

INDUCED DIRAC STRUCTURES ON ISOTROPY-TYPE MANIFOLDS

M. JOTZ*

T. S. RATIU**

Section de Mathématiques
Ecole Polytechnique Fédérale de
Lausanne
1015 Lausanne, Switzerland
madeleine.jotz@epfl.ch

Section de Mathématiques and
Bernoulli Center
Ecole Polytechnique Fédérale de
Lausanne
1015 Lausanne, Switzerland
tudor.ratiu@epfl.ch

Abstract. A new method of singular reduction is extended from Poisson to Dirac manifolds. Then it is shown that the Dirac structures on the strata of the quotient coincide with those of the only other known singular Dirac reduction method.

1. Introduction

Dirac structures were introduced in [CW88] and studied systematically for the first time in [Cou90]. The past years have seen remarkable applications of Dirac manifolds in geometry and theoretical physics. Dirac structures include 2-forms, Poisson structures, and foliations. They turn out to provide the right geometric framework for nonholonomic systems and circuits. If a Lie group action is compatible with the Dirac structure on a manifold, one has all the ingredients for reduction; see [Cou90], [Bla00], [BvdS01], [BCG07], [BC05], [JR08], [Zam08], [JRZ10], [YM07], [YM09] for the regular case, [BR04], [JRS10] for the singular situation, and [JR10] for optimal reduction. All these reduction procedures are in the spirit of Poisson reduction ([MR86], [Śni03], [FOR09], [JR09]).

Consider a Dirac structure D on a manifold M which is invariant under the proper action of a Lie group G . The most general singular Dirac reduction method known today was introduced in [JRS10] and is in the same spirit as [Śni03]. Certain mild regularity conditions are required to construct a specific subspace of the set of pairs of *vector fields* with *1-forms* on the stratified space $\bar{M} := M/G$ which is then shown to naturally induce a Dirac structure on each stratum of \bar{M} . These regularity conditions are automatically satisfied for Poisson manifolds.

However, in the case of Poisson manifolds, there is an alternative singular reduction method presented in [FOR09]. If $(M, \{\cdot, \cdot\})$ is a Poisson manifold with

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a Lie group G acting in a proper canonical way on it, then there is an induced Poisson structure on each isotropy-type manifold M_H (see also [JR09]) that is invariant under the induced action of $N(H)/H$; here H is an isotropy subgroup of the action and $N(H)$ is the normalizer of H in G . Therefore, these Poisson structures descend to the quotient $M_H/(N(H)/H)$ whose connected components are strata in \bar{M} .

The first goal of this paper is to extend this reduction method to symmetric Dirac manifolds. In a first step, the G -invariant Dirac structure D on M is shown to induce a Dirac structure D_Q on each connected component Q of an isotropy-type manifold M_H . Then D_Q is shown to satisfy the conditions for regular Dirac reduction (as in [JRZ10]), thus descending to a Dirac structure on the stratum $Q/N(H)$ of \bar{M} . The second goal of the paper is to show that the reduced manifolds obtained this way correspond exactly to the singularly reduced manifolds described previously (those obtained in [JRS10]). More precisely, the Dirac structures induced on the connected components of $M_{(H)}/G$ and of $M_H/N(H)$ are forward and backward Dirac images of each other under a canonical diffeomorphism between the two reduced manifolds.

Notation and conventions. The manifold M is always assumed to be paracompact. The sheaf of local functions on M is denoted by $C^\infty(M)$; that is, an element $f \in C^\infty(M)$ is a smooth function $f : U \rightarrow \mathbb{R}$, with U an open subset of M . Similarly, if E is a vector bundle or a generalized distribution with U an open subset of M , $\Gamma(E)$ denotes the set of local sections of E . In particular, $\mathfrak{X}(M)$ and $\Omega^1(M)$ are the sets of local vector fields and 1-forms on M , respectively. The open domain of the definition of $\sigma \in \Gamma(E)$ is denoted by $\text{Dom}(\sigma)$.

The Lie group G is always assumed to be connected; \mathfrak{g} denotes the Lie algebra of G . Let $\Phi : (g, m) \in G \times M \mapsto gm = g \cdot m = \Phi_g(m) \in M$ be a smooth left action. If $\xi \in \mathfrak{g}$, then $\xi_M \in \mathfrak{X}(M)$, defined by $\xi_M(m) := (d/dt)|_{t=0} \exp(t\xi) \cdot m$, is called the *infinitesimal generator* or *fundamental vector field* defined by ξ .

A section X of TM (resp. α of T^*M) is called *G -invariant* if $\Phi_g^*X = X$ (resp. $\Phi_g^*\alpha = \alpha$) for all $g \in G$. Recall that $\Phi_g^*X := T\Phi_{g^{-1}} \circ X \circ \Phi_g$, that is, $(\Phi_g^*X)(m) = T_{gm}\Phi_{g^{-1}}X(gm)$ for all $m \in M$.

We write $TM \oplus T^*M$ for the direct sum of the vector bundles TM and T^*M and use the same notation for the sum of a tangent (a distribution in TM) and cotangent distribution (a distribution in T^*M ; see Section 2.1 for the definitions of these objects). We choose this notation because we want to distinguish these direct sums from direct sums of distributions in the same vector bundle, denoted, as usual, by \oplus .

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2. Generalities on Dirac structures and distributions

2.1. Generalized distributions and orthogonal spaces

Let E be a vector bundle over M . A *generalized distribution* Δ in E is a subset Δ of E such that for each $m \in M$, the set $\Delta(m) := \Delta \cap E(m)$ is a vector subspace of E_m . The number $\dim \Delta(m)$ is called the *rank* of Δ at $m \in M$. A point $m \in M$ is a *regular* point of the generalized distribution Δ if there exists a neighborhood U of m such that the rank of Δ is constant on U . Otherwise, m is a *singular* point of the generalized distribution.

A local *differentiable section* of Δ is a smooth section $\sigma \in \Gamma(E)$ defined on some open subset $U \subset M$ such that $\sigma(u) \in \Delta(u)$ for each $u \in U$. We denote with $\Gamma(\Delta)$ the space of the local sections of Δ . A generalized distribution is said to be *differentiable* or *smooth* if, for every point $m \in M$ and every vector $v \in \Delta(m)$, there is a differentiable section $\sigma \in \Gamma(\Delta)$ defined on an open neighborhood U of m such that $\sigma(m) = v$.

A smooth generalized distribution in the tangent bundle TM (i.e., $E = TM$) is called a *smooth tangent distribution*. A smooth generalized distribution in the cotangent bundle T^*M (i.e., $E = T^*M$) is called a *smooth cotangent distribution*. We will work most of the time with smooth generalized distributions in the *Pontryagin bundle* $P_M := TM \oplus T^*M$ which we will call *smooth generalized distributions*, for simplicity.

Generalized smooth distributions and annihilators. Assume that E is a vector bundle on M that is endowed with a smooth nondegenerate symmetric bilinear map $\langle \cdot, \cdot \rangle_E$. In the special case where E is the *Pontryagin bundle* $P_M = TM \oplus T^*M$ of a smooth manifold M , we will always consider the nondegenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by

$$\langle (u_m, \alpha_m), (v_m, \beta_m) \rangle := \beta_m(u_m) + \alpha_m(v_m) \tag{1}$$

for all $u_m, v_m \in T_mM$ and $\alpha_m, \beta_m \in T_m^*M$.

If $\Delta \subset E$ is a smooth distribution in E , its *smooth orthogonal* distribution is the smooth generalized distribution Δ^\perp in E defined by

$$\Delta^\perp(m) := \left\{ \tau(m) \left| \begin{array}{l} \tau \in \Gamma(E) \text{ with } m \in \text{Dom}(\tau) \text{ is such that for all} \\ \sigma \in \Gamma(\Delta) \text{ with } m \in \text{Dom}(\sigma), \\ \text{we have } \langle \sigma, \tau \rangle_E = 0 \text{ on } \text{Dom}(\tau) \cap \text{Dom}(\sigma) \end{array} \right. \right\}.$$

We have $\Delta \subset \Delta^{\perp\perp}$, in general strict. Note that the smooth orthogonal distribution of a smooth generalized distribution is smooth by construction. If the distribution Δ is a vector subbundle of E , then its smooth orthogonal distribution is also a vector subbundle of E . The proof of the following proposition can be found in [JRS10].

