

FOLIATED GROUPOIDS AND INFINITESIMAL IDEAL SYSTEMS

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ABSTRACT. The main goal of this work is to introduce a natural notion of ideal in a Lie algebroid, the “infinitesimal ideal systems”. Ideals in Lie algebras and the Bott connection associated to involutive subbundles of tangent bundles are special cases. The definition of these objects is motivated by the infinitesimal description of involutive multiplicative distributions on Lie groupoids. In the Lie group case, such distributions correspond to ideals. Several examples of infinitesimal ideal systems are presented, and (under suitable regularity conditions) the quotient of a Lie algebroid by an infinitesimal ideal system is shown to be a Lie algebroid.

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1. INTRODUCTION

Lie algebras and tangent bundles are the corner cases of Lie algebroids. In both cases, equivalence relations compatible with the structure are well understood.

In the first case, the quotient of a Lie algebra by an ideal is again a Lie algebra. If TM is the tangent bundle of a manifold M and F_M is an involutive subbundle of TM , then parallel transport relative to the Bott connection associated to F_M defines an equivalence relation on TM/F_M . If the involutive subbundle is simple, i.e. if the space of leaves of the corresponding foliation on the underlying manifold is a smooth manifold, then the quotient by this equivalence relation is the tangent bundle of the leaf space.

In these two reduction processes, one quotients a Lie algebroid by a compatible equivalence relation to construct a new Lie algebroid of the same type. The usual definition of an ideal of a Lie algebroid only makes sense for Lie algebra bundles, but is useless in the case of tangent bundles of manifolds. To be more explicit, let $(q: A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. An ideal in A is simply a subbundle $I \subseteq A$ over M , such that the space of sections $\Gamma(I)$ is an ideal in $\Gamma(A)$ endowed with the Lie bracket $[\cdot, \cdot]$. The first immediate consequence of this definition is the inclusion $I \subseteq \ker(\rho)$, which shows that I is totally intransitive. The main goal of this work is to present a more permissive notion of ideal in a Lie algebroid, encompassing both ideals in Lie algebras and the Bott connection associated to involutive subbundles.

We first encountered these *infinitesimal ideal systems* when we found them to correspond to multiplicative involutive distributions on Lie groupoids. We briefly describe this problem. Let us start with a Lie group G with Lie algebra $\mathfrak{g} = T_e G$ and multiplication map $m: G \times G \rightarrow G$. Then the tangent space TG of G is also a Lie group with unit $0_e \in \mathfrak{g}$ and multiplication map $Tm: TG \times TG \rightarrow TG$. A multiplicative distribution $S \subseteq TG$ is a distribution on G – that is, $S(g) := S \cap T_g G$ is a vector subspace of $T_g G$ for all $g \in G$ – which is in addition a subgroup of TG . Since at each $g \in G$, $S(g)$ is a vector subspace of $T_g G$, the zero section of TG is contained in S . Thus, using $T_{(g,h)}m(0_g, v_h) = T_h L_g v_h$ for any $g, h \in G$ and $v_h \in T_h G$, where $L_g: G \rightarrow G$ is the left translation by g , we find that the distribution S is left invariant. It follows that S is a smooth left invariant subbundle of TG defined by $S(g) = \mathfrak{s}^L(g)$, where \mathfrak{s} the vector subspace $S(e) = S \cap \mathfrak{g}$ of \mathfrak{g} . In the same manner, S is right invariant and we find thus that \mathfrak{s} is invariant under the adjoint action of G on \mathfrak{g} . Hence, \mathfrak{s} is an ideal in \mathfrak{g} and the subbundle $S \subseteq TG$ is completely integrable in the sense of Frobenius. Its leaf N through the unit element e of G is a normal subgroup of G and the leaf space G/S of S is the quotient group G/N .

In summary, we make two observations. On the one hand, the leaf space of a multiplicative (and hence involutive of constant rank) distribution on a Lie group is automatically a group. On the other hand, multiplicative distributions on a Lie group are infinitesimally the same as ideals in its Lie algebra.

