= \operatorname{span}_{\mathbb{R}} \left\{ \begin{array}{c} (\partial_{z_1}, 0), (0, \mathbf{d}z_2), (2\sigma\partial_\delta - 2\delta\partial_\sigma, \mathbf{d}f_1), \\ \left(0, \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\mathbf{d}f_1 - 2\sigma\mathbf{d}\sigma - 2\delta\mathbf{d}\delta\right), \\ \left(-2\sigma\partial_{f_1} - \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma, \mathbf{d}\delta\right), \\ \left(2\delta\partial_{f_1} + \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta, \mathbf{d}\sigma\right) \end{array} \right\} (f_1, \sigma, \delta, z_1, z_2)$$

for all $(f_1, \sigma, \delta, z_1, z_2)$ in U. Since

$$\begin{aligned} &\left(0, \left(f_1 + \frac{\sigma^2 + \delta^2}{f_1}\right)\mathbf{d}f_1 - 2\sigma\mathbf{d}\sigma - 2\delta\mathbf{d}\delta\right) \\ &= \left(f_1 + \frac{\sigma^2 + \delta^2}{f_1}\right)(2\sigma\partial_\delta - 2\delta\partial_\sigma, \mathbf{d}f_1) \\ &\quad - 2\delta\left(-2\sigma\partial_{f_1} - \left(f_1 + \frac{\sigma^2 + \delta^2}{f_1}\right)\partial_\sigma, \mathbf{d}\delta\right) \\ &\quad - 2\sigma\left(2\delta\partial_{f_1} + \left(f_1 + \frac{\sigma^2 + \delta^2}{f_1}\right)\partial_\delta, \mathbf{d}\sigma\right), \end{aligned} \tag{6.3}$$

this leads to

$$D_{\bar{M}_1}(f_1,\sigma,\delta,z_1,z_2) = \operatorname{span}_{\mathbb{R}}\left\{\begin{array}{c}(\partial_{z_1},0),(0,\mathbf{d}z_2),(2\sigma\partial_\delta-2\delta\partial_\sigma,\mathbf{d}f_1),\\ \left(-2\sigma\partial_{f_1}-\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma,\mathbf{d}\delta\right),\\ \left(2\delta\partial_{f_1}+\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta,\mathbf{d}\sigma\right)\end{array}\right\}(f_1,\sigma,\delta,z_1,z_2). \tag{6.4}$$

In the same manner, we get in the chart $(\psi_2(U),\psi_2^{-1})$:

$$D_{\bar{M}_1}(f_2,\sigma,\delta,z_1,z_2) = \operatorname{span}_{\mathbb{R}}\left\{\begin{array}{c}(\partial_{z_1},0),(0,\mathbf{d}z_2),(2\sigma\partial_\delta-2\delta\partial_\sigma,\mathbf{d}f_2),\\ \left(-2\sigma\partial_{f_2}-\left(f_2+\frac{\sigma^2+\delta^2}{f_2}\right)\partial_\sigma,\mathbf{d}\delta\right),\\ \left(2\delta\partial_{f_2}+\left(f_2+\frac{\sigma^2+\delta^2}{f_2}\right)\partial_\delta,\mathbf{d}\sigma\right)\end{array}\right\}(f_2,\sigma,\delta,z_1,z_2).$$

It is easy to verify in both charts that this indeed defines a Dirac bundle on $\bar{M}_1$; it is constant $\dim\bar{M}_1$-dimensional and Lagrangian relative to the pairing on $T\bar{M}_1\oplus T^*\bar{M}_1$.

Since the Dirac structure D on M is integrable, we check that the reduced Dirac manifolds $(\bar{M}_0,D_{\bar{M}_0})$ and $(\bar{M}_1,D_{\bar{M}_1})$ are also integrable. For $(\bar{M}_0,D_{\bar{M}_0})$, this is obvious. For $(\bar{M}_1,D_{\bar{M}_1})$, we have to compute several Courant brackets. Since the expressions for $D_{\bar{M}_1}$ are the same in both charts $(\psi_1(U),\psi_1^{-1})$ and $(\psi_2(U),\psi_2^{-1})$, it suffices to carry out these computations only in the first chart. Denote by (X_i,α_i), $i=1,\ldots,5$, the five spanning sections of $D_{\bar{M}_1}$ in the chart $(\psi_1(U),\psi_1^{-1})$ in the order of formula (6.4). We only have to check that $[(X_i,\alpha_i),(X_j,\alpha_j)]\in\Gamma(D_{\bar{M}_1})$ for all $i,j\in\{1,\ldots,5\}$.

To see this, we begin by noting that

$$[(X_1,\alpha_1),(X_j,\alpha_j)]=[(X_2,\alpha_2),(X_j,\alpha_j)]=0\in\Gamma(D_{\bar{M}_1})$$

for $j=1,\ldots,5$. Next, we compute

$$\begin{aligned}
&[(X_3,\alpha_3),(X_4,\alpha_4)]\\
&\quad=\left[(2\sigma\partial_\delta-2\delta\partial_\sigma,\mathbf{d}f_1),\left(-2\sigma\partial_{f_1}-\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma,\mathbf{d}\delta\right)\right]\\
&\quad=\Bigg(2\sigma\cdot\frac{-2\delta}{f_1}\partial_\sigma-2\delta\cdot(-2)\partial_{f_1}-2\delta\cdot\frac{-2\sigma}{f_1}\partial_\sigma+\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\cdot 2\partial_\delta,\\
&\qquad\qquad\mathbf{d}\left(\mathbf{d}\delta(2\sigma\partial_\delta-2\delta\partial_\sigma)\right)\Bigg)\\
&\quad=2\left(2\delta\partial_{f_1}+\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta,\mathbf{d}\sigma\right)=2(X_5,\alpha_5)\in\Gamma(D_{\bar{M}_1}).
\end{aligned}$$