Proposition 1. *Let Δ_1 and Δ_2 be smooth subbundles of the vector bundle $(E, \langle \cdot, \cdot \rangle_E)$. Since both Δ_1 and Δ_2 have constant rank on M , their smooth orthogonal Δ_1^\perp and Δ_2^\perp are also smooth subbundles of E and equal to the pointwise orthogonal of Δ_1 and Δ_2 . The following are equivalent:*

- (i) *The intersection $\Delta_1^\perp \cap \Delta_2^\perp$ is smooth.*
- (ii) $(\Delta_1 + \Delta_2)^\perp = \Delta_1^\perp \cap \Delta_2^\perp$.
- (iii) $(\Delta_1^\perp \cap \Delta_2^\perp)^\perp = \Delta_1 + \Delta_2$.
- (iv) $\Delta_1^\perp \cap \Delta_2^\perp$ has constant rank on M .

A tangent (resp. cotangent) distribution $\mathcal{J} \subseteq TM$ (resp. $\mathcal{C} \subseteq T^*M$) can be identified with the smooth generalized distribution $\mathcal{J} \oplus \{0\}$ (resp. $\{0\} \oplus \mathcal{C}$). The smooth orthogonal distribution of $\mathcal{J} \oplus \{0\}$ in $TM \oplus T^*M$ is easily computed to be $(\mathcal{J} \oplus \{0\})^\perp = TM \oplus \mathcal{J}^\circ$, where

$$\mathcal{J}^\circ(m) = \left\{ \alpha(m) \mid \begin{array}{l} \alpha \in \Omega^1(M), m \in \text{Dom}(\alpha) \text{ and } \alpha(X) = 0 \\ \text{on } \text{Dom}(\alpha) \cap \text{Dom}(X) \text{ for all } X \in \Gamma(\mathcal{J}) \end{array} \right\} \subseteq T_m^*M$$

for all $m \in M$. This smooth cotangent distribution is called the *smooth annihilator* of \mathcal{J} . Analogously, we define the smooth annihilator $\mathcal{C}^\circ \subseteq TM$ of a cotangent distribution \mathcal{C} . Then \mathcal{C}° is a smooth tangent distribution and we have $(\{0\} \oplus \mathcal{C})^\perp = \mathcal{C}^\circ \oplus T^*M$.

The tangent distribution \mathcal{V} spanned by the fundamental vector fields of the action of a Lie group G on a manifold M will be of great importance later on. At every point $m \in M$ it is defined by

$$\mathcal{V}(m) = \{ \xi_M(m) \mid \xi \in \mathfrak{g} \}.$$

If the action is not free, the rank of the fibers of \mathcal{V} can vary on M . The smooth annihilator \mathcal{V}° of \mathcal{V} is given by

$$\mathcal{V}^\circ(m) = \{ \alpha(m) \mid \alpha \in \Omega^1(M), m \in \text{Dom}(\alpha), \text{ such that } \alpha(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g} \}.$$

We will also use the smooth generalized distribution $\mathcal{K} := \mathcal{V} \oplus \{0\}$ and its smooth orthogonal space $\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ$.

We will need the smooth codistribution \mathcal{V}_G° spanned by the G -invariant sections of \mathcal{V}° . It is shown in [JRS10] that $\mathcal{V}_G^\circ(m) = \{ \mathbf{d}f_m \mid f \in C^\infty(M)^G \}$ for all $m \in M$. Thus, \mathcal{V}_G° is spanned by the exact G -invariant sections of \mathcal{V}° .

2.2. Dirac structures

Recall that the Pontryagin bundle $\mathbf{P}_M = TM \oplus T^*M$ is endowed with a natural pairing $\langle \cdot, \cdot \rangle$ given by (1). A *Dirac structure* (see [Cou90]) on M is a Lagrangian subbundle $\mathbf{D} \subset TM \oplus T^*M$. That is, \mathbf{D} coincides with its orthogonal relative to (1) and so its fibers are necessarily $\dim M$ -dimensional.

The space $\Gamma(\mathbf{P}_M)$ of local sections of the Pontryagin bundle is also endowed with a skew-symmetric bracket, the *Courant bracket*, given by

$$\begin{aligned} [(X, \alpha), (Y, \beta)] &:= ([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} \mathbf{d}(\alpha(Y) - \beta(X))) \\ &= ([X, Y], \mathcal{L}_X \beta - \mathbf{i}_Y \mathbf{d}\alpha - \frac{1}{2} \mathbf{d}\langle (X, \alpha), (Y, \beta) \rangle) \end{aligned} \tag{2}$$

for all $(X, \alpha), (Y, \beta) \in \Gamma(\mathbf{P}_M)$ (see [Cou90]). This bracket is \mathbb{R} -bilinear (in the sense that $[a_1(X_1, \alpha_1) + a_2(X_2, \alpha_2), (Y, \beta)] = a_1[(X_1, \alpha_1), (Y, \beta)] + a_2[(X_2, \alpha_2), (Y, \beta)]$ for all $a_1, a_2 \in \mathbb{R}$ and $(X_1, \alpha_1), (X_2, \alpha_2), (Y, \beta) \in \Gamma(TM \oplus T^*M)$ on the common

domain of definition of the three sections), skew-symmetric, but does not satisfy the Jacobi identity.

The Dirac structure is *integrable* (or *closed*) if $[\Gamma(\mathbf{D}), \Gamma(\mathbf{D})] \subset \Gamma(\mathbf{D})$. Since $\langle (X, \alpha), (Y, \beta) \rangle = 0$ if $(X, \alpha), (Y, \beta) \in \Gamma(\mathbf{D})$, integrability of the Dirac structure is often expressed in the literature relative to a nonskew-symmetric bracket that differs from (2) by eliminating in the second formula the third term of the second component. This truncated expression, which satisfies the Jacobi identity if and only if the Dirac structure is integrable, but is no longer skew-symmetric, appears in the literature sometimes also as the *Courant*, *Courant–Dorfman*, or *Dorfman bracket*:

$$[(X, \alpha), (Y, \beta)] := ([X, Y], \mathcal{L}_X \beta - \mathbf{i}_Y \mathbf{d}\alpha). \quad (3)$$

If the Dirac structure (M, \mathbf{D}) is integrable, then \mathbf{D} has the structure of a Lie algebroid over M with anchor map the projection $\mathbf{P}_M \rightarrow TM$ and bracket the Courant bracket.

Maps in the Dirac category. Let (M, \mathbf{D}_M) and (N, \mathbf{D}_N) be smooth Dirac manifolds. A smooth map $\phi : (M, \mathbf{D}_M) \rightarrow (N, \mathbf{D}_N)$ is a *forward Dirac map* if for all $(Y, \beta) \in \Gamma(\mathbf{D}_N)$ there exist $X \in \mathfrak{X}(M)$ such that $X \sim_\phi Y$ and $(X, \phi^* \beta) \in \Gamma(\mathbf{D}_M)$. It is a *backward Dirac map* if for all $(X, \alpha) \in \Gamma(\mathbf{D}_M)$ there exist $(Y, \beta) \in \Gamma(\mathbf{D}_N)$ such that $X \sim_\phi Y$ and $\alpha = \phi^* \beta$.

If ϕ is a diffeomorphism, then it is easy to check that it is a backward Dirac map if and only if it is a forward Dirac map.

Let M and N be smooth manifolds and let $\phi : M \rightarrow N$ be a smooth map. Assume that N is endowed with a Dirac structure \mathbf{D}_N . The *pullback* $\phi^* \mathbf{D}_N$ of \mathbf{D}_N is the subdistribution of \mathbf{P}_M defined by

$$(\phi^* \mathbf{D}_N)(m) = \left\{ (v_m, \alpha_m) \in \mathbf{P}_M(m) \left| \begin{array}{l} \exists (w_{\phi(m)}, \beta_{\phi(m)}) \in \mathbf{D}_N(\phi(m)) \\ \text{such that } T_m \phi(v_m) = w_{\phi(m)} \\ \text{and } \alpha_m = (T_m \phi)^* \beta_{\phi(m)} \end{array} \right. \right\} \quad (4)$$

for all $m \in M$.

