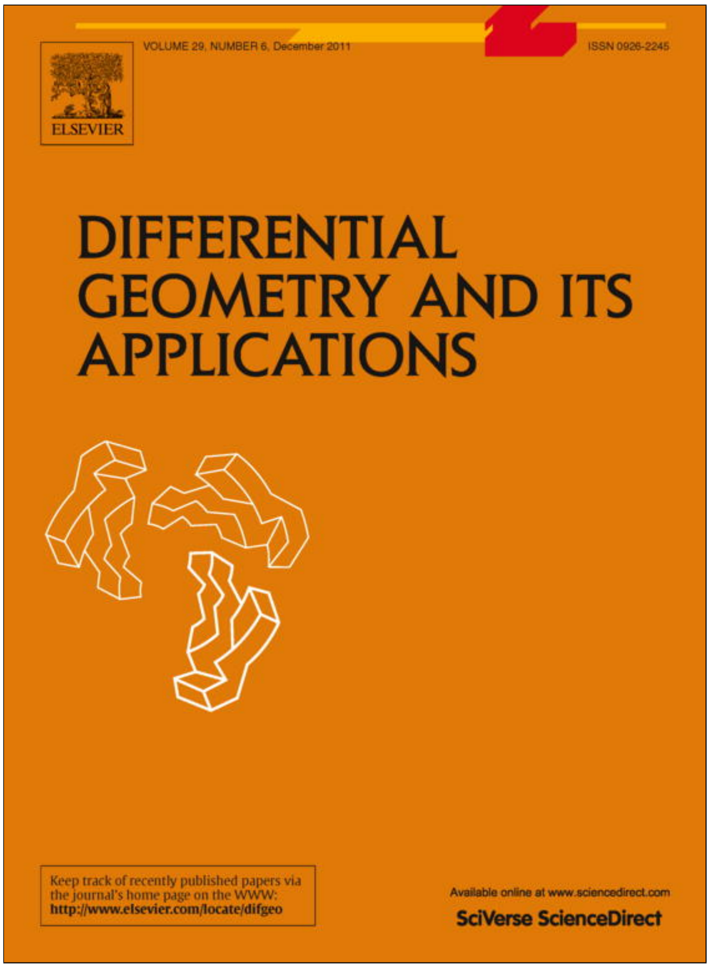
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Invariant generators for generalized distributions ☆

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ABSTRACT

The existence of invariant generators for distributions satisfying a compatibility condition with the symmetry algebra is proved.

1. Introduction

If a smooth manifold M is acted upon in a proper way by a Lie group G, the space of orbits M/G has the structure of a stratified space. If the action is free or with conjugated isotropy subgroups, the quotient M/G is known to be a smooth manifold and the quotient map $\pi : M \to M/G$ a smooth surjective submersion. In the first case, a free and proper action is induced on the tangent space TM and on the cotangent space T^*M, but in the case of conjugated isotropies, the isotropy subgroups of the induced action on the tangent space are not necessarily conjugated (see [7] for a complete characterization of the isotropy lattice of the lifted action).

More generally, if a subdistribution of the Pontryagin bundle $\mathsf{P}_M := TM \oplus T^*M$ is invariant under an action by an involutive subbundle $\mathcal{I}$ of TM how can we decide if this distribution has invariant sections? The existence of invariant generators would imply that the generalized distribution, assumed that its cotangent part annihilates the vertical space of the action, pushes forward to a smooth generalized distribution on the space of leaves $M/\mathcal{I}$. In this note, we present a theorem giving sufficient conditions for a locally finitely generated generalized distribution to be spanned by sections pushing forward to the quotient. This property has its origins in control theory and the first results in this direction were obtained in [6] and [2]. Results about invariant generators for a certain class of invariant subbundles of exact Courant algebroids are shown to hold in [5], but the techniques used there cannot be applied in a straightforward manner to the present situation.

The paper is organized as follows. Background on smooth generalized distributions is reviewed in Section 2. Special emphasis is given to the notion of pointwise and smooth annihilators since these are essential tools for the rest of the

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paper. The main theorem of the paper, guaranteeing, under certain assumptions, the existence of generators of generalized distributions that push forward to the quotient, is proved in Section 3.

Conventions and notations. If M is a smooth manifold, $C^\infty(M)$ denotes the sheaf of *local* functions on M, that is, an element $f \in C^\infty(M)$ is, by definition, a smooth function $f : U \to \mathbb{R}$, where the domain of definition U of f is an open subset of M. Similarly, if E is a vector bundle over M, or a generalized distribution on M, $\Gamma(E)$ denotes the set of smooth *local* sections of E. In particular, the sets of smooth local vector fields and one-forms on M are denoted by $\mathfrak{X}(M)$ and $\Omega^1(M)$, respectively. The open domain of definition of the local section σ of E is denoted by $\operatorname{Dom}(\sigma)$.

2. Generalities on distributions and smooth annihilators

2.1. Smooth and pointwise annihilators

The *Pontryagin bundle* $\mathsf{P} := TM \oplus T^*M$ of a smooth manifold M is naturally endowed with a non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by

$$\big\langle (u_m, \alpha_m), (v_m, \beta_m) \big\rangle := \langle \beta_m, u_m \rangle + \langle \alpha_m, v_m \rangle \tag{1}$$

for all $u_m, v_m \in T_mM$ and $\alpha_m, \beta_m \in T_m^*M$.

A *generalized distribution* is a subset $\Delta \subseteq \mathsf{P} = TM \oplus T^*M$ such that for each $m \in M$, the set $\Delta(m) := \Delta \cap \mathsf{P}(m)$ is a vector subspace of $\mathsf{P}(m) = T_mM \oplus T_m^*M$. The number $\dim \Delta(m)$ is called the *rank* of Δ at $m \in M$. A point $m \in M$ is a *regular* point of the distribution Δ if there exists a neighborhood U of m such that the rank of Δ is constant on U. Otherwise, m is a *singular* point of the distribution.

A local *differentiable section* of Δ is a smooth section $\sigma \in \Gamma(\mathsf{P}) = \mathfrak{X}(M) \times \Omega^1(M)$ defined on some open subset $U \subset M$ such that $\sigma(u) \in \Delta(u)$ for each $u \in U$; $\Gamma(\Delta)$ denotes the space of local differentiable sections of Δ. A generalized distribution is said to be *differentiable* or *smooth* if for every point $m \in M$ and every $v \in \Delta(m)$, there is a differentiable section $\sigma \in \Gamma(\Delta)$ defined on an open neighborhood U of m such that $\sigma(m) = v$.

If $\Delta \subset \mathsf{P}$ is a smooth distribution, its *smooth orthogonal* distribution (or simply its *smooth orthogonal*) is the smooth generalized distribution $\Delta^\perp \subseteq \mathsf{P}$ defined by

$$\Delta^\perp(m) := \left\{ \big(X(m), \alpha(m)\big) \;\middle|\; \begin{array}{c} (X, \alpha) \in \mathfrak{X}(M) \times \Omega^1(M) \\ \text{with } m \in \operatorname{Dom}(X) \cap \operatorname{Dom}(\alpha) \\ \text{such that for all } (Y, \beta) \in \mathfrak{X}(M) \times \Omega^1(M) \\ \text{with } m \in \operatorname{Dom}(Y) \cap \operatorname{Dom}(\beta), \\ \text{we have } \langle (X, \alpha), (Y, \beta) \rangle = 0 \\ \text{on } \operatorname{Dom}(X) \cap \operatorname{Dom}(Y) \cap \operatorname{Dom}(\alpha) \cap \operatorname{Dom}(\beta) \end{array} \right\}.$$

In general, the inclusion $\Delta \subset \Delta^{\perp\perp}$ is strict. The smooth orthogonal of a smooth generalized distribution is smooth by construction. If the distribution Δ is a vector subbundle of P, then its smooth orthogonal distribution is also a vector subbundle of P. Note also that the smooth orthogonal of a smooth generalized distribution Δ is, in general, different from the *pointwise* orthogonal distribution of Δ, defined by

$$\Delta^{\perp_\mathrm{p}}(m) := \big\{ \sigma_m \in \mathsf{P}(m) \;\big|\; \langle \sigma_m, \tau_m \rangle = 0 \text{ for all } \tau_m \in \Delta(m) \big\},$$

where the subscript p stands for "pointwise". The pointwise orthogonal of a smooth generalized distribution Δ is not smooth, in general. The following statement is an immediate consequence of the definitions.