Let G be a Lie groupoid over M . Applying the tangent functor to each of the structural maps defining $G \rightrightarrows M$, we get a Lie groupoid structure on the tangent bundle TG over TM – the **tangent groupoid**. A **multiplicative distribution** on $G \rightrightarrows M$ is a subbundle $F_G \subseteq TG$ that is also a Lie subgroupoid of the tangent groupoid over a subbundle $F_M \subseteq TM$. If F_G is also involutive, then, for simplicity, the pair $(G \rightrightarrows M, F_G)$ is said to be a **foliated groupoid**. This paper and [19] investigate the counterparts of the two observations made above in the more general situation of foliated groupoids. The paper [19] studies the leaf space of foliations associated to multiplicative involutive distributions¹ on Lie groupoids. Here, we introduce infinitesimal ideal systems as the infinitesimal counterpart of foliated groupoids.

¹Note that in the following, distributions will always be subbundles of *constant rank* of the tangent bundle.

Just as a Lie algebroid is the infinitesimal version of a Lie groupoid, a foliated groupoid corresponds at the infinitesimal level to a **foliated Lie algebroid**. Let us be more concrete. Take a Lie algebroid $(q: A \rightarrow M, \rho, [\cdot, \cdot])$; the tangent bundle TA inherits a Lie algebroid structure over TM [26]. If G is a Lie groupoid with Lie algebroid A , then there exists a one-to-one correspondence between multiplicative involutive subbundles $F_G \subseteq TG$ and **morphic** involutive distributions on A , i.e. involutive subbundles $F_A \subseteq TA$ which are also Lie subalgebroids of TA [30] (for simplicity, we call such pairs (A, F_A) “foliated algebroids”). It would nevertheless be rather cumbersome to infinitesimally describe a foliated group as a morphic distribution on its Lie algebra.

The “ideals” of the following definition infinitesimally describe foliated groupoids as the ideals in Lie algebras describe foliated groups.

Definition 1.1. *Let $(q: A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $K \subseteq A$ a subalgebroid over M such that $\rho(K) \subseteq F_M$ and ∇ a flat F_M -connection on A/K with the following properties:*

- (1) *If $a \in \Gamma(A)$ is ∇ -flat, then $[a, b] \in \Gamma(K)$ for all $b \in \Gamma(K)$.*
- (2) *If $a, b \in \Gamma(A)$ are ∇ -flat, then $[a, b]$ is also ∇ -flat.*
- (3) *If $a \in \Gamma(A)$ is ∇ -flat, then $\rho(a)$ is ∇^{F_M} -flat, where*

$$\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \rightarrow \Gamma(TM/F_M)$$

is the Bott connection associated to F_M .

The triple (F_M, K, ∇) is an infinitesimal ideal system² in A .

Note that this is an infinitesimal version of the **ideal systems** in [15] (see also [25]), which are described there to be the kernels of fibrations of Lie algebroids (see Section 6). Note also that infinitesimal ideal systems already appear (not under this name) in [14] in connection with geometric quantization as the infinitesimal version of polarizations on Lie groupoids, where Eli Hawkins already finds that they correspond to foliated algebroids and groupoids. Finally, let us mention that the special case where $F_M = TM$ has been studied independently in [8] in relation with a modern approach to Cartan’s work on pseudogroups.

We claim that the infinitesimal ideal systems are the objects that should be considered as the ideal objects in Lie algebroids. In the second part of the paper, we describe several examples of infinitesimal ideal systems and we show that, under regularity conditions, one can take the quotient of a Lie algebroid by an ideal system to define a new (reduced) Lie algebroid. More concretely, we prove that if (A, F_A) is a foliated algebroid with corresponding infinitesimal ideal system (F_M, K, ∇) in A , then, modulo regularity conditions, the leaf space A/F_A inherits a natural Lie algebroid structure over the leaf space M/F_M . The projections $A \rightarrow A/F_A$ and $M \rightarrow M/F_M$ form a Lie algebroid morphism. The Lie algebroid structure on the leaf space is realized as the quotient of A by the ideal system (in the sense of [15]) that integrates (F_M, K, ∇) . In particular, infinitesimal ideal systems arise as the kernels of fibrations of Lie algebroids. We also show that if a foliated groupoid (G, F_G) – with associated foliated algebroid (A, F_A) – is such that the leaf space $G/F_G \rightrightarrows M/F_M$ is a Lie groupoid, then its Lie algebroid is isomorphic to the reduced Lie algebroid structure on $A/F_A \rightarrow M/F_M$.