Each fiber of the subdistribution $\phi^* \mathbf{D}_N$ is Lagrangian, that is, $((\phi^* \mathbf{D}_N)(m))^\perp = (\phi^* \mathbf{D}_N)(m)$ in $T_m M \times T_m^* M$ for all $m \in M$. This fact is well known; we will prove it here for the sake of completeness. The inclusion $(\phi^* \mathbf{D}_N)(m) \subseteq ((\phi^* \mathbf{D}_N)(m))^\perp$ is easy. For the converse inclusion, choose $(u_m, \gamma_m) \in ((\phi^* \mathbf{D}_N)(m))^\perp \subseteq T_m M \times T_m^* M$. Since $\ker(T_m \phi : T_m M \rightarrow T_{\phi(m)} N) \times \{0_m\} \subseteq (\phi^* \mathbf{D}_N)(m)$ by definition, we have $\gamma_m(v_m) = 0$ for all $v_m \in \ker T_m \phi$. Thus, $\gamma_m \in (\ker T_m \phi)^\circ = \text{range}(T_m \phi)^*$ and there exists $\delta_{\phi(m)} \in T_{\phi(m)}^* N$ such that $\gamma_m = (T_m \phi)^* \delta_{\phi(m)}$. Then, the equality

$$\begin{aligned} 0 &= \langle (u_m, \gamma_m), (v_m, (T_m \phi)^* \beta_{\phi(m)}) \rangle \\ &= \langle (u_m, (T_m \phi)^* \delta_{\phi(m)}), (v_m, (T_m \phi)^* \beta_{\phi(m)}) \rangle \\ &= \langle (T_m \phi(u_m), \delta_{\phi(m)}), (T_m \phi(v_m), \beta_{\phi(m)}) \rangle \end{aligned}$$

holds for all pairs $(v_m, (T_m \phi)^* \beta_{\phi(m)}) \in (\phi^* \mathbf{D}_N)(m)$, that is, for all $(T_m \phi v_m, \beta_{\phi(m)}) \in \mathbf{D}_N(\phi(m))$. This leads to

$$\begin{aligned} (T_m \phi(u_m), \delta_{\phi(m)}) &\in (\mathbf{D}_N(\phi(m)) \cap (\text{range}(T_m \phi) \times T_{\phi(m)}^* N))^\perp \\ &= \mathbf{D}_N(\phi(m)) + \{0_{\phi(m)}\} \times \ker(T_m \phi)^*. \end{aligned}$$

Thus, there exists $\delta'_{\phi(m)} \in T_{\phi(m)}^*N$ such that $(T_m\phi)^*\delta'_{\phi(m)} = (T_m\phi)^*\delta_{\phi(m)} = \gamma_m$ and $(T_m\phi u_m, \delta'_m) \in \mathbf{D}_N(\phi(m))$. This shows that $(u_m, \gamma_m) \in (\phi^*\mathbf{D}_N)(m)$.

Hence, if $\phi^*\mathbf{D}_N$ is smooth, it is a Dirac structure on M such that ϕ is a backward Dirac map. The Dirac structure $\phi^*\mathbf{D}_N$ is then the *backward Dirac image* of \mathbf{D}_N under ϕ .

Symmetries of Dirac manifolds. Let G be a Lie group and $\Phi : G \times M \rightarrow M$ a smooth left action. Then G is called a *symmetry Lie group of \mathbf{D}* if for every $g \in G$ the condition $(X, \alpha) \in \Gamma(\mathbf{D})$ implies that $(\Phi_g^*X, \Phi_g^*\alpha) \in \Gamma(\mathbf{D})$. In other words, $\Phi_g : (M, \mathbf{D}) \rightarrow (M, \mathbf{D})$ is a forward and backward Dirac map for all $g \in G$. We say then that the Lie group G acts *canonically* or that the action of G on M is *Dirac*.

Let \mathfrak{g} be a Lie algebra and let $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ be a smooth left Lie algebra action, that is, the map

$$(x, \xi) \in M \times \mathfrak{g} \mapsto \xi_M(x) \in TM$$

is smooth and $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a Lie algebra antihomomorphism. The Lie algebra \mathfrak{g} is said to be a *symmetry Lie algebra of \mathbf{D}* if for every $\xi \in \mathfrak{g}$ the condition $(X, \alpha) \in \Gamma(\mathbf{D})$ implies that $(\mathcal{L}_{\xi_M}X, \mathcal{L}_{\xi_M}\alpha) \in \Gamma(\mathbf{D})$. Of course, if \mathfrak{g} is the Lie algebra of G and $\xi \mapsto \xi_M$ the Lie algebra antihomomorphism, then if G is a symmetry Lie group of \mathbf{D} it follows that \mathfrak{g} is a symmetry Lie algebra of \mathbf{D} .

Regular reduction of Dirac manifolds. Let (M, \mathbf{D}) be a smooth Dirac manifold with a proper smooth Dirac action of a Lie group G on it, such that all isotropy subgroups of the action are conjugated. Then the space $\bar{M} := M/G$ of orbits of the action is a smooth manifold and the orbit map $\pi : M \rightarrow \bar{M}$ is a smooth surjective submersion. We have the following theorem (see [JRZ10]).

Theorem 2. *Let G be a connected Lie group acting in a proper way on the manifold M , such that all isotropy subgroups are conjugated. Assume that $\mathbf{D} \cap \mathcal{K}^\perp$ has constant rank on M , where \mathcal{K}^\perp is defined as the direct sum $TM \oplus \mathcal{V}^\circ$. Then the Dirac structure \mathbf{D} on M induces a Dirac structure $\bar{\mathbf{D}}$ on the quotient $\bar{M} = M/G$ given by*

$$\bar{\mathbf{D}}(\bar{m}) = \left\{ (\bar{X}(\bar{m}), \bar{\alpha}(\bar{m})) \in T_{\bar{m}}\bar{M} \times T_{\bar{m}}^*\bar{M} \mid \begin{array}{l} \exists X \in \mathfrak{X}(M) \text{ such that } X \sim_\pi \bar{X} \\ \text{and } (X, \pi^*\bar{\alpha}) \in \Gamma(\mathbf{D}) \end{array} \right\} \quad (5)$$

for all $\bar{m} \in \bar{M}$. If (M, \mathbf{D}) is integrable, then $(\bar{M}, \bar{\mathbf{D}})$ is also integrable.

The Dirac structure $\bar{\mathbf{D}}$ is then the *forward Dirac image* $\pi(\mathbf{D})$ of \mathbf{D} under π .

3. Proper actions and orbit-type manifolds

3.1. Orbits of a proper action

In this section we consider a left smooth proper action

$$\begin{aligned} \Phi : G \times M &\rightarrow M, \\ (g, m) &\mapsto \Phi(g, m) \equiv \Phi_g(m) \equiv gm \equiv g \cdot m, \end{aligned} \quad (6)$$

of a Lie group G on a manifold M . Let $\pi : M \rightarrow \bar{M}$ be the natural projection on the orbit space.

For each closed Lie subgroup H of G we define the *isotropy-type* set

$$M_H = \{m \in M \mid G_m = H\},$$

where $G_m = \{g \in G \mid gm = m\}$ is the isotropy subgroup of $m \in M$. Since the action is proper, all isotropy subgroups are compact. The sets M_H , where H ranges over the set of closed Lie subgroups of G for which M_H is nonempty, form a partition of M and therefore they are the equivalence classes of an equivalence relation in M . Define the normalizer of H in G ,

$$N(H) = \{g \in G \mid gHg^{-1} = H\}$$

which is a closed Lie subgroup of G . Since H is a normal subgroup of $N(H)$, the quotient $N(H)/H$ is a Lie group. If $m \in M_H$, we have $G_m = H$ and $G_{gm} = gHg^{-1}$ for all $g \in G$. Therefore, $gm \in M_H$ if and only if $g \in N(H)$. The action of G on M restricts to an action of $N(H)$ on M_H which induces a free and proper action of $N(H)/H$ on M_H .