Proposition 1. *Let Δ be a smooth generalized distribution. Then we have*

$$\Delta^\perp \subseteq \Delta^{\perp_\mathrm{p}}, \qquad \Delta = \Delta^{\perp_\mathrm{p}\perp_\mathrm{p}}, \quad \text{and} \quad \Delta \subseteq \Delta^{\perp\perp}.$$

If Δ is itself a vector bundle over M, the smooth orthogonal distribution $\Delta^\perp$ of Δ is also a vector subbundle of P, and we have $\Delta^\perp = \Delta^{\perp_\mathrm{p}}$.

This implies the following property of the smooth annihilator of a sum of vector subbundles of P; its proof is easy and can be found in [4].

Proposition 2. *Let Δ_1 and Δ_2 be vector subbundles of the vector bundle $(\mathsf{P}, \langle\,,\,\rangle)$. Since Δ_1 and Δ_2 have constant rank on M, their smooth orthogonals $\Delta_1^\perp$ and $\Delta_2^\perp$ are also vector subbundles of P and equal to the pointwise orthogonals of Δ_1 and Δ_2. The intersection $\Delta_1^\perp \cap \Delta_2^\perp$ is smooth if and only if it has locally constant rank.*

A tangent (respectively cotangent) distribution $\mathcal{T} \subseteq TM$ (respectively $\mathcal{C} \subseteq T^*M$) can be identified with the smooth generalized distribution $\mathcal{T} \oplus \{0\}$ (respectively $\{0\} \oplus \mathcal{C}$). The smooth orthogonal distribution of $\mathcal{T} \oplus \{0\}$ in $TM \oplus T^*M$ is easily computed to be $(\mathcal{T} \oplus \{0\})^\perp = TM \oplus \mathcal{T}^\circ$, where

$$\mathcal{T}^\circ(m) = \left\{ \alpha(m) \,\middle|\, \begin{array}{l} \alpha \in \Omega^1(M), m \in \mathrm{Dom}(\alpha) \text{ and } \alpha(X) = 0 \\ \text{on } \mathrm{Dom}(\alpha) \cap \mathrm{Dom}(X) \text{ for all } X \in \Gamma(\mathcal{T}) \end{array} \right\}$$

for all $m \in M$. This smooth cotangent distribution is called the smooth *annihilator* of $\mathcal{T}$. Analogously, we define the smooth annihilator $\mathcal{C}^\circ$ of a cotangent distribution $\mathcal{C}$. Then $\mathcal{C}^\circ$ is a smooth tangent distribution and we have $(\{0\} \oplus \mathcal{C})^\perp = \mathcal{C}^\circ \oplus T^*M$.

2.2. Examples

Example 1. If a Lie group G with Lie algebra $\mathfrak{g}$ acts on the manifold M, the tangent distribution $\mathcal{V}$ whose value at each point $m \in M$ is given by

$$\mathcal{V}(m) = \{\xi_M(m) \mid \xi \in \mathfrak{g}\}$$

is called the *vertical distribution*. Let $(\xi^1, \ldots, \xi^d)$ be a basis of the Lie algebra $\mathfrak{g}$. The smooth distribution $\mathcal{V}$ is tangent and globally finitely generated by the fundamental vector fields $\xi_M^1, \ldots, \xi_M^d$, but not of constant rank, in general, unless the action is free or with conjugated isotropy subgroups.

If the G-action has conjugated isotropy groups, the vertical distribution is a vector subbundle of TM. The smooth annihilator $\mathcal{V}^\circ$ of $\mathcal{V}$ is given by

$$\mathcal{V}^\circ(m) = \{\alpha(m) \mid \alpha \in \Omega^1(M), m \in \mathrm{Dom}(\alpha), \text{ such that } \alpha(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g}\}. \tag{2}$$

In the following, we will also need the smooth generalized distribution $\mathcal{K} := \mathcal{V} \oplus \{0\}$ and its smooth orthogonal $\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ$.

Example 2. Consider the tangent distribution $\mathcal{D} \subseteq T\mathbb{R}^2$ defined at every $(x, y) \in \mathbb{R}^2$ by

$$\mathcal{D}(x, y) = \begin{cases} \mathrm{span}(\partial_y, \partial_x), & x \geqslant 0, \\ \mathrm{span}(\partial_y), & x < 0. \end{cases}$$

Assume that X is a vector field on $\mathbb{R}^2$ such that $X(0,0) = \partial_x$ and $X(x, y) \in \mathcal{D}(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Write $X = a\partial_x + b\partial_y$ with $a, b \in C^\infty(\mathbb{R}^2)$. Then $a(0,0) = 1$ and $b(0,0) = 0$. Since a is smooth, there exists a neighborhood U of $(0,0)$ in $\mathbb{R}^2$ such that a doesn't vanish on U. This implies that X does not take values in $\mathcal{D}$, a contradiction. Hence, $\mathcal{D}$ is not a smooth distribution.

Example 3. Consider the tangent distribution $\mathcal{D} \subseteq T\mathbb{R}^2$ defined at every $(x, y) \in \mathbb{R}^2$ by

$$\mathcal{D}(x, y) = \begin{cases} \mathrm{span}(\partial_y, \partial_x), & x > 0, \\ \mathrm{span}(\partial_y), & x \leqslant 0. \end{cases}$$

This distribution is smooth.

3. Invariant generators for distributions

3.1. The theorem

We present here a theorem which can help to decide if a locally finitely generated distribution is spanned by its descending sections. The proof is inspired by [2].

The space $\Gamma(TM \oplus T^*M)$ of local sections of the Pontryagin bundle is endowed with a skew-symmetric bracket given by

$$\begin{aligned} \big[(X, \alpha), (Y, \beta)\big] &:= \left([X, Y], \boldsymbol{\pounds}_X \beta - \boldsymbol{\pounds}_Y \alpha + \frac{1}{2}\mathbf{d}\big(\alpha(Y) - \beta(X)\big)\right) \\ &= \left([X, Y], \boldsymbol{\pounds}_X \beta - \mathbf{i}_Y \mathbf{d}\alpha - \frac{1}{2}\mathbf{d}\big\langle (X, \alpha), (Y, \beta)\big\rangle\right) \end{aligned} \tag{3}$$

(see [1]). This bracket is $\mathbb{R}$-bilinear (in the sense that $[a_1(X_1, \alpha_1) + a_2(X_2, \alpha_2), (Y, \beta)] = a_1[(X_1, \alpha_1), (Y, \beta)] + a_2[(X_2, \alpha_2), (Y, \beta)]$ for all $a_1, a_2 \in \mathbb{R}$ and $(X_1, \alpha_1), (X_2, \alpha_2), (Y, \beta) \in \Gamma(TM \oplus T^*M)$ on the common domain of definition of the three sections) and does not satisfy the Jacobi identity.

Let $\mathcal{I} \subseteq TM$ be a smooth tangent distribution. If $(Y, 0)$ is a local section of $\mathcal{I} \oplus \{0\} \subseteq TM \oplus T^*M$ and (X, α) a local section of $TM \oplus \mathcal{I}^\circ$, using $\alpha(Y) = 0$ we get

$$\left[(Y,0),(X,\alpha)\right]=\left([Y,X],\boldsymbol{\pounds}_Y\alpha+\frac{1}{2}\,\mathbf{d}\big(-\alpha(Y)\big)\right)=(\boldsymbol{\pounds}_Y X,\boldsymbol{\pounds}_Y\alpha). \tag{4}$$

Note also that

$$\left[(Y,0),f(X,\alpha)\right]=(\boldsymbol{\pounds}_Y f)(X,\alpha)+f\left[(Y,0),(X,\alpha)\right] \tag{5}$$

for all $f\in C^\infty(M)$.