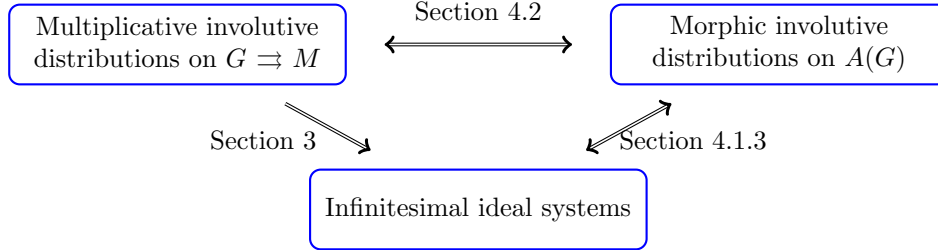
Outline of the paper. This paper is organized as follows. In Section 2, we recall the definitions of the tangent Lie groupoid associated to a Lie groupoid, and of the tangent Lie algebroid defined by a Lie algebroid. We then recall some facts about flat partial connections, as well as the definition of the Bott connection associated to an involutive subbundle of the tangent bundle of a manifold.

²Infinitesimal ideal systems were called “IM-foliations” in an earlier version of this work, in analogy to the “IM-2-forms” of [6], but we find this new terminology more adequate.

In Section 3, we give the definition of foliated groupoids. The first main result of this paper (Theorem 3.6) states that a foliated groupoid $(G \rightrightarrows M, F_G)$ defines an infinitesimal ideal system (F_M, K, ∇) in the Lie algebroid A of $G \rightrightarrows M$. Then, we examine how the involutivity of the multiplicative distribution is encoded by the properties of the infinitesimal ideal system.

Infinitesimal ideal systems were found in [14] to encode morphic involutive distributions on Lie algebroids. In Section 4 we summarize the approach of [9] to this result and we show how infinitesimal ideal systems are in one-to-one correspondence with foliated algebroids. The approach in [9] is an application of a result on the correspondence between morphisms of representations up to homotopy and morphisms of VB-algebroids (see [13] for the correspondence between representations up to homotopy and VB-algebroids). To avoid unnecessary technicalities in this paper, we do not introduce these objects, but describe explicitly our special situation. Yet, the reader who knows these concepts should note that one of the results in [9] is that an infinitesimal ideal system in A is equivalent to a pair of subrepresentations up to homotopy: one of the adjoint and one of the double representations up to homotopy defined by the Lie algebroid A . This shows that our proposed notion of ideal is compatible with the definition of an ideal in a Lie algebra via subrepresentations of the adjoint representation.

We prove that the infinitesimal ideal system defined by a foliated groupoid and the infinitesimal ideal system defined by the corresponding foliated algebroid coincide (Theorem 4.9). Then, we show that there exists a one-to-one correspondence between source-simply connected foliated groupoids and infinitesimal ideal systems on integrable Lie algebroids.



The examples of infinitesimal ideal systems in Section 5 explain why they can be seen as the ideals in Lie algebroids. We show that kernels of Dirac structures, usual ideals in Lie algebroids, Bott connections associated to involutive subbundle and kernels of transitive Lie algebroid morphisms are examples of infinitesimal ideal systems. We compare also our notion of foliated algebroids with the ones of [37], as well as the infinitesimal descriptions in both approaches.

Section 6 finally shows that the quotient of a Lie algebroid by an infinitesimal ideal system inherits a natural Lie algebroid structure such that the quotient map is a Lie algebroid morphism (Theorem 6.8).