Define the *orbit-type* set

$$M_{(H)} = \{m \in M \mid G_m \text{ is conjugate to } H\}. \quad (7)$$

Then

$$M_{(H)} = \{gm \mid g \in G, m \in M_H\} = \pi^{-1}(\pi(M_H)).$$

Connected components of M_H and $M_{(H)}$ are embedded submanifolds of M ; therefore, M_H is called an *isotropy-type manifold* and $M_{(H)}$ an *orbit-type manifold*. Moreover,

$$\pi(M_{(H)}) = \{gm \mid m \in M_H\}/G = M_H/N(H) = M_H/(N(H)/H).$$

Since the action of $N(H)/H$ on M_H is free and proper, it follows that the quotient $M_H/(N(H)/H)$ is a quotient manifold of M_H . Hence, $\pi(M_{(H)})$ is a manifold contained in the orbit space $\bar{M} = M/G$.

For a connected component Q of M_H , we denote by N_Q the subgroup of $N(H)$ leaving Q invariant, i.e.,

$$N_Q := \{g \in G \mid g \cdot q \in Q \text{ for all } q \in Q\}. \quad (8)$$

Thus N_Q is a union of connected components of $N(H)$ and is equal to $N(H)$ if $N(H)$ is connected.

Partitions of the orbit space $\bar{M} = M/G$ by connected components of $\pi(M_{(H)})$ is a decomposition of the differential space \bar{M} . The corresponding stratification of \bar{M} is called the orbit-type stratification of the orbit space (see [DK00], [Pf01]). It is a minimal stratification in the partial order discussed above (see [Bie75]). This implies that the strata $\pi(M_{(H)})$ of the orbit-type stratification are accessible sets of the family of all vector fields on \bar{M} (see [LS08]).

The smooth distribution $\mathcal{J}_G \subseteq TM$ spanned by the G -invariant vector fields, i.e.,

$$\mathcal{J}_G(m) := \{X(m) \mid X \in \mathfrak{X}(M)^G, \text{ such that } m \in \text{Dom}(X)\} \quad \text{for all } m \in M, \quad (9)$$

is shown in [OR04] to be completely integrable in the sense of Stefan and Sussmann. Its leaves are the connected components of the isotropy-types.

The smooth distribution $\mathcal{J} \subseteq TM$ is defined as the span of the descending vector fields, that is, the vector fields $X \in \mathfrak{X}(M)$ satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$. A descending vector field X can be written as a sum $X = X^G + X^\mathcal{V}$, with $X^G \in \mathfrak{X}(M)^G$ and $X^\mathcal{V} \in \Gamma(\mathcal{V})$. Therefore,

$$\mathcal{J} := \mathcal{J}_G + \mathcal{V}, \quad (10)$$

\mathcal{J} is completely integrable in the sense of Stefan and Sussmann, and its integral leaves are the connected components of the orbit-types. For the proofs of these statements see [JRS10] and [OR04].

A local section (X, α) of $TM \oplus \mathcal{V}^\circ = \mathcal{X}^\perp$ satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ and $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ is called a *descending section* of \mathbb{P}_M .

3.2. Tube theorem and G -invariant average

If the action of the Lie group G on M is proper, we can find for each point $m \in M$ a G -invariant neighborhood of m such that the action can be described easily on this neighborhood. The proof of the following theorem can be found, for example, in [OR04].

Theorem 3 (Tube Theorem). *Let M be a manifold and G a Lie group acting properly on M . For a given point $m \in M$ denote $H := G_m$. Then there exists a G -invariant open neighborhood U of the orbit $G \cdot m$, called tube at m , and a G -equivariant diffeomorphism $G \times_H B \xrightarrow{\sim} U$. The set B is an open H -invariant neighborhood of 0 in an H -representation space H -equivariantly isomorphic to $T_m M / T_m(G \cdot m)$. The H -representation on $T_m M / T_m(G \cdot m)$ is given by $h \cdot (v + T_m(G \cdot m)) := T_m \Phi_h(v) + T_m(G \cdot m)$, $h \in H$, $v \in T_m M$. The smooth manifold $G \times_H B$ is the quotient of the smooth free and proper (twisted) action Ψ of H on $G \times B$ given by $\Psi(h, (g, b)) := (gh^{-1}, h \cdot b)$, $g \in G$, $h \in H$, $b \in B$. The G -action on $G \times_H B$ is given by $k \cdot [g, b] := [kg, b]_H$, where $k, g \in G$, $b \in B$, and $[g, b]_H \in G \times_H B$ is the equivalence class (i.e., H -orbit) of (g, b) .*

G -invariant average. Let $m \in M$ and $H := G_m$. If the action of G on M is proper, the isotropy subgroup H of m is a compact Lie subgroup of G . Hence, there exists a Haar measure λ^H on H , that is, a G -invariant measure on H satisfying $\int_H d\lambda^H = 1$ (see, e.g., [DK00]). Left G -invariance of λ^H is equivalent to right G -invariance of λ^H and $R_h^* d\lambda^H = d\lambda^H = L_h^* d\lambda^H$ for all $h \in H$, where $L_h : H \rightarrow H$ (resp. $R_h : H \rightarrow H$) denotes left (resp. right) translation by h on H .

Let $X \in \mathfrak{X}(M)$ be defined on the tube U at $m \in M$ for the proper action of the Lie group G on M . As in the Tube Theorem, we write the points of U as equivalence classes $[g, b]_H$ with $g \in G$ and $b \in B$. Recall that for all $h \in H$

we have $[g, b]_H = [gh^{-1}, h \cdot b]_H$. Furthermore, the action of G on U is given by $\Phi_{g'}([g, b]_H) = [g'b, b]_H$, for $g' \in G$. Define $X_G \in \mathfrak{X}(G \times_H B)$ by

$$\begin{aligned} X_G([g, b]_H) &:= \left(\Phi_{g^{-1}}^* \left(\int_H \Phi_h^* X d\lambda^H \right) \right) ([g, b]_H) \\ &= T_{[e, b]_H} \Phi_g \left(\int_H (T_{[h, b]_H} \Phi_{h^{-1}} X([h, b]_H)) d\lambda^H \right) \end{aligned}$$

for each point $m' = [g, b]_H \in U$. This definition doesn't depend on the choice of the representative $[g, b]_H$ for the point m' . The vector field X_G is an element of $\mathfrak{X}(M)^G$ (see [JRS10]) and is called the G -invariant average of the vector field X . Note that X_G is, in general, not equal to X (at any point); it can even vanish.

Similarly, if $\alpha \in \Omega^1(M)$, define the G -invariant average $\alpha_G \in \Omega^1(M)^G$ of α by

$$\alpha_G([g, b]_H) := \left(\Phi_{g^{-1}}^* \left(\int_H \Phi_h^* \alpha d\lambda^H \right) \right) ([g, b]_H) \quad (11)$$

$$\begin{aligned} &= \left(\int_H \Phi_h^* \alpha d\lambda^H \right)_{[e, b]_H} \circ T_{[g, b]_H} \Phi_{g^{-1}} \\ &= \left(\int_H (\alpha([h, b]_H) \circ T_{[e, b]_H} \Phi_h) d\lambda^H \right) \circ T_{[g, b]_H} \Phi_{g^{-1}} \quad (12) \end{aligned}$$

for each point $m' = [g, b]_H \in U$.

If (X, α) is a section of a G -invariant generalized distribution \mathcal{D} , the section (X_G, α_G) is a G -invariant section of \mathcal{D} .

If f is a smooth function defined in the tube U of the action of G at $m \in M$, define its G -invariant average f_G by

$$f_G([g, b]_H) := \int_{h \in H} f([h, b]_H) d\lambda^H.$$

Because the action of G on M is proper, there exists a G -invariant Riemannian metric ϱ on M (see [DK00]). Let Q be a connected component of an isotropy-type manifold M_H , H compact subgroup of G . Then Q is an embedded submanifold of M . We write $TM|_Q = TQ \oplus TQ^\varrho$ with TQ^ϱ the subbundle of $TM|_Q$ orthogonal to TQ (seen as a subbundle of $TM|_Q$) relative to ϱ .