Theorem 1. *Let $\mathcal{J}\subseteq TM$ be an involutive vector subbundle of TM and $\mathcal{D}\subseteq TM\oplus\mathcal{J}^\circ$ a generalized distribution on M. Let $\Theta:=\mathcal{J}\oplus\{0\}\subseteq TM\oplus T^*M$. Assume that for each $m\in M$ there exist an open set $U\subseteq M$ with $m\in U$ and smooth sections $d_1,\dots,d_r\in\Gamma(\mathcal{D})$ such that*

$$\mathcal{D}\big(m'\big)=\operatorname{span}_{\mathbb{R}}\big\{d_1\big(m'\big),\dots,d_r\big(m'\big)\big\}$$

for all $m'\in U$ and

$$\big[d_i,\Gamma(\Theta)\big]\subseteq\operatorname{span}_{C^\infty(U)}\{d_1,\dots,d_r\}+\Gamma(\Theta) \tag{6}$$

for $i=1,\dots,r$. Let (X,α) be a local section of $TM\oplus\mathcal{J}^\circ$ satisfying

$$\big[(X,\alpha),\Gamma(\Theta)\big]\subseteq\operatorname{span}_{C^\infty(U)}\{d_1,\dots,d_r\}+\Gamma(\Theta). \tag{7}$$

Then there exist smooth sections $d,d_1',\dots,d_r'$ in $\operatorname{span}_{C^\infty(U)}\{d_1,\dots,d_r\}$ satisfying

(i) *$\mathcal{D}(m')=\operatorname{span}\{d_1'(m'),\dots,d_r(m')\}$ for all $m'\in U$,*
(ii) *$[d_i',\Gamma(\Theta)]\subseteq\Gamma(\Theta)$ on U for all $i=1,\dots,r$, and*
(iii) *$[(X,\alpha)+d,\Gamma(\Theta)]\subseteq\Gamma(\Theta)$ on U.*

Remark 1. Note that if $\mathcal{D}$ has constant rank on U, then $\Gamma(\mathcal{D})=\operatorname{span}_{C^\infty(U)}\{d_1,\dots,d_r\}$. For a generalized distribution this is not necessarily true.

3.2. The proof

This is a long proof and so it will be broken up in four steps. Since the statement is local, we work in a foliated chart of $\mathcal{J}$. We choose a set of local sections as in the hypothesis of the theorem and write its elements as a sum of a component tangent to the leaves of $\mathcal{J}$ and the rest. The main work is the analysis of this second component. The r local sections spanning pointwise $\mathcal{D}$ in the theorem are constructed from an initially chosen set of local spanning sections of $\mathcal{D}$ using in an essential way the information gathered about their second component. In the first step we construct a family of linear systems for the derivatives (along the leaves of $\mathcal{J}$) of these second components. Using specific properties of this system, in the second step, we linearly transform the second components in order to get pairs formed by a vector field and a one-form that are independent of the coordinates of the leaves of $\mathcal{J}$. In the third step, we extend this linear transformation to the set of local spanning sections of $\mathcal{D}$ and, using the property found in the previous step, r sections of $\mathcal{D}$ are constructed that satisfy the first two properties in the statement. In the fourth step an additional section of $\mathcal{D}$ is constructed that satisfies the third property in the statement.

Step 1 *(Construction of the linear system).* Let $n:=\dim M$ and $k:=\dim\mathcal{J}(x)$, for $x\in M$. Since the vector subbundle $\mathcal{J}$ is involutive, it is integrable by the Frobenius Theorem and thus any $m\in M$ lies in a foliated chart domain U_1 described by coordinates $(x^1,\dots,x^n)$ such that the first k among them define the local integral submanifold containing m. Thus, for any $m'\in U_1$ the basis vector fields $\partial_{x^1},\dots,\partial_{x^k}$ evaluated at m' span $\mathcal{J}(m')$.

Because $\mathcal{D}$ is locally finitely generated, we can find on a sufficiently small neighborhood U of m in U_1, smooth sections $d_1=(X_1,\alpha^1),\dots,d_r=(X_r,\alpha^r)$ spanning $\Gamma_U(\mathcal{D})$ as a $C^\infty(U)$-module. Write, for $i=1,\dots,r$,

$$d_i=\big(X_i,\alpha^i\big)=\sum_{j=1}^n\big(X_i^j\partial_{x^j},\alpha_j^i\,\mathbf{d}x^j\big),$$

with X_i^j and α_j^i smooth local functions defined on U for $j=1,\dots,n$. Note that $\alpha_1^i=\dots=\alpha_k^i=0$ for $i=1,\dots,r$, since $\alpha^i\in\Gamma(\mathcal{J}^\circ)$. By hypothesis (6) and with $\partial_{x^l}\in\Gamma(\mathcal{J})$ for $l=1,\dots,k$, we get for all $i=1,\dots,r$ and $l=1,\dots,k$:

$$\begin{aligned}\partial_{x^l}\big(X_i,\alpha^i\big)&:=\big[(\partial_{x^l},0),\big(X_i,\alpha^i\big)\big]=\big([\partial_{x^l},X_i],\boldsymbol{\pounds}_{\partial_{x^l}}\alpha^i\big)\\&=\sum_{j=1}^n\big(\partial_{x^l}\big(X_i^j\big)\partial_{x^j},\partial_{x^l}\big(\alpha_j^i\big)\,\mathbf{d}x^j\big)\in\operatorname{span}_{C^\infty(U)}\{d_1,\dots,d_r\}+\Gamma(\Theta).\end{aligned}$$

Hence we can write

$$\partial_{x^l}\big(X_i,\alpha^i\big)=\sum_{j=1}^{n}\big(\partial_{x^l}\big(X_i^j\big)\partial_{x^j},\partial_{x^l}\big(\alpha_j^i\big)\,\mathbf{d}x^j\big)=\sum_{j=1}^{k}A_{li}^j(\partial_{x^j},0)+\sum_{s=1}^{r}B_{ls}^i\big(X_s,\alpha^s\big)$$

with $A_{li}^j, B_{ls}^i\in C^\infty(U)$ for $i,s=1,\dots,r$ and $l,j=1,\dots,k$. Setting

$$\big(\widetilde{X}_i,\tilde{\alpha}^i\big):=\sum_{j=k+1}^{n}\big(X_i^j\partial_{x^j},\alpha_j^i\,\mathbf{d}x^j\big)$$

for $i=1,\dots,r$, we get

$$\partial_{x^l}\big(\widetilde{X}_i,\tilde{\alpha}^i\big):=\big[(\partial_{x^l},0),\big(\widetilde{X}_i,\tilde{\alpha}^i\big)\big]=\sum_{j=k+1}^{n}\big(\partial_{x^l}\big(X_i^j\big)\partial_{x^j},\partial_{x^l}\big(\alpha_j^i\big)\,\mathbf{d}x^j\big)=\sum_{s=1}^{r}B_{ls}^i\big(\widetilde{X}_s,\tilde{\alpha}^s\big). \tag{8}$$