In the same manner,

$$\begin{aligned}
&[(X_3,\alpha_3),(X_5,\alpha_5)]\\
&\quad=\left[(2\sigma\partial_\delta-2\delta\partial_\sigma,\mathbf{d}f_1),\left(2\delta\partial_{f_1}+\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta,\mathbf{d}\sigma\right)\right]\\
&\quad=-2\left(-2\sigma\partial_{f_1}-\left(f_1+\frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma,\mathbf{d}\delta\right)=-2(X_4,\alpha_4)\in\Gamma(D_{\bar{M}_1})
\end{aligned}$$

and

$$\begin{aligned}
&[(X_4, \alpha_4), (X_5, \alpha_5)] \\
&= \left[\left(-2\sigma\partial_{f_1} - \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\sigma, \mathbf{d}\delta\right), \left(2\delta\partial_{f_1} + \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\partial_\delta, \mathbf{d}\sigma\right)\right] \\
&= \left(-2\sigma\cdot\left(1 - \frac{\sigma^2+\delta^2}{f_1^2}\right)\partial_\delta - \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\cdot\frac{2\sigma}{f_1}\partial_\delta\right. \\
&\qquad \left. +2\delta\cdot\left(1 - \frac{\sigma^2+\delta^2}{f_1^2}\right)\partial_\sigma + \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\cdot\frac{2\delta}{f_1}\partial_\sigma, -\mathbf{d}\left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\right) \\
&= \left(-4\sigma\partial_\delta + 4\delta\partial_\sigma, -\mathbf{d}f_1 + \frac{\sigma^2+\delta^2}{f_1^2}\mathbf{d}f_1 - \frac{2\sigma\mathbf{d}\sigma}{f_1} - \frac{2\delta\mathbf{d}\delta}{f_1}\right) \\
&= -2(2\sigma\partial_\delta - 2\delta\partial_\sigma, \mathbf{d}f_1) + \frac{1}{f_1}\left(0, \left(f_1 + \frac{\sigma^2+\delta^2}{f_1}\right)\mathbf{d}f_1 - 2\sigma\mathbf{d}\sigma - 2\delta\mathbf{d}\delta\right).
\end{aligned}$$

This is a section of $D_{\bar{M}_1}$ since the first summand is the pair (X_3, α_3) and the second summand is shown in (6.3) to be a section of $D_{\bar{M}_1}$.

Thus, we have directly verified in this example the conclusions of Theorems 6.5 and 6.6.

6.3. **Reduction of dynamics.**

Definition 6.11. The function $f \in C^\infty(M)$ will be called *admissible* if there exists a vector field $X_f \in \mathfrak{X}(M)$ such that

$$(X_f, \mathbf{d}f) \in \Gamma(D). \tag{6.5}$$

Note that we have $(X_f + Y, \mathbf{d}f) \in \Gamma(D)$ for all sections Y of G_0. Hence if the distribution G_0 is not trivial, equation (6.5) does not define a unique vector field X_f.

Theorem 6.12. *Let $f \in C^\infty(M)^G$ be admissible. Then there exists $X_f \in \mathfrak{X}(M)^G$ such that $(X_f, \mathbf{d}f)$ is a section of D. Hence, for each stratum $\bar{P}$ of $\bar{M}$ satisfying the conditions of Theorem 6.5, there exists a section $(X_{\bar{P}}, \alpha_{\bar{P}})$ such that $X_f \sim_\pi \bar{X} \in \mathfrak{X}(\bar{M})$, $X_{\bar{P}} \sim_{\iota_{\bar{P}}} \bar{X}$, and $\alpha_{\bar{P}} = \iota_{\bar{P}}^*\bar{\alpha}$, $\pi^*\bar{\alpha} = \mathbf{d}f$. The vector field $X_{\bar{P}}$ is a solution of the implicit Hamiltonian system*

$$(X_{\bar{P}}, \mathbf{d}f_{\bar{P}}) \in \Gamma(D_{\bar{P}}),$$

where $f_{\bar{P}} \in C^\infty(\bar{P})$ is the function defined by $f_{\bar{P}} = \iota_{\bar{P}}^*\bar{f}$, with $\bar{f} \in C^\infty(\bar{M})$ defined by $\pi^*(\bar{f}) = f$. If $X'_{\bar{P}}$ is another solution of this equation, there exists an element Y of $\Gamma(\mathsf{G}_0)^G$ such that $X_f + Y$ descends to a vector field on $\bar{M}$ that restricts to $X'_{\bar{P}}$.

Proof. Let $f \in C^\infty(M)^G$ be admissible. Let X be a vector field satisfying $(X, \mathbf{d}f) \in \Gamma(D)$, and consider the average of this pair. Since $\mathbf{d}f$ is already G-invariant, it remains unchanged and the G-invariant average of $(X, \mathbf{d}f)$ is $(X_G, \mathbf{d}f)$ with a G-invariant vector field X_G. Since D is G-invariant, the section $(X_G, \mathbf{d}f)$ is also a section of D. If X_G disappears, the solutions of the implicit Hamiltonian system span the generalized tangent distribution G_0. Set $X_G =: X_f$. Then the first statement is proved, and $(X_f, \mathbf{d}f) \in \mathcal{D}^G$. Then remainder of the theorem follows immediately. ☐

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