Lemma 4. *The ϱ -orthogonal projection of $v_m \in T_m M$ onto $T_m Q$ is equal to its G -invariant average $\int_H T_m \Phi_h(v_m) d\lambda^H$ at $m \in Q$. The composition of $\alpha_m \in T_m^* Q$ with $\text{pr}_{T_m Q}$ is equal to the average $\int_H (T_m \Phi_h)^*(\alpha_m) d\lambda^H$ at $m \in Q$.*

Proof. Recall that \mathcal{T}_G is (completely integrable in the sense of Stefan and Sussmann) the tangent distribution on M that is spanned by the G -invariant vector fields on M and that Q is a leaf of \mathcal{T}_G ; see (9) and the considerations following it.

The vector v_m can be written as an orthogonal sum $v_m = v_m^\top + v_m^\varrho$ with $v_m^\top \in T_m Q$ and $v_m^\varrho \in T_m Q^\varrho$. Then we find smooth (without loss of generality global) vector fields $X^\top \in \Gamma(\mathcal{T}_G)$ and $X^\varrho \in \mathfrak{X}(M)$ such that $X^\varrho|_Q \in \Gamma(TQ^\varrho)$ ($X^\top|_Q \in \Gamma(TQ)$) by the properties of \mathcal{T}_G , and with values $X^\top(m) = v_m^\top$ and $X^\varrho(m) = v_m^\varrho$.

Consider the G -invariant averages X_G^\top and X_G^ϱ of X^\top and X^ϱ in a tube centered at m . Then X_G^ϱ is G -invariant, $X_G^\varrho \in \Gamma(\mathcal{J}_G)$, and we get $X_G^\varrho|_Q \in \Gamma(TQ)$. But since the metric ϱ is G -invariant, the orthogonal space TQ^ϱ is G -invariant and the average $X_G^\varrho|_Q$ remains a section of TQ^ϱ . Hence, it must be the zero section. In the same manner, we have $X_G^\top \in \mathfrak{X}(M)^G$ and thus, $X_G^\top|_Q$ remains tangent to Q . In particular, we get at the point $m = [e, 0]_H$,

$$X_G^\top(m) = \int_H T_m \Phi_{h^{-1}} X^\top(m) d\lambda^H = \int_H T_m \Phi_{h^{-1}}(v_m^\top) d\lambda^H = \int_H v_m^\top d\lambda^H = v_m^\top.$$

The third equality is proved in the following way. Since v_m^\top is tangent to Q , there exists a curve $c : (-\varepsilon, \varepsilon) \rightarrow Q \subseteq M_H$ such that $c(0) = m$ and $\dot{c}(0) = v_m^\top$. We have then $T_m \Phi_{h^{-1}}(v_m^\top) = (d/dt)|_{t=0}(\Phi_{h^{-1}} \circ c)(0) = \dot{c}(0) = v_m^\top$ for all $h \in H$. This leads to $\int_H T_m \Phi_h(v_m) d\lambda^H = X_G^\top(m) + X_G^\varrho(m) = v_m^\top + 0 = \text{pr}_{T_m Q}(v_m)$.

Choose now $\alpha_m \in T_m^* M$. Then the G -invariant average of α_m at m is equal to

$$\alpha_m = \int_H \alpha_m \circ T_m \Phi_h d\lambda^H = \alpha_m \circ \int_H T_m \Phi_h d\lambda^H = \alpha_m \circ \text{pr}_{T_m Q},$$

as follows from the first statement. □

4. Induced Dirac structures on the isotropy-type manifolds

Let (M, D) be a smooth Dirac manifold with a smooth proper Dirac action of a Lie group G on it. Let Q be a connected component of an isotropy-type M_H , for a compact subgroup $H \subseteq G$. Then Q is an embedded submanifold of M . We denote by $\iota_Q : Q \hookrightarrow M$ the inclusion.

4.1. Dirac structures on connected components of isotropy-type submanifolds

We show in this subsection that D induces a Dirac structure D_Q on Q . Then, we will study in the next subsection the induced action of N_Q (defined by (8)) on Q .

Theorem 5. *Define $D_Q \subseteq P_Q$ by*

$$D_Q(q) = \left\{ (\tilde{v}_q, \tilde{\alpha}_q) \in P_Q(q) \mid \begin{array}{l} \exists (v_q, \alpha_q) \in D(q) \text{ such that} \\ T_q \iota_Q \tilde{v}_q = v_q \text{ and } (T_q \iota_Q)^* \alpha_q = \tilde{\alpha}_q \end{array} \right\} \quad (13)$$

for all $q \in Q$, i.e., D_Q is the backward Dirac image of D under ι_Q . Then D_Q is a Dirac structure on Q . If (M, D) is integrable, then (Q, D_Q) is integrable.

Proof. Since D_Q is, by definition, equal to $\iota_Q^* D$ (see (4) and the considerations following this equation), we have only to check that (13) defines a smooth generalized distribution in P_Q . Choose $q \in Q$ and $(\tilde{v}_q, \tilde{\alpha}_q) \in D_Q(q)$. For simplicity, we also write q for $\iota_Q(q) \in M$. Then we find $(v_q, \alpha_q) \in D(q)$ such that $T_q \iota_Q \tilde{v}_q = v_q \in T_q Q \subseteq T_q M$ and $(T_q \iota_Q)^* \alpha_q = \tilde{\alpha}_q$. Since D is a smooth vector bundle on M , there exists $(X, \alpha) \in \Gamma(D)$ with $q \in \text{Dom}(X, \alpha)$ such that $(X, \alpha)(q) = (v_q, \alpha_q)$. Consider the G -invariant average (X_G, α_G) of the pair (X, α) in a tube in M centered at q . Since D is invariant under the action of G on M , we

have $(X_G, \alpha_G) \in \Gamma(\mathbb{D})$ and since it is G -invariant, $X_G|_Q$ is tangent to the connected component Q of the isotropy-type manifold M_H . This shows that $X_G \in \mathfrak{X}(M)$ is such that there exists $\tilde{X} \in \mathfrak{X}(Q)$ with $\tilde{X} \sim_{\iota_Q} X_G$. By definition of \mathbb{D}_Q , the pair $(\tilde{X}, \iota_Q^* \alpha_G)$ is a section of \mathbb{D}_Q . Furthermore, we have

$$\begin{aligned} X_G(q) &= \int_H (\Phi_h^* X)(q) d\lambda^H = \int_H T_q \Phi_{h^{-1}} X(q) d\lambda^H \\ &= \int_H T_q \Phi_{h^{-1}}(v_q) d\lambda^H = \text{pr}_{T_q Q}(v_q) = v_q \end{aligned}$$

by Lemma 4 since $v_q \in T_q Q$. This leads to $\tilde{X}(q) = \tilde{v}_q$. In the same manner, $\alpha_G(q) = \alpha_q \circ \text{pr}_{T_q Q} = \alpha_q$ since $\alpha_q \in ((T_q Q)^\circ)^\circ$ and thus $(\iota_Q^* \alpha_G)(q) = \alpha_G(q) \circ T_q \iota_Q = \alpha_q \circ T_q \iota_Q = \tilde{\alpha}_q$. Hence, we have found a smooth section $(\tilde{X}, \iota_Q^* \alpha_G)$ of \mathbb{D}_Q whose value at q is $(\tilde{v}_q, \tilde{\alpha}_q)$.

Since $\iota_Q : (Q, \mathbb{D}_Q) \rightarrow (M, \mathbb{D})$ is a backward Dirac map, we know using for instance Lemma 2.2 in [SX08] that \mathbb{D}_Q is integrable if \mathbb{D} is integrable. \square

Remark 1. In the situation of the previous theorem one can show, with the same methods as in the proof of the smoothness of \mathbb{D}_Q , that the intersection

$$\mathbb{D} \cap (TQ \oplus (TQ^\circ)^\circ) \tag{14}$$

is smooth. It has hence constant rank on Q by Proposition 1. This intersection is then a Dirac structure in $TQ \oplus (TQ^\circ)^\circ$. The Dirac structure \mathbb{D}_Q can be seen as the pullback of this intersection via the identification of \mathbb{P}_Q with $TQ \oplus (TQ^\circ)^\circ$.