The first appendix recalls how the Lie algebroid $A(TG) \rightarrow TM$ of the tangent groupoid $TG \rightrightarrows TM$ is isomorphic to the tangent Lie algebroid $T(A(G)) \rightarrow TM$. The second appendix proves some useful results on invariance of subbundles of a vector bundle under the flow of a vector field on it.

Notation. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sets of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $q_E: E \rightarrow M$, the set of (local) sections of E will be written $\Gamma(E)$. We will write $\text{Dom}(\sigma)$ for the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(E)$ is defined. The linear function on E associated to a section $\xi \in \Gamma(E^*)$ will be

written ℓ_ξ . For any $\varepsilon \in \Gamma(E)$, the vertical (or “core”) vector field $\varepsilon^\uparrow \in \mathfrak{X}(E)$ is defined by

$$(1.1) \quad \varepsilon^\uparrow(e_m) = \left. \frac{d}{dt} \right|_{t=0} (e_m + t\varepsilon(m))$$

for all $e_m \in E$. In other words, $\varepsilon^\uparrow(\ell_\xi) = q_E^* \langle \xi, \varepsilon \rangle$ for all $\xi \in \Gamma(E^*)$, and $X(q_E^* f) = 0$ for all $f \in C^\infty(M)$. The Lie bracket of two such vector fields vanishes.

The flow of a vector field X will be written ϕ^X , unless specified otherwise.

Let $f : M \rightarrow N$ be a smooth map between two smooth manifolds M and N . Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be **f -related** if $Tf \circ X = Y \circ f$ on $\text{Dom}(X) \cap f^{-1}(\text{Dom}(Y))$. We write then $X \sim_f Y$.

The pullback or restriction of a vector bundle $E \rightarrow M$ to an embedded submanifold N of M will be written $E|_N$. In the special case of the tangent and cotangent spaces of M , we will write $T_N M$ and $T_N^* M$. If $f : M \rightarrow N$ is a smooth surjective submersion, we write $T^f M$ for the kernel of $Tf : TM \rightarrow TN$.

The projection map of $TM \rightarrow M$ is finally denoted by p_M .

A **groupoid** G over the **units** M will be written $G \rightrightarrows M$. The **source** and **target** maps are denoted by $s, t : G \rightarrow M$ respectively, the **unit section** $\epsilon : M \rightarrow G$, the inversion map $i : G \rightarrow G$ and the multiplication $m : G_{(2)} \rightarrow G$, where $G_{(2)} = \{(g, h) \in G \times G \mid t(h) = s(g)\}$ is the set of composable groupoid pairs. A groupoid G over M is called a **Lie groupoid** if both G and M are smooth Hausdorff manifolds, the source and target maps $s, t : G \rightarrow M$ are surjective submersions, and all the other structural maps are smooth. Throughout this work we only consider Lie groupoids.

The Lie algebroid of $G \rightrightarrows M$ is defined in this paper to be $AG = T_M^s G$, with anchor $\rho_{AG} := Tt|_{AG}$ and bracket $[\cdot, \cdot]_{AG}$ defined by using right invariant vector fields.

We will write $A(\cdot)$ for the functor that sends Lie groupoids to Lie algebroids and Lie groupoid morphisms to Lie algebroid morphisms. *For simplicity, $(AG, \rho_{AG}, [\cdot, \cdot]_{AG})$ will be written $(A, \rho, [\cdot, \cdot])$ in the following.*

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2. BACKGROUND

2.1. Tangent and cotangent groupoids. Let G be a Lie groupoid over M with Lie algebroid A . The tangent bundle TG has a natural Lie groupoid structure over TM , which is obtained by applying the tangent functor to each of the structure maps defining G . That is, the set of composable pairs $(TG)_{(2)}$ of this groupoid is equal to $T(G_{(2)})$ and for $(g, h) \in G_{(2)}$ and a pair $(v_g, w_h) \in (TG)_{(2)}$, the multiplication is

$$v_g \star w_h := Tm(v_g, w_h).$$

We refer to TG with the groupoid structure over TM as the **tangent groupoid** of G [25].