We verify the last equality. Since $\alpha_j^s=0$ for $j=1,\dots,k$ and $s=1,\dots,r$, we have for any $i=1,\dots,r$, $l=1,\dots,k$,

$$\begin{aligned}\partial_{x^l}\big(X_i,\alpha^i\big)&=\sum_{m=1}^{k}\big(\partial_{x^l}\big(X_i^m\big)\partial_{x^m},0\big)+\sum_{j=k+1}^{n}\big(\partial_{x^l}\big(X_i^j\big)\partial_{x^j},\partial_{x^l}\big(\alpha_j^i\big)\,\mathbf{d}x^j\big)\\&=\sum_{m=1}^{k}A_{li}^m(\partial_{x^m},0)+\sum_{s=1}^{r}B_{ls}^i\big(X_s,\alpha^s\big)\\&=\sum_{m=1}^{k}A_{li}^m(\partial_{x^m},0)+\sum_{s=1}^{r}B_{ls}^i\sum_{j=1}^{n}\big(X_s^j\partial_{x^j},\alpha_j^s\,\mathbf{d}x^j\big)\\&=\sum_{m=1}^{k}A_{li}^m(\partial_{x^m},0)+\sum_{s=1}^{r}B_{ls}^i\sum_{m=1}^{k}\big(X_s^m\partial_{x^m},0\big)+\sum_{s=1}^{r}B_{ls}^i\sum_{j=k+1}^{n}\big(X_s^j\partial_{x^j},\alpha_j^s\,\mathbf{d}x^j\big)\\&=\sum_{m=1}^{k}\left(A_{li}^m+\sum_{s=1}^{r}B_{ls}^iX_s^m\right)(\partial_{x^m},0)+\sum_{j=k+1}^{n}\left(\left(\sum_{s=1}^{r}B_{ls}^iX_s^j\right)\partial_{x^j},\left(\sum_{s=1}^{r}B_{ls}^i\alpha_j^s\right)\mathbf{d}x^j\right)\end{aligned}$$

which is equivalent to

$$\partial_{x^l}\big(X_i^m\big)=A_{li}^m+\sum_{s=1}^{r}B_{ls}^iX_s^m,\quad \text{for all } l,m=1,\dots,k,\ i=1,\dots,r, \tag{9}$$

$$\partial_{x^l}\big(X_i^j\big)=\sum_{s=1}^{r}B_{ls}^iX_s^j\quad \text{for all } l=1,\dots,k,\ i=1,\dots,r,\ j=k+1,\dots n, \tag{10}$$

$$\partial_{x^l}\big(\alpha_j^i\big)=\sum_{s=1}^{r}B_{ls}^i\alpha_j^s\quad \text{for all } l=1,\dots,k,\ i=1,\dots,r,\ j=k+1,\dots,n. \tag{11}$$

Using (10) and (11) we get

$$\begin{aligned}\partial_{x^l}\big(\widetilde{X}_i,\tilde{\alpha}^i\big)&=\sum_{j=k+1}^{n}\big(\partial_{x^l}\big(X_i^j\big)\partial_{x^j},\partial_{x^l}\big(\alpha_j^i\big)\,\mathbf{d}x^j\big)\\&=\sum_{j=k+1}^{n}\left(\left(\sum_{s=1}^{r}B_{ls}^iX_s^j\right)\partial_{x^j},\left(\sum_{s=1}^{r}B_{ls}^i\alpha_j^s\right)\mathbf{d}x^j\right)\\&=\sum_{j=k+1}^{n}\sum_{s=1}^{r}B_{ls}^i\big(X_s^j\partial_{x^j},\alpha_j^s\,\mathbf{d}x^j\big)=\sum_{s=1}^{r}B_{ls}^i\sum_{j=k+1}^{n}\big(X_s^j\partial_{x^j},\alpha_j^s\,\mathbf{d}x^j\big)\\&=\sum_{s=1}^{r}B_{ls}^i\big(\widetilde{X}_s,\tilde{\alpha}^s\big)\end{aligned}$$

which proves (8). Note that (9) gives a formula for $\partial_{x^l}(X_i^m)$ for all $l,m=1,\dots,k$ and $i=1,\dots,r$.

Step 2 *(Construction of an $r \times r$-matrix B such that $((\widetilde{X}_1, \tilde{\alpha}^1), \dots, (\widetilde{X}_r, \tilde{\alpha}^r))B$ does not depend on $x^1, \dots, x^k$).* We rewrite the system (8) in the form

$$\left(\partial_{x^l}\left(\widetilde{X}_1, \tilde{\alpha}^1\right), \dots, \partial_{x^l}\left(\widetilde{X}_r, \tilde{\alpha}^r\right)\right) = \left(\left(\widetilde{X}_1, \tilde{\alpha}^1\right), \dots, \left(\widetilde{X}_r, \tilde{\alpha}^r\right)\right) B_l, \tag{12}$$

where $B_l := [B^i_{ls}]$ is the $r \times r$ matrix whose entry $B^i_{ls} \in C^\infty(U)$ is the intersection of the i-th column and the s-th row ($i, j = 1, \dots, r$). In view of (10), (11), this system can be explicitly written as

$$\partial_{x^l}\begin{bmatrix} X_1^{k+1} & \dots & X_r^{k+1} \\ \vdots & \vdots & \vdots \\ X_1^n & \dots & X_r^n \\ \alpha^1_{k+1} & \dots & \alpha^r_{k+1} \\ \vdots & \vdots & \vdots \\ \alpha^1_n & \dots & \alpha^r_n \end{bmatrix} = \begin{bmatrix} X_1^{k+1} & \dots & X_r^{k+1} \\ \vdots & \vdots & \vdots \\ X_1^n & \dots & X_r^n \\ \alpha^1_{k+1} & \dots & \alpha^r_{k+1} \\ \vdots & \vdots & \vdots \\ \alpha^1_n & \dots & \alpha^r_n \end{bmatrix} \begin{bmatrix} B^1_{l1} & B^2_{l1} & \dots & B^r_{l1} \\ B^1_{l2} & B^2_{l2} & \dots & B^r_{l2} \\ \vdots & \vdots & \vdots & \vdots \\ B^1_{lr} & B^2_{lr} & \dots & B^r_{lr} \end{bmatrix},$$

where

$$\left(\left(\widetilde{X}_1, \tilde{\alpha}^1\right), \dots, \left(\widetilde{X}_r, \tilde{\alpha}^r\right)\right) = \begin{bmatrix} X_1^{k+1} & \dots & X_r^{k+1} \\ \vdots & \vdots & \vdots \\ X_1^n & \dots & X_r^n \\ \alpha^1_{k+1} & \dots & \alpha^r_{k+1} \\ \vdots & \vdots & \vdots \\ \alpha^1_n & \dots & \alpha^r_n \end{bmatrix}. \tag{13}$$

Equivalently, taking the transpose of this system, we get

$$\partial_{x^l}\begin{bmatrix} X_1^{k+1} & \dots & X_1^n & \alpha^1_{k+1} & \dots & \alpha^1_n \\ X_2^{k+1} & \dots & X_2^n & \alpha^2_{k+1} & \dots & \alpha^2_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_r^{k+1} & \dots & X_r^n & \alpha^r_{k+1} & \dots & \alpha^r_n \end{bmatrix} = B_l^\top \begin{bmatrix} X_1^{k+1} & \dots & X_1^n & \alpha^1_{k+1} & \dots & \alpha^1_n \\ X_2^{k+1} & \dots & X_2^n & \alpha^2_{k+1} & \dots & \alpha^2_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_r^{k+1} & \dots & X_r^n & \alpha^r_{k+1} & \dots & \alpha^r_n \end{bmatrix}. \tag{14}$$

Now fix $j \in \{1, \dots, k\}$, think of x^j as a time variable and all the other x^i, $i \neq j$, as parameters. Consider the following linear ordinary differential equation on $\mathbb{R}^r$, where the solutions Y depend smoothly on the parameters $x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^n$:

$$\partial_{x^j} Y = B_j^\top Y. \tag{15}$$

Let $Y^j_1, \dots, Y^j_r$ be r linearly independent solutions of (15). The $r \times r$ matrix W_j whose columns are $Y^j_1, \dots, Y^j_r$, that is, $W_j := (Y^j_1, \dots, Y^j_r)$, is invertible. However, by (14) with $l = j$, the $2(n-k)$ columns of the matrix in this system also satisfy (15) and therefore these columns are linear combinations of $Y^j_1, \dots, Y^j_r$, that is, there exists a matrix L_j having r rows and $2(n-k)$ columns such that

$$\left(\left(\widetilde{X}_1, \tilde{\alpha}^1\right), \dots, \left(\widetilde{X}_r, \tilde{\alpha}^r\right)\right)^\top = W_j L_j, \quad j = 1, \dots, k.$$

The entries of L_j are smooth functions of $(x^1, \dots, x^n)$ and are independent of the variable x^j (the "time" in the differential equation (15)).