Assume that N is an embedded submanifold of a manifold M endowed with a Dirac structure \mathbb{D}_M . It was already shown in [Cou90] that if $\mathbb{D}_M \cap (TN \oplus T^*M|_N)$ has constant rank, then there is an induced Dirac structure on N defined as in (13) and such that the inclusion map $N \hookrightarrow M$ is a backward Dirac map. The hypothesis in [Cou90] ensures that the bundle defined by (13) is smooth. In the present situation we cannot use this known result of [Cou90] for the proof of Theorem 5. On the other hand, averaging techniques prove that the bundle defined by (13) is smooth. We could also have shown first that (14) is smooth (and has hence constant rank) and then that its pullback to Q is a Dirac structure on Q .

4.2. Induced Dirac structures on the quotients

Let G act on the Dirac manifold (M, \mathbb{D}) properly and canonically. Let Q be a connected component of the orbit-type manifold M_H , for H a compact subgroup of G . In Theorem 5 we have shown that there is an induced Dirac structure defined by (13) on Q such that $\iota_Q : (Q, \mathbb{D}_Q) \rightarrow (M, \mathbb{D})$ is a backward Dirac map. We will show here that the action of N_Q on (Q, \mathbb{D}_Q) is a proper Dirac action that satisfies the conditions for regular reduction.

From the proof of Theorem 5, we can see that each pair $(\tilde{v}_q, \tilde{\alpha}_q) \in \mathbb{D}_Q(q)$ corresponds to a unique pair $(v_q, \alpha_q) \in \mathbb{D}(q) \cap (T_q Q \cap (T_q Q^\circ)^\circ)$ such that $T_q \iota_Q \tilde{v}_q = v_q$ and $\tilde{\alpha}_q = (T_q \iota_Q)^* \alpha_q$. The converse is also true by definition of $\mathbb{D}_Q(q)$. Thus, the map

$$I_q : \mathbb{D}_Q(q) \rightarrow \mathbb{D}(q) \cap (T_q Q \times (T_q Q^\circ)^\circ) \quad (\subseteq T_q M \times T_q^* M)$$

sending each $(\tilde{v}_q, \tilde{\alpha}_q)$ to the corresponding (v_q, α_q) is an isomorphism of vector spaces for all $q \in Q$.

We will use the maps $I_q, q \in Q$, as a technical tool in the proof of the following lemma. Recall that \mathcal{J} is the smooth tangent distribution defined as the sum of \mathcal{J}_G and \mathcal{V} (see (9) and (10)).

Lemma 6. *Let N_Q be the subgroup of $N(H)$ leaving Q invariant (see (8)), denote by \mathcal{V}_Q the vertical distribution of the induced proper action of N_Q on Q , and set $\mathcal{K}_Q := \mathcal{V}_Q \oplus \{0\} \subseteq \mathbb{P}_Q$. If $\mathbb{D} \cap (\mathcal{J} \oplus \mathcal{V}_G^\circ)$ is smooth, then the intersection $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$ has constant rank on Q .*

Proof. Consider the restriction of the map I_q defined above to the set $\mathbb{D}_Q(q) \cap \mathcal{K}_Q^\perp(q)$. We show that if $\tilde{\alpha}_q \in (\mathcal{V}_Q)^\circ(q)$, then $\alpha_q \in \mathcal{V}_G^\circ(q)$. Since the action of N_Q on Q is proper with fixed isotropy H , the vertical space \mathcal{V}_Q is a smooth vector bundle on Q , and we have $(\mathcal{V}_Q)^\circ(q) = (\mathcal{V}_Q(q))^\circ = (T_q(N_Q \cdot q))^\circ$. But since the action of H on Q is trivial, we also have $((T_q(N_Q \cdot q))^\circ)^H = (T_q(N_Q \cdot q))^\circ$. Using Theorem 2.5.10 in [OR04] we then find

$$(\mathcal{V}_Q)^\circ(q) = ((T_q(N_Q \cdot q))^\circ)^H = \{\mathbf{d}f(q) \mid f \in C^\infty(Q)^{N_Q}\}.$$

If $\tilde{\alpha}_q \in (\mathcal{V}_Q)^\circ(q)$, we hence find a smooth N_Q -invariant function $\tilde{f} \in C^\infty(Q)^{N_Q}$ such that $\mathbf{d}\tilde{f}_q = \tilde{\alpha}_q$. Since Q is an embedded submanifold of M , there then exists a smooth function $f \in C^\infty(M)$ such that $\iota_Q^* f = \tilde{f}$. Consider the G -invariant average F of f at q . Since \tilde{f} was N_Q -invariant, we still have $\iota_Q^* F = \tilde{f}$. Therefore, $\mathbf{d}F_q \circ T_q \iota_Q = (\iota_Q^* \mathbf{d}F)(q) = \mathbf{d}\tilde{f}_q$ and $\mathbf{d}F \in \Gamma(\mathcal{V}^\circ)^G$, that is, $\mathbf{d}F_q \in \mathcal{V}_G^\circ(q)$. Also, since $\mathbf{d}F$ is G -invariant, we have $\mathbf{d}F_q = \mathbf{d}F_q \circ \text{pr}_{T_q Q}$ by Lemma 4 and thus $\mathbf{d}F_q \in (T_q Q^\ell)^\circ$. The covector $\mathbf{d}F_q$ is hence the unique element of $(T_q Q^\ell)^\circ$ satisfying $\tilde{\alpha}_q = \mathbf{d}F_q \circ T_q \iota_Q$. Thus $\mathbf{d}F_q$ is equal to the covector α_q and we have shown that $\alpha_q \in \mathcal{V}_G^\circ(q)$. Hence, we get

$$I_q|_{\mathbb{D}_Q(q) \cap \mathcal{K}_Q^\perp(q)} : \mathbb{D}_Q(q) \cap \mathcal{K}_Q^\perp(q) \rightarrow \mathbb{D}(q) \cap (T_q Q \times \mathcal{V}_G^\circ(q)).$$

This map is obviously surjective since $\mathcal{V}_G^\circ|_Q \subseteq (TQ^\ell)^\circ$ because \mathcal{V}_G° is spanned by G -invariant sections.

Since $\mathbb{D} \cap (\mathcal{J} \oplus \mathcal{V}_G^\circ)$ is smooth by hypothesis, it follows by G -invariant averaging that $\mathbb{D} \cap (\mathcal{J}_G \oplus \mathcal{V}_G^\circ)$ is also smooth (see [JR10]).

We use this to show that $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$ is smooth. Choose $q \in Q$ and $(v_q, \alpha_q) \in \mathbb{D}_Q(q) \cap \mathcal{K}_Q^\perp(q)$. Then $I_q(v_q, \alpha_q) \in \mathbb{D}(q) \cap (T_q Q \oplus \mathcal{V}_G^\circ(q))$. Since $\mathbb{D} \cap (TQ \oplus \mathcal{V}_G^\circ|_Q) = \mathbb{D} \cap (\mathcal{J}_G \oplus \mathcal{V}_G^\circ)|_Q$, we find a smooth section $(X, \alpha) \in \Gamma(\mathbb{D} \cap (\mathcal{J}_G \oplus \mathcal{V}_G^\circ))$ defined on a neighborhood U of q in M such that $(X, \alpha)(q) = I_q(v_q, \alpha_q)$. Since (X, α) is a section of \mathbb{D} restricting to a section of $\mathbb{D} \cap (TQ \oplus \mathcal{V}_G^\circ|_Q) \subseteq \mathbb{D} \cap (TQ \oplus (TQ^\ell)^\circ)$ on Q we get, by the proof of Theorem 5, the existence of a smooth section $(\tilde{X}, \tilde{\alpha})$ of \mathbb{D}_Q such that $\tilde{X} \sim_{\iota_Q} X$ and $\tilde{\alpha} = \iota_Q^* \alpha$. But for all $q' \in Q \cap U$, we get from the considerations above that $(\tilde{X}, \tilde{\alpha})(q') = I_{q'}^{-1}((X, \alpha)(q')) \in \mathbb{D}_Q(q') \cap \mathcal{K}_Q^\perp(q')$. Thus, $(\tilde{X}, \tilde{\alpha})$ is a smooth section of $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$, taking the value $(\tilde{X}, \tilde{\alpha})(q) = (v_q, \alpha_q)$ at q .