As in [24], we define **star vector fields on G** or **star sections of TG** to be vector fields $X \in \mathfrak{X}(G)$ such that there exists $\bar{X} \in \mathfrak{X}(M)$ with $X \sim_s \bar{X}$ and $\bar{X} \sim_\epsilon X$, i.e. X and \bar{X} are

\mathfrak{s} -related and \bar{X} and X are ϵ -related, i.e. X restricts to \bar{X} on M . We then write $X \overset{\star}{\sim}_{\mathfrak{s}} \bar{X}$. In the same manner, we can define \mathfrak{t} -star sections, $X \overset{\star}{\sim}_{\mathfrak{t}} \bar{X}$ with $\bar{X} \in \mathfrak{X}(M)$ and $X \in \mathfrak{X}(G)$. It is easy to see that the tangent space TG is spanned by star vector fields at each point in $G \setminus M$. Note also that the Lie bracket of two star sections of TG is again a star section.

We call a vector field $X \in \mathfrak{X}(G)$ a **t-section** if there exists $\bar{X} \in \mathfrak{X}(M)$ such that $X \sim_{\mathfrak{t}} \bar{X}$.

We will also need the cotangent groupoid $T^*G \rightrightarrows A^*$ in the proof of Theorem 3.8. It was shown in [7], that T^*G is a Lie groupoid over A^* . The source and target of $\alpha_g \in T_g^*G$ are defined by

$$\tilde{\mathfrak{s}}(\alpha_g) \in A_{\mathfrak{s}(g)}^*, \quad \tilde{\mathfrak{s}}(\alpha_g)(a) = \alpha_g(Tl_g(a - T\mathfrak{t}(a))) \quad \text{for all } a \in A_{\mathfrak{s}(g)}$$

and

$$\tilde{\mathfrak{t}}(\alpha_g) \in A_{\mathfrak{t}(g)}^*, \quad \tilde{\mathfrak{t}}(\alpha_g)(b) = \alpha_g(Tr_g(b)) \quad \text{for all } b \in A_{\mathfrak{t}(g)}.$$

A one-form $\eta \in \Omega^1(G)$ is a **t-section** of T^*G if $\tilde{\mathfrak{t}} \circ \eta = \bar{\eta} \circ \mathfrak{t}$ for some $\bar{\eta} \in \Gamma(A^*)$.

2.2. The tangent Lie algebroid $TA \rightarrow TM$. Consider a vector bundle $q_A: A \rightarrow M$. Then the tangent space TA of A has two vector bundle structures. First, the usual vector bundle structure $p_A: TA \rightarrow A$ and second the vector bundle structure $Tq_A: TA \rightarrow TM$, with the addition defined as follows. If x_{a_m} and $x_{a'_m}$ are such that $Tq_A(x_{a_m}) = Tq_A(x_{a'_m}) =: x_m \in TM$, then there exist curves $c, c': (-\epsilon, \epsilon) \rightarrow A$ such that $\dot{c}(0) = x_{a_m}$, $\dot{c}'(0) = x_{a'_m}$ and $q_A \circ c = q_A \circ c'$. The sum $x_{a_m} +_{Tq_A} x_{a'_m}$ is then defined by

$$x_{a_m} +_{Tq_A} x_{a'_m} = \left. \frac{d}{dt} \right|_{t=0} (c(t) +_{q_A} c'(t)) \in T_{a_m+a'_m} A.$$

We get a double vector bundle

$$\begin{array}{ccc} TA & \xrightarrow{p_A} & A \\ Tq_A \downarrow & & \downarrow q_A \\ TM & \xrightarrow{p_M} & M \end{array},$$

that is, the structure maps of each vector bundle structure are vector bundle morphisms relative to the other structure [25]. Note that a subbundle H of TA over A is said to be **linear** if it is closed under the addition of TA as a vector bundle over TM .