Since this argument holds for any $j = 1, \dots, k$, this implies

$$W_1 L_1 = W_2 L_2 = \dots = W_k L_k. \tag{16}$$

Because W_2 is nonsingular, we have

$$L_2\left(x^1, \dots, x^n\right) = \left(W_2^{-1} W_1 L_1\right)\left(x^1, \dots, x^n\right).$$

Since L_2 is independent of x^2 and L_1 is independent of x^1, we get

$$\begin{aligned}L_2(x^1,x^2,\dots,x^n)&=L_2(x^1,0,x^3,\dots,x^n)\\&=(W_2^{-1}W_1)(x^1,0,x^3,\dots,x^n)L_1(x^1,0,x^3,\dots,x^n)\\&=(W_2^{-1}W_1)(x^1,0,x^3,\dots,x^n)L_1(0,0,x^3,\dots,x^n).\end{aligned}\tag{17}$$

In the same manner, we have $L_3(x^1,\dots,x^n)=(W_3^{-1}W_2L_2)(x^1,\dots,x^n)$, and hence

$$\begin{aligned}L_3(x^1,\dots,x^n)&=L_3(x^1,x^2,0,x^4,\dots,x^n)\\&=(W_3^{-1}W_2)(x^1,x^2,0,x^4,\dots,x^n)L_2(x^1,x^2,0,x^4,\dots,x^n)\\&\overset{(17)}{=}(W_3^{-1}W_2)(x^1,x^2,0,x^4,\dots,x^n)(W_2^{-1}W_1)(x^1,0,0,x^4,\dots,x^n)L_1(0,0,0,x^4,\dots,x^n).\end{aligned}$$

By induction, we get

$$\begin{aligned}L_k(x^1,\dots,x^n)&=L_k(x^1,\dots,x^{k-1},0,x^{k+1},\dots,x^n)\\&=(W_k^{-1}W_{k-1})(x^1,\dots,x^{k-1},0,x^{k+1},\dots,x^n)\\&\quad\times L_{k-1}(x^1,\dots,x^{k-1},0,x^{k+1},\dots,x^n)\\&=(W_k^{-1}W_{k-1})(x^1,\dots,x^{k-1},0,x^{k+1},\dots,x^n)\\&\quad\times(W_{k-1}^{-1}W_{k-2})(x^1,\dots,x^{k-2},0,0,x^{k+1},\dots,x^n)\\&\quad\times\cdots\times(W_3^{-1}W_2)(x^1,x^2,0,\dots,0,x^{k+1},\dots,x^n)\\&\quad\times(W_2^{-1}W_1)(x^1,0,\dots,0,x^{k+1},\dots,x^n)L_1(0,\dots,0,x^{k+1},\dots,x^n).\end{aligned}\tag{18}$$

Define the smooth $r\times r$ nonsingular matrix

$$\begin{aligned}H(x^1,\dots,x^n):=&\,W_k(x^1,\dots,x^n)(W_k^{-1}W_{k-1})(x^1,\dots,x^{k-1},0,x^{k+1},\dots,x^n)\\&\times\cdots\times(W_3^{-1}W_2)(x^1,x^2,0,\dots,0,x^{k+1},\dots,x^n)\\&\times(W_2^{-1}W_1)(x^1,0,\dots,0,x^{k+1},\dots,x^n)\end{aligned}\tag{19}$$

and the smooth $r\times(2n-2k)$ matrix

$$L(x^1,\dots,x^n):=L_1(0,\dots,0,x^{k+1},\dots,x^n).$$

Note that L does not depend on $x^1,\dots,x^k$. Using Eqs. (18) and (19), we get

$$\begin{aligned}(HL)(x^1,\dots,x^n)&=H(x^1,\dots,x^n)L_1(0,\dots,0,x^{k+1},\dots,x^n)\\&=W_k(x^1,\dots,x^n)L_k(x^1,\dots,x^n)\\&=\big((\widetilde{X}_1,\tilde{\alpha}^1),\dots,(\widetilde{X}_r,\tilde{\alpha}^r)\big)^\top.\end{aligned}$$

Define the $r\times r$ nonsingular matrix B depending smoothly on $(x^1,\dots,x^n)$ by

$$B(x^1,\dots,x^n):=\big(H(x^1,\dots,x^n)^\top\big)^{-1}=\big(H(x^1,\dots,x^n)^{-1}\big)^\top.\tag{20}$$

Then $((\widetilde{X}_1,\tilde{\alpha}^1),\dots,(\widetilde{X}_r,\tilde{\alpha}^r))B=L^\top$ does *not* depend on $x^1,\dots,x^k$.

Step 3 *(Construction of the local sections $(Z_1,\gamma_1),\dots,(Z_r,\gamma_r)$).* We want to better understand the columns of this matrix. In view of (13), we can write

$$\big((\widetilde{X}_1,\tilde{\alpha}^1),\dots,(\widetilde{X}_r,\tilde{\alpha}^r)\big)B=\begin{bmatrix}\widetilde{C}\\\widetilde{D}\end{bmatrix},$$

where

$$\widetilde{C}=[\widetilde{C}_{jl}]_{\substack{j=k+1,\dots,n\\l=1,\dots,r}}\quad\text{and}\quad\widetilde{D}=[\widetilde{D}_{jl}]_{\substack{j=k+1,\dots,n\\l=1,\dots,r}}$$

are $(n-k)\times r$-matrices whose entries are smooth functions of only $x^{k+1},\dots,x^n$ (that is, they do not depend on $x^1,\dots,x^k$). Thus the i-th column of the matrix $((\widetilde{X}_1,\tilde{\alpha}^1),\dots,(\widetilde{X}_r,\tilde{\alpha}^r))B$ equals

$$\Big(\big((\widetilde{X}_1,\tilde{\alpha}^1),\ldots,(\widetilde{X}_r,\tilde{\alpha}^r)\big)B\Big)_i := \sum_{j=k+1}^{n}\big(\widetilde{C}_{ji}\partial_{x^j},\widetilde{D}_{ji}\,\mathbf{d}x^j\big) \tag{21}$$

and so, by (4), we get for any $l = 1,\ldots,k$ and $i = 1,\ldots,r$,

$$\Big[(\partial_{x^l},0),\Big(\big((\widetilde{X}_1,\tilde{\alpha}^1),\ldots,(\widetilde{X}_r,\tilde{\alpha}^r)\big)B\Big)_i\Big] = 0. \tag{22}$$