Since $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$ is smooth and \mathbb{D}_Q and \mathcal{K}_Q^\perp are the smooth orthogonal bundles of the vector subbundles \mathbb{D}_Q and \mathcal{K}_Q of $(\mathbb{P}_Q, \langle \cdot, \cdot \rangle_{\mathbb{P}_Q})$, respectively, we can conclude using Proposition 1 that $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$ has constant rank on Q . \square

Lemma 7. *Let (M, \mathbb{D}) be a smooth Dirac manifold with a proper smooth Dirac action of a Lie group G on it such that the intersection $\mathbb{D} \cap (\mathcal{J} \oplus \mathcal{V}_G^\circ)$ is smooth. The Dirac structure \mathbb{D}_Q defined by \mathbb{D} on Q , as in Theorem 5, is invariant under the induced action of N_Q (see(8)) on Q and has the property that $\mathbb{D}_Q \cap \mathcal{K}_Q^\perp$ has constant rank on Q .*

Proof. Recall the construction of \mathbb{D}_Q . Since \mathbb{D} is G -invariant, it is N_Q -invariant. The connected component Q of the isotropy-type manifold M_H is an accessible set of the family of the G -invariant vector fields on M . Since ϱ is also G -invariant, the spaces TQ and $(TQ^e)^\circ$ are N_Q -invariant and so the induced action of N_Q on (Q, \mathbb{D}_Q) is canonical.

The second claim has been shown in Lemma 6. □

Theorem 8. *Let (M, \mathbb{D}) be a smooth Dirac manifold with a proper smooth Dirac action of a Lie group G on it such that the intersection $\mathbb{D} \cap (\mathcal{J} \oplus \mathcal{V}_G^\circ)$ is smooth. Let \mathbb{D}_Q be the induced Dirac structure on the connected component Q of the isotropy-type manifold M_H and $q_Q : Q \rightarrow Q/N_Q$ the projection (a smooth surjective submersion). The forward Dirac image $q_Q(\mathbb{D}_Q)$ (as in Theorem 2) is a Dirac structure on Q/N_Q . If \mathbb{D} is integrable, then $q_Q(\mathbb{D}_Q)$ is integrable.*

Proof. By Lemma 7 all the hypotheses for regular reduction are satisfied for the smooth proper Dirac action of N_Q on (Q, \mathbb{D}_Q) with fixed isotropies. Thus the quotient space Q/N_Q inherits a smooth Dirac structure defined by the forward Dirac image $q_Q(\mathbb{D}_Q)$ of \mathbb{D}_Q on Q/N_Q (see the paragraph about regular Dirac reduction at the end of Section 2). □

4.3. Comparison with the Dirac strata of the reduced space $(\bar{M}, \bar{\mathbb{D}})$

We want to compare the Dirac manifolds $(Q/N_Q, q_Q(\mathbb{D}_Q))$ obtained above with the Dirac manifolds induced by the singular Dirac reduction method in [JRS10].

We present a short review of the Dirac reduction methods in [JRS10]. Let (M, \mathbb{D}) be a smooth Dirac manifold acted upon in a smooth proper and Dirac manner by a Lie group G such that the intersection $\mathbb{D} \cap (\mathcal{J} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections (recall that \mathcal{J} is defined by (10)). Let $\pi : M \rightarrow M/G$ be the projection.

Consider the subset \mathcal{D}^G of $\Gamma(\mathbb{D})$ defined by

$$\mathcal{D}^G := \{(X, \alpha) \in \Gamma(\mathbb{D}) \mid \alpha \in \Gamma(\mathcal{V}^\circ)^G \text{ and } [X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})\},$$

that is, the set of the descending sections of \mathbb{D} .

Each vector field X satisfying $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$ pushes-forward to a “vector field” \bar{X} on \bar{M} . (Since we will not need these objects in the rest of the paper, we will not give more details about what we call the “vector fields” and “1-forms” on the stratified space $\bar{M} = M/G$ and refer to [JRS10] for more information.) For each stratum \bar{P} of \bar{M} , the restriction of \bar{X} to points of \bar{P} is a vector field $X_{\bar{P}}$ on \bar{P} . On the other hand, if $(X, \alpha) \in \mathcal{D}^G$, then we have $\alpha \in \Gamma(\mathcal{V}^\circ)^G$ which pushes-forward to the 1-form $\bar{\alpha} := \pi_*\alpha$ such that, for every $\bar{Y} \in \mathfrak{X}(\bar{M})$ and every vector field $Y \in \mathfrak{X}(M)$ satisfying $Y \sim_\pi \bar{Y}$, we have

$$\pi^*(\bar{\alpha}(\bar{Y})) = \alpha(Y).$$

Moreover, for each stratum \bar{P} of \bar{M} , the restriction of $\bar{\alpha}$ to points of \bar{P} defines a 1-form $\alpha_{\bar{P}}$ on \bar{P} . Let

$$\bar{\mathcal{D}} = \{(\bar{X}, \bar{\alpha}) \mid (X, \alpha) \in \mathcal{D}^G\}$$

and for each stratum \bar{P} of \bar{M} , set

$$\mathcal{D}_{\bar{P}} = \{(X_{\bar{P}}, \alpha_{\bar{P}}) \mid (\bar{X}, \bar{\alpha}) \in \bar{\mathcal{D}}\}.$$

Define the smooth distribution $D_{\bar{P}}$ on \bar{P} by

$$D_{\bar{P}}(\bar{p}) = \{(X_{\bar{P}}(\bar{p}), \alpha_{\bar{P}}(\bar{p})) \in T_{\bar{p}}\bar{P} \times T_{\bar{p}}^*\bar{P} \mid (X_{\bar{P}}, \alpha_{\bar{P}}) \in \mathcal{D}_{\bar{P}}\} \tag{15}$$

for all $\bar{p} \in \bar{P}$. Note that $\Gamma(D_{\bar{P}}) = \mathcal{D}_{\bar{P}}$. We have the following theorem (see [JRZ10] for the regular case).

Theorem 9. *Let (M, D) be a Dirac manifold with a proper Dirac action of a connected Lie group G on it. Let \bar{P} be a stratum of the quotient space \bar{M} . If $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections, then $D_{\bar{P}}$ defined in (15) is a Dirac structure on \bar{P} . If (M, D) is integrable, then $(\bar{P}, D_{\bar{P}})$ is integrable.*

Note that if $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections, it is smooth and all the hypotheses of Theorem 8 are satisfied. We can thus compare the reduction methods in Theorems 9 and 8.

We have the following theorem, comparing the Dirac manifolds $(Q/N_Q, q_Q(D_Q))$ and $(\bar{P}, D_{\bar{P}})$ if Q is a connected component of M_H , H a compact subgroup of G , and P is the connected component of $M_{(H)}$ containing Q .

Theorem 10. *Let (M, D) be a smooth Dirac manifold with a proper smooth Dirac action of a Lie group G on it, such that the intersection $D \cap (\mathcal{T} \oplus \mathcal{V}_G^\circ)$ is spanned by its descending sections.*

Let Q be a connected component of M_H , H a compact subgroup of G , and P the connected component of $M_{(H)}$ containing Q . Consider the Dirac manifolds $(Q/N_Q, q_Q(D_Q))$ as in Theorem 8 and $(\bar{P}, D_{\bar{P}})$ as in Theorem 9, and define the map $\Phi : Q/N_Q \rightarrow \bar{P}$ by $(\Phi \circ q_Q)(q) = \pi(q)$ for all $q \in Q \subseteq P$. Then the map Φ is a diffeomorphism preserving the Dirac structure, i.e., it is a forward and, equivalently, a backward Dirac map.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \iota_Q & & \\
 & & \curvearrowright & & \\
 Q & \xrightarrow{\iota_{Q,P}} & P & \xrightarrow{\iota_P} & M \\
 q_Q \downarrow & & \pi|_P \downarrow & & \pi \downarrow \\
 Q/N_Q & \xrightarrow{\Phi} & \bar{P} & \xrightarrow{\iota_{\bar{P}}} & \bar{M}
 \end{array}$$

where $\iota_{Q,P}, \iota_P, \iota_{\bar{P}}$ are inclusions. We show that Φ is bijective. Let $q_Q(m_1), q_Q(m_2) \in Q/N_Q$ be such that $\pi(m_1) = \Phi(q_Q(m_1)) = \Phi(q_Q(m_2)) = \pi(m_2)$. Then there exists $g \in G$ such that $gm_1 = m_2$. Since $m_1, m_2 \in Q$, we have $gHg^{-1} = H$

and thus $g \in N(H)$. However, g maps Q onto a connected component of M_H containing $m_2 \in Q$, so it follows that $g \in N_Q$. We then have $q_Q(m_1) = q_Q(m_2)$ thereby showing that Φ is injective. Choose now $\pi(m) \in \bar{P}$ with $m \in P$. Since $P \subseteq G \cdot Q$, there exists $g \in G$ and $m' \in Q \subseteq P$ such that $gm' = m$. We get $(\Phi \circ q_Q)(m') = \pi(m') = \pi(m)$ and hence Φ is surjective.

To show that Φ is a diffeomorphism, choose a smooth function $\bar{f} \in C^\infty(\bar{P})$. Then the G_P -invariant function $(\pi|_P)^* \bar{f} = f$ satisfies $f \in C^\infty(P)$, where G_P is the Lie subgroup of G that leaves P invariant. Since Q is a smooth submanifold of P , the function $\iota_{Q,P}^* f$ is an element of $C^\infty(Q)$. But since $f \in C^\infty(P)^{G_P}$, we have $\iota_{Q,P}^* f \in C^\infty(Q)^{N_Q}$ and there exists $\tilde{f} \in C^\infty(Q/N_Q)$ such that $q_Q^* \tilde{f} = \iota_{Q,P}^* f$. We then have $q_Q^*(\Phi^* \bar{f}) = (\Phi \circ q_Q)^* \bar{f} = (\pi|_P \circ \iota_{Q,P})^* \bar{f} = \iota_{Q,P}^* f = q_Q^* \tilde{f}$ and thus $\Phi^* \bar{f} = \tilde{f} \in C^\infty(Q/N_Q)$ which shows that Φ is smooth.

Now choose a smooth function $\tilde{f} \in C^\infty(Q/N_Q)$. Then the pullback $f_Q = q_Q^* \tilde{f}$ is an N_Q -invariant element of $C^\infty(Q)$ and we find, by the same method as in the proof of Lemma 6, a smooth G -invariant function $f \in C^\infty(M)^G$ such that $\iota_Q^* f = f_Q$. We then have $f_P := \iota_P^* f \in C^\infty(P)^{G_P}$ and $\iota_{Q,P}^* f_P = \iota_{Q,P}^*(\iota_P^* f) = \iota_Q^* f = f_Q$. There exists $\bar{f} \in C^\infty(\bar{P})$ such that $(\pi|_P)^* \bar{f} = f_P$ and we get $q_Q^*(\Phi^* \bar{f}) = (\Phi \circ q_Q)^* \bar{f} = (\pi|_P \circ \iota_{Q,P})^* \bar{f} = \iota_{Q,P}^* f_P = f_Q = q_Q^* \tilde{f}$, that is, $\Phi^* \bar{f} = \tilde{f}$ and thus $(\Phi^{-1})^* \tilde{f} = \bar{f} \in C^\infty(\bar{P})$. Therefore Φ^{-1} is smooth.

We now show that $D_{\bar{P}}$ is the Φ -forward Dirac image of $q_Q(D_Q)$. The fact that $q_Q(D_Q)$ is the Φ -backward Dirac image of $D_{\bar{P}}$ follows because Φ is a diffeomorphism.

Consider the forward Dirac image $\Phi(q_Q(D_Q))$ of $q_Q(D_Q)$ under Φ , defined on \bar{P} by

$$\Phi(q_Q(D_Q))(\bar{p}) = \left\{ (v_{\bar{p}}, \alpha_{\bar{p}}) \in \mathbf{P}_{\bar{P}}(\bar{p}) \mid \begin{array}{l} \exists (v_{\bar{q}}, \alpha_{\bar{q}}) \in q(D_Q)(\bar{q}) \text{ such that} \\ T_{\bar{q}} \Phi v_{\bar{q}} = v_{\bar{p}} \text{ and } \alpha_{\bar{q}} = (T_{\bar{q}} \Phi)^* \alpha_{\bar{p}} \end{array} \right\}$$

for all $\bar{p} \in \bar{P}$ and $\bar{q} = \Phi^{-1}(\bar{p})$. Since Φ is a diffeomorphism and $q_Q(D_Q)$ is a Dirac structure on Q/N_Q , the forward Dirac image $\Phi(q_Q(D_Q))$ is a Dirac structure on \bar{P} . Hence, it is sufficient to show the inclusion $D_{\bar{P}} \subseteq \Phi(q_Q(D_Q))$.

Choose $(\bar{X}, \bar{\alpha}) \in \Gamma(D_{\bar{P}})$. Then there exists a pair $(X_{\bar{M}}, \alpha_{\bar{M}}) \in \bar{\mathcal{D}}$ such that $X_{\bar{M}}|_{\bar{P}} = \bar{X}$ and $\alpha_{\bar{M}}|_{\bar{P}} = \bar{\alpha}$. Thus we find $(X, \alpha) \in \mathcal{D}^G$ satisfying $X \sim_\pi X_{\bar{M}}$ and $\pi^* \alpha_{\bar{M}} = \alpha$. The 1-form α is a G -invariant section of \mathcal{V}° and since $[X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V})$, the vector field X can be written as a sum $X = X^G + V$ with $X^G \in \mathfrak{X}(M)^G$ and $V \in \Gamma(\mathcal{V})$ (see [JRS10]). Let (X_G, α_G) be the G -invariant average of (X, α) in a tube U centered at a point $m \in P$. We then have $\alpha_G = \alpha$ and $X_G = X^G + V_G$, that is, we still have $X_G \sim_\pi X_{\bar{M}}$. Since (X_G, α) is a G -invariant section of $\mathcal{D} \cap (\mathcal{T}_G \oplus \mathcal{V}_G^\circ)$, we have $(X_G, \alpha)|_Q \in \Gamma(\mathcal{D} \cap (TQ \oplus (TQ^\circ)^\circ))$ and, by definition of D_Q , we find $(X_Q, \alpha_Q) \in \Gamma(D_Q)$ such that $X_Q \sim_{\iota_Q} X_G$ and $\alpha_Q = \iota_Q^* \alpha$. The pair (X_Q, α_Q) is then automatically an N_Q -invariant section of $D_Q \cap \mathcal{K}_Q^\perp$ (by the proof of Lemma 6) and thus descends to a section $(\tilde{X}, \tilde{\alpha})$ of $q_Q(D_Q)$, that is, we have $X_Q \sim_{q_Q} \tilde{X}$

and $q_Q^* \tilde{\alpha} = \alpha_Q$. Therefore, for all $m' \in Q$, we have

$$\begin{aligned} T_{q_Q(m')} \Phi(\tilde{X}(q_Q(m'))) &= T_{q_Q(m')} \Phi(T_{m'} q_Q(X_Q(m'))) \\ &= T_{m'}(\pi|_P \circ \iota_{Q,P})(X_Q(m')) \\ &= T_{m'}(\pi \circ \iota_Q)(X_Q(m')) = T_{m'} \pi(X_G(m')) \\ &= X_{\bar{M}}(\pi(m')) = \bar{X}(\pi(m')), \end{aligned}$$

that is, $\tilde{X} \sim_{\Phi} \bar{X}$. In the same manner, we show the equality $q_Q^* \tilde{\alpha} = q_Q^*(\Phi^* \bar{\alpha})$ which implies that $\tilde{\alpha} = \Phi^* \bar{\alpha}$ since $q_Q : Q \rightarrow Q/N_Q$ is a surjective submersion. Thus, $(\tilde{X}, \tilde{\alpha})$ is a section of $\Phi(q_Q(D_Q))$. \square

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