Let $d'_1 = (Z_1,\gamma_1),\ldots,d'_r = (Z_r,\gamma_r)$ be the sections of $\mathcal{D} \subseteq TM \oplus \mathcal{I}^\circ$ on U defined by

$$d'_i = (Z_i,\gamma_i) := \Big(\big((X_1,\alpha^1),\ldots,(X_r,\alpha^r)\big)B\Big)_i \quad \text{for } i = 1,\ldots,r,$$

where, as before, if we write

$$\big((X_1,\alpha^1),\ldots,(X_r,\alpha^r)\big)B = \begin{bmatrix} X_1^1 & \ldots & X_r^1 \\ \vdots & \vdots & \vdots \\ X_1^k & \ldots & X_r^k \\ X_1^{k+1} & \ldots & X_r^{k+1} \\ \vdots & \vdots & \vdots \\ X_1^n & \ldots & X_r^n \\ \alpha_1^1 & \ldots & \alpha_1^r \\ \vdots & \vdots & \vdots \\ \alpha_k^1 & \ldots & \alpha_k^r \\ \alpha_{k+1}^1 & \ldots & \alpha_{k+1}^r \\ \vdots & \vdots & \vdots \\ \alpha_n^1 & \ldots & \alpha_n^r \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1r} \\ B_{21} & B_{22} & \ldots & B_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ B_{r1} & B_{r2} & \ldots & B_{rr} \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix},$$

with

$$C = [C_{jl}]_{\substack{j=1,\ldots,n \\ l=1,\ldots,r}} \quad \text{and} \quad D = [D_{jl}]_{\substack{j=1,\ldots,n \\ l=1,\ldots,r}},$$

$n \times r$-matrices whose entries depend smoothly on all coordinates $x^1,\ldots,x^n$, we have

$$\Big(\big((X_1,\alpha^1),\ldots,(X_r,\alpha^r)\big)B\Big)_i = \sum_{j=1}^{n}\big(C_{ji}\partial_{x^j},D_{ji}\,\mathbf{d}x^j\big).$$

Note that since $\alpha_j^i = 0$ for $i = 1,\ldots,r$ and $j = 1,\ldots,k$, we get $D_{ji} = 0$ for all $i = 1,\ldots,r$ and $j = 1,\ldots,k$. Using (21), we conclude

$$Z_i = \xi_i + \mathbb{P}_{TM}\Big(\big((\widetilde{X}_1,\ldots,\widetilde{X}_r)B\big)_i\Big) = \xi_i + \sum_{j=k+1}^{n} C_{ji}\partial_{x^j},$$

$$\gamma_i = \mathbb{P}_{T^*M}\Big(\big((\widetilde{\alpha}_1,\ldots,\tilde{\alpha}_r)B\big)_i\Big) = \sum_{j=k+1}^{n} D_{ji}\,\mathbf{d}x^j,$$

where $\mathbb{P}_{TM}$ and $\mathbb{P}_{T^*M}$ are the projections on the vector field and one-form factors, respectively, and

$$\xi_i = \sum_{j=1}^{k} C_{ji}\partial_{x^j} \in \Gamma(\mathcal{I}).$$

Thus, we have for all $l = 1,\ldots,k$ and $i = 1,\ldots,r$,

$$\begin{aligned}\big[(\partial_{x^l},0),(Z_i,\gamma_i)\big] &= \big[(\partial_{x^l},0),(\xi_i,0)\big] + \Big[(\partial_{x^l},0),\Big(\big((\widetilde{X}_1,\tilde{\alpha}_1),\ldots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)\cdot B\Big)_i\Big] \\ &\overset{(22)}{=} \big([\partial_{x^l},\xi_i],0\big) + 0 \in \Gamma(\Theta)\end{aligned} \tag{23}$$

since $\mathcal{I} \subset TM$ is an involutive vector subbundle, by hypothesis. Hence, if we write an arbitrary section $(\eta,0) \in \Gamma(\Theta)$ as

$$(\eta, 0) = \left(\sum_{j=1}^{k} \eta_j \partial_{x^j}, 0\right),$$

where $\eta_1, \dots, \eta_k$ are smooth functions of $x^1, \dots, x^n$, we get for $i = 1, \dots, r$

$$\begin{aligned}\big[(\eta, 0), (Z_i, \gamma_i)\big] &= \left[\left(\sum_{j=1}^{k} \eta_j \partial_{x^j}, 0\right), (Z_i, \gamma_i)\right] \\ &= \sum_{j=1}^{k} \eta_j \big[(\partial_{x^j}, 0), (Z_i, \gamma_i)\big] - \sum_{j=1}^{k} Z_i[\eta_j](\partial_{x^j}, 0) \in \Gamma(\Theta).\end{aligned}$$

Indeed, both terms are elements of $\Gamma(\Theta)$: the first summand by (23) and the second summand because of its form.

Thus, since by construction, $(Z_1, \gamma_1), \dots, (Z_r, \gamma_r)$ span $\mathcal{D}$ on U (because B is an invertible $r \times r$ matrix), these smooth sections of $\mathcal{D} \subseteq TM \oplus \mathcal{T}^\circ$ satisfy the first two statements of the proposition.

Step 4 *(Construction of the local section d).* We use the sections $d_1 = (X_1, \alpha^1), \dots, d_r = (X_r, \alpha^r)$ spanning $\mathcal{D}$ on the open subset $U \subseteq M$. Since $(X, \alpha) \in \Gamma(TM \oplus \mathcal{T}^\circ)$, it can be written in the form

$$(X, \alpha) = \sum_{j=1}^{n} a^j (\partial_{x^j}, 0) + \sum_{j=k+1}^{n} b_j \big(0, \mathbf{d}x^j\big),$$

where $a^1, \dots, a^n, b_{k+1}, \dots, b_n$ are C^∞-functions of $x^1, \dots, x^n$. By hypothesis (7),

$$\big[(\partial_{x^l}, 0), (X, \alpha)\big] = \sum_{j=1}^{n} \partial_{x^l}\big(a^j\big)(\partial_{x^j}, 0) + \sum_{j=k+1}^{n} \partial_{x^l}(b_j)\big(0, \mathbf{d}x^j\big) \in \operatorname{span}_{C^\infty(U)}\{d_1, \dots, d_r\} + \Gamma(\Theta)$$

for all $l = 1, \dots, k$. Thus, for each $l = 1, \dots, k$, there exist functions $\beta_l^1, \dots, \beta_l^r$ and $\sigma_l^1, \dots, \sigma_l^k$ depending smoothly on $x^1, \dots, x^n$ such that

$$\sum_{j=1}^{n} \partial_{x^l}\big(a^j\big)(\partial_{x^j}, 0) + \sum_{j=k+1}^{n} \partial_{x^l}(b_j)\big(0, \mathbf{d}x^j\big) = \sum_{j=1}^{k} \sigma_l^j (\partial_{x^j}, 0) + \sum_{j=1}^{r} \beta_l^j \big(X_j, \alpha^j\big).$$

Hence, if we define

$$(\widetilde{X}, \tilde{\alpha}) := \sum_{j=k+1}^{n} \big(a^j \partial_{x^j}, b_j\, \mathbf{d}x^j\big),$$

then proceeding as in the proof of (8), we get for each $l = 1, \dots, k$,

$$\begin{aligned}\big[(\partial_{x^l}, 0), (\widetilde{X}, \tilde{\alpha})\big] &= \sum_{j=k+1}^{n} \big(\partial_{x^l}\big(a^j\big)\partial_{x^j}, \partial_{x^l}(b_j)\, \mathbf{d}x^j\big) \\ &= \sum_{j=1}^{r} \beta_l^j \big(\widetilde{X}_j, \tilde{\alpha}^j\big) = \big((\widetilde{X}_1, \tilde{\alpha}^1), \dots, (\widetilde{X}_r, \tilde{\alpha}^r)\big)\beta_l \end{aligned} \tag{24}$$

where β_l is the $r \times 1$ matrix with entries $\beta_l^1, \dots, \beta_l^r$.

Consider the $r \times 1$ matrix of derivatives $\partial_{x^l}(H^\top \beta_j)$ for fixed $j, l = 1, \dots, k$. We consider below the product of the $2(n-k) \times r$ matrix $((\widetilde{X}_1, \tilde{\alpha}^1), \dots, (\widetilde{X}_r, \tilde{\alpha}^r))B$ with the $r \times 1$ matrix $\partial_{x^l}(H^\top \beta_j)$. We need the conclusion of Step 2: $((\widetilde{X}_1, \tilde{\alpha}^1), \dots, (\widetilde{X}_r, \tilde{\alpha}^r))B$ does not depend on $x^1, \dots, x^k$, that is,

$$0 = \partial_{x^l}\big(\big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)B\big) = \big[(\partial_{x^l}, 0), \big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)B\big]. \tag{25}$$

Therefore, the definition (20) of the matrix B, (25), and the Leibniz rule yield

$$\begin{aligned}&\big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)B\partial_{x^l}\big(H^\top \beta_j\big) \\ &\quad = \big[(\partial_{x^l}, 0), \big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)B\big]\big(H^\top \beta_j\big) + \big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)B\partial_{x^l}\big(H^\top \beta_j\big) \\ &\quad = \big[(\partial_{x^l}, 0), \big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)BH^\top \beta_j\big] \\ &\quad = \big[(\partial_{x^l}, 0), \big((\widetilde{X}_1, \tilde{\alpha}_1), \dots, (\widetilde{X}_r, \tilde{\alpha}_r)\big)\beta_j\big] = \big[(\partial_{x^l}, 0), \big[(\partial_{x^j}, 0), (\widetilde{X}, \tilde{\alpha})\big]\big].\end{aligned} \tag{26}$$

However,

$$\begin{aligned}\big[(\partial_{x^l},0),\big[(\partial_{x^j},0),(\widetilde{X},\tilde{\alpha})\big]\big]&=\big[(\partial_{x^l},0),\big([\partial_{x^j},\widetilde{X}],\boldsymbol{\pounds}_{\partial_{x^j}}\tilde{\alpha}\big)\big]\\&=\big(\big[\partial_{x^l},[\partial_{x^j},\tilde{X}]\big],\boldsymbol{\pounds}_{\partial_{x^l}}\boldsymbol{\pounds}_{\partial_{x^j}}\tilde{\alpha}\big)\\&=\big(-\big[\tilde{X},[\partial_{x^l},\partial_{x^j}]\big]-\big[\partial_{x^j},[\tilde{X},\partial_{x^l}]\big],\boldsymbol{\pounds}_{\partial_{x^l}}\boldsymbol{\pounds}_{\partial_{x^j}}\tilde{\alpha}\big)\end{aligned}$$

by the Jacobi identity. Since $[\partial_{x^l},\partial_{x^j}]=0$ and $\boldsymbol{\pounds}_{\partial_{x^j}}\boldsymbol{\pounds}_{\partial_{x^l}}\tilde{\alpha}=\boldsymbol{\pounds}_{\partial_{x^l}}\boldsymbol{\pounds}_{\partial_{x^j}}\tilde{\alpha}$, we conclude from (26),

$$\begin{aligned}\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}\big(H^\top\beta_j\big)&=\big(\big[\partial_{x^j},[\partial_{x^l},\tilde{X}]\big],\boldsymbol{\pounds}_{\partial_{x^j}}\boldsymbol{\pounds}_{\partial_{x^l}}\tilde{\alpha}\big)\\&=\big[(\partial_{x^j},0),\big[(\partial_{x^l},0),(\widetilde{X},\tilde{\alpha})\big]\big]\\&=\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^j}\big(H^\top\beta_l\big).\end{aligned}\tag{27}$$

Define the $r\times 1$ matrix with C^∞-entries in the variables $x^1,\dots,x^n$,

$$\begin{aligned}\Pi=(\Pi_1,\dots,\Pi_r)^\top:=-B\Bigg[&\int_0^{x^k}\big(H^\top\beta_k\big)\big(x^1,\dots,x^{k-1},\tau,x^{k+1},\dots,x^n\big)\,d\tau\\&+\int_0^{x^{k-1}}\big(H^\top\beta_{k-1}\big)\big(x^1,\dots,x^{k-2},\tau,0,x^{k+1},\dots,x^n\big)\,d\tau\\&+\dots+\int_0^{x^1}\big(H^\top\beta_1\big)\big(\tau,0,\dots,0,x^{k+1},\dots,x^n\big)\,d\tau\Bigg]\\=-(BR)\big(x^1,\dots,x^n\big),&\end{aligned}$$

where $R(x^1,\dots,x^n)$ is the $r\times 1$ matrix in the parenthesis. Then for $l=1,\dots,k$, we get

$$\begin{aligned}&\big[(\partial_{x^l},0),\big((\widetilde{X},\tilde{\alpha})+\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)\Pi\big)\big]\\&\quad=\big[(\partial_{x^l},0),(\widetilde{X},\tilde{\alpha})\big]+\big[(\partial_{x^l},0),\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)\Pi\big]\\&\quad=\big[(\partial_{x^l},0),(\widetilde{X},\tilde{\alpha})\big]-\big[(\partial_{x^l},0),\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)BR\big]\\&\quad=\big[(\partial_{x^l},0),(\widetilde{X},\tilde{\alpha})\big]-\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B(\partial_{x^l}R)\end{aligned}\tag{28}$$

by (5) and using the fact that $((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r))B$ is independent of $x^1,\dots,x^k$ (see the conclusion of Step 2). For any $l=1,\dots,k$, we prove the identity

$$\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}R=\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)\beta_l.\tag{29}$$

Indeed, since we can move freely $((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r))B$ (a matrix depending smoothly on $x^{k+1},\dots,x^n$) in and out of the integral signs in the computation below, we get

$$\begin{aligned}&\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}R\\&\quad=\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}\Bigg[\int_0^{x^k}\big(H^\top\beta_k\big)\big(x^1,\dots,x^{k-1},\tau,x^{k+1},\dots,x^n\big)\,d\tau\\&\qquad+\int_0^{x^{k-1}}\big(H^\top\beta_{k-1}\big)\big(x^1,\dots,x^{k-2},\tau,0,x^{k+1},\dots,x^n\big)\,d\tau+\dots+\int_0^{x^1}\big(H^\top\beta_1\big)\big(\tau,0,\dots,0,x^{k+1},\dots,x^n\big)\,d\tau\Bigg]\\&\quad=\int_0^{x^k}\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}\big(H^\top\beta_k\big)\big(x^1,\dots,x^{k-1},\tau,x^{k+1},\dots,x^n\big)\,d\tau\\&\qquad+\int_0^{x^{k-1}}\big((\widetilde{X}_1,\tilde{\alpha}_1),\dots,(\widetilde{X}_r,\tilde{\alpha}_r)\big)B\partial_{x^l}\big(H^\top\beta_{k-1}\big)\big(x^1,\dots,x^{k-2},\tau,0,x^{k+1},\dots,x^n\big)\,d\tau\end{aligned}$$

$$+\cdots+\int_0^{x^{l+1}}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^l}\big(H^\top\beta_{l+1}\big)\big(x^1,\ldots,x^l,\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau$$
$$+\partial_{x^l}\int_0^{x^l}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{l-1},\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau$$
$$+\int_0^{x^{l-1}}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^l}\big(H^\top\beta_{l-1}\big)\big(x^1,\ldots,x^{l-2},\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau$$
$$+\cdots+\int_0^{x^1}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^l}\big(H^\top\beta_1\big)\big(\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau.$$

Since $(H^\top\beta_{m+1})(x^1,\ldots,x^m,\tau,0,\ldots,0,x^{k+1},\ldots,x^n)$ doesn't depend on x^l for $m<l-1$, the last $l-1$ integrals in this expression vanish. Using (27) in the first $k-l$ integrals we get

$$\begin{aligned}
&\int_0^{x^k}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^k}\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{k-1},\tau,x^{k+1},\ldots,x^n\big)\,d\tau\\
&\quad+\int_0^{x^{k-1}}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^{k-1}}\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{k-2},\tau,0,x^{k+1},\ldots,x^n\big)\,d\tau\\
&\quad+\cdots+\int_0^{x^{l+1}}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\partial_{x^{l+1}}\big(H^\top\beta_l\big)\big(x^1,\ldots,x^l,\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau\\
&\quad+\partial_{x^l}\int_0^{x^l}\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{l-1},\tau,0,\ldots,0,x^{k+1},\ldots,x^n\big)\,d\tau\\
&=\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\big[\big(\big(H^\top\beta_l\big)\big(x^1,\ldots,x^k,\ldots,x^n\big)-\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{k-1},0,x^{k+1},\ldots,x^n\big)\big)\\
&\quad+\big(\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{k-1},0,x^{k+1},\ldots,x^n\big)-\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{k-2},0,0,x^{k+1},\ldots,x^n\big)\big)\\
&\quad+\cdots+\big(\big(H^\top\beta_l\big)\big(x^1,\ldots,x^{l+1},0,\ldots,0,x^{k+1},\ldots,x^n\big)-\big(H^\top\beta_l\big)\big(x^1,\ldots,x^l,0,\ldots,0,x^{k+1},\ldots,x^n\big)\big)\\
&\quad+\big(H^\top\beta_l\big)\big(x^1,\ldots,x^l,0,\ldots,0,x^{k+1},\ldots,x^n\big)\big]\\
&=\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)B\big(x^1,\ldots,x^n\big)H^\top\big(x^1,\ldots,x^n\big)\beta_l\big(x^1,\ldots,x^n\big)\\
&=\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)\beta_l\big(x^1,\ldots,x^n\big)
\end{aligned}$$

by (20). This proves (29).

From (29), (28), and (24) we conclude

$$\begin{aligned}
&\big[(\partial_{x^l},0),\big((\widetilde{X},\tilde\alpha)+\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)\Pi\big)\big]\\
&\quad=\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)\beta_l-\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)\beta_l=0.
\end{aligned}\tag{30}$$

This identity suggests that the required section $d=(Z,\gamma)\in\Gamma(\mathcal{D})$ satisfying the third condition in the statement of the proposition is

$$d=(Z,\gamma):=\big((X_1,\alpha_1),\ldots,(X_r,\alpha_r)\big)\Pi=\sum_{l=1}^r(X_l,\alpha_l)\Pi_l\in\Gamma(\mathcal{D}).$$

We have

$$\begin{aligned}
(Z,\gamma)&=\big((\bar{X}_1,0),\ldots,(\bar{X}_r,0)\big)\Pi+\big((\widetilde{X}_1,\tilde\alpha_1),\ldots,(\widetilde{X}_r,\tilde\alpha_r)\big)\Pi\\
&=\sum_{l=1}^r(\bar{X}_l,0)\Pi_l+\sum_{l=1}^r(\widetilde{X}_l,\tilde\alpha_l)\Pi_l,
\end{aligned}$$

where $\bar{X} := \sum_{j=1}^{k} a^j \partial_{x^j}$ for $i = 1, \ldots, r$. Set in the same manner $\bar{X}_i = \sum_{j=1}^{k} X_i^j \partial_{x^j} \in \Gamma(\mathcal{I})$ for $i = 1, \ldots, r$, and verify (iii) in the statement of the theorem. For any $l = 1, \ldots, k$, since $\partial_{x^1}, \ldots, \partial_{x^k}$ is a basis of the space of sections of $\mathcal{I}$ over U, we get

$$\begin{aligned}\big[(X+Z, \alpha+\gamma), (\partial_{x^l}, 0)\big] &= \left[(\bar{X}, 0) + \sum_{k=1}^{r} (\bar{X}_k, 0)\Pi_k, (\partial_{x^l}, 0)\right] + \left[(\tilde{X}, \tilde{\alpha}) + \sum_{k=1}^{r} (\tilde{X}_k, \tilde{\alpha}_k)\Pi_k, (\partial_{x^l}, 0)\right] \\ &\overset{(30)}{=} \left[(\bar{X}, 0) + \sum_{k=1}^{r} (\bar{X}_k, 0)\Pi_k, (\partial_{x^l}, 0)\right] + 0 \in \Gamma(\Theta)\end{aligned}$$

since, by construction, $\bar{X}, \bar{X}_i \in \Gamma(\mathcal{I})$, $i = 1, \ldots, r$, so that $\bar{X} + \sum_{k=1}^{r} \bar{X}_k \Pi_k \in \Gamma(\mathcal{I})$. Since $(\partial_{x^1}, 0), \ldots, (\partial_{x^k}, 0)$ span the distribution Θ over U, we conclude that $[(X+Z, \alpha+\gamma), \Gamma(\Theta)] \subseteq \Gamma(\Theta)$ on U and (iii) in the statement is proved.

3.3. Example

Let $(G \rightrightarrows M, \mathsf{t}, \mathsf{s}, \mathsf{m}, \epsilon)$ be a t-connected Lie groupoid with Lie algebroid $A \to M$. The Pontryagin bundle $TG \oplus T^*G$ inherits the structure of a Lie groupoid over the vector bundle $TM \oplus A^*$ (see for instance [3]). The source and target maps of this groupoid are written $\mathbb{T}\mathsf{s}$ and $\mathbb{T}\mathsf{t}$. Consider a smooth subdistribution $\mathcal{D}$ of the kernel $\ker \mathbb{T}\mathsf{s} = T^{\mathsf{s}}G \oplus (T^{\mathsf{t}}G)^\circ$, where $T^{\mathsf{s}}G$ is the involutive subbundle tangent to the s-fibers and $T^{\mathsf{t}}G \subseteq TG$ is the involutive subbundle tangent to the t-fibers.

For any left-invariant section X^l of $T^{\mathsf{t}}G$ and for any right-invariant section $(Y^r, \mathsf{t}^*\alpha)$ of $\ker \mathbb{T}\mathsf{s}$, we find that $[(X^l, 0), (Y^r, \mathsf{t}^*\alpha)] = 0$ (see for instance [3]). Since these sections span $T^{\mathsf{t}}G$ and $\ker \mathbb{T}\mathsf{s}$, respectively, we find that $[\Gamma(\ker \mathbb{T}\mathsf{s}), \Gamma(T^{\mathsf{t}}G \oplus \{0\})] \subseteq \Gamma(\ker \mathbb{T}\mathsf{s} + T^{\mathsf{t}}G \oplus \{0\})$.

Choose $g \in G$ and assume that there exist smooth sections $d_1, \ldots, d_r$ of $\mathcal{D}$, defined on a neighborhood U of g in G and such that

$$\mathcal{D}(g') = \operatorname{span}_{\mathbb{R}}\{d_1(g'), \ldots, d_r(g')\}$$

for all $g' \in U$ and

$$\big[d_i, \Gamma\big(T^{\mathsf{t}}G \oplus \{0\}\big)\big] \subseteq \operatorname{span}_{C^\infty(U)}\{d_1, \ldots, d_r\} + \Gamma\big(T^{\mathsf{t}}G \oplus \{0\}\big) \tag{31}$$

for $i = 1, \ldots, r$.

Then, using Theorem 1, we conclude that, on U, $\mathcal{D}$ is spanned pointwise by sections of the type $(X, \mathsf{t}^*\alpha) \in \Gamma(\ker \mathbb{T}\mathsf{s})$ with $X \sim_{\mathsf{t}} \bar{X}$ for some $\bar{X} \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$, i.e., such that $\mathbb{T}\mathsf{t} \circ (X, \mathsf{t}^*\alpha)$ is constant on t-fibers.

Furthermore, if $p \in \Gamma(\mathsf{P})$ is defined on U and such that $[p, \Gamma(T^{\mathsf{t}}G \oplus \{0\})] \subseteq \operatorname{span}_{C^\infty(U)}\{d_1, \ldots, d_r\} + \Gamma(T^{\mathsf{t}}G \oplus \{0\})$, then there exists $k \in \Gamma(\mathcal{D})$ such that $p + k$ is $T^{\mathsf{t}}G$-invariant and $\mathbb{T}\mathsf{s} \circ (p + k) = \mathbb{T}\mathsf{s} \circ p$.

For another example from control theory, see [2].

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