Assume now that $q_A: A \rightarrow M$ has a Lie algebroid structure with anchor map $\rho: A \rightarrow TM$ and Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$. Then there is a Lie algebroid structure on TA over TM . In order to describe it explicitly, we recall first that there exists a **canonical involution**

$$(2.2) \quad \begin{array}{ccc} TTM & \xrightarrow{J_M} & TTM \\ p_{TM} \downarrow & & \downarrow T p_M \\ TM & \xrightarrow{\text{Id}_{TM}} & TM \end{array}$$

which is given as follows [25, 36]. Elements $(\xi; v, x; m) \in TTM$, that is, with $p_{TM}(\xi) = v \in T_m M$ and $T p_M(\xi) = x \in T_m M$, are considered as second derivatives

$$\xi = \frac{\partial^2 \sigma}{\partial t \partial u}(0, 0),$$

where $\sigma : \mathbb{R}^2 \rightarrow M$ is a smooth square of elements of M^3 . The canonical involution $J_M : TTM \rightarrow TTM$ is defined by

$$J_M(\xi) := \frac{\partial^2 \sigma}{\partial u \partial t}(0, 0).$$

We can apply the tangent functor to the anchor map $\rho : A \rightarrow TM$, and then compose with the canonical involution to obtain a bundle map $\rho_{TA} : TA \rightarrow TTM$ defined by

$$\rho_{TA} = J_M \circ T\rho.$$

This defines the tangent anchor map. In order to define the tangent Lie bracket, we observe that every section $a \in \Gamma_M(A)$ induces two types of sections of $TA \rightarrow TM$. The first type of section is simply $Ta : TM \rightarrow TA$, and the second type of section are the **core** sections $a^\dagger : TM \rightarrow TA$, which are defined by

$$(2.3) \quad a^\dagger(v_m) = T_m 0^A(v_m) +_{p_A} \left. \frac{d}{dt} \right|_{t=0} ta(m)$$

where $0^A : M \rightarrow A$ denotes the zero section. As observed in [27], sections of the form Ta and a^\dagger generate the module of sections $\Gamma_{TM}(TA)$. Therefore, the tangent Lie bracket $[\cdot, \cdot]_{TA}$ is completely determined by

$$[Ta, Tb]_{TA} = T[a, b], \quad [Ta, b^\dagger]_{TA} = [a, b]^\dagger, \quad [a^\dagger, b^\dagger]_{TA} = 0$$

for all $a, b \in \Gamma(A)$, the extension to general sections is done using the Leibniz rule with respect to the tangent anchor ρ_{TA} .

Note that $(TA \rightarrow TM; A \rightarrow M)$ and $(TA \rightarrow A; TM \rightarrow M)$ are two examples of VB-algebroids, which we now quickly define. Let $(D; A, B; M)$ be a double vector bundle. A section ξ of $D \rightarrow B$ is linear if it is a vector bundle morphism over a section of $A \rightarrow M$. We write $\Gamma_l(B, D)$ for the set of linear sections of $D \rightarrow B$ and $\Gamma_c(B, D)$ for the set of *core* sections of $D \rightarrow B$. Recall that the core sections of $TA \rightarrow A$ are defined by (1.1) and the core sections of $TA \rightarrow TM$ by (2.3).

Definition 2.1. *We say that $(D \rightarrow B; A \rightarrow M)$ is a **VB-algebroid** if $D \rightarrow B$ is a Lie algebroid, the anchor $\rho_D : D \rightarrow TB$ is a bundle morphism over $\rho_A : A \rightarrow TM$ and the three Lie bracket conditions below are satisfied:*

- (i) $[\Gamma_l(B, D), \Gamma_l(B, D)]_D \subset \Gamma_l(B, D)$;
- (ii) $[\Gamma_l(B, D), \Gamma_c(B, D)]_D \subset \Gamma_c(B, D)$;
- (iii) $[\Gamma_c(B, D), \Gamma_c(B, D)]_D = 0$.

2.3. Flat partial connections.

Definition 2.2. ([2]) *Let M be a smooth manifold and $F \subseteq TM$ a smooth involutive vector subbundle of the tangent bundle. Let $E \rightarrow M$ be a vector bundle over M . A **partial F-connection** is a map $\nabla : \Gamma(F) \times \Gamma(E) \rightarrow \Gamma(E)$, written $\nabla(X, e) =: \nabla_X e$ for $X \in \Gamma(F)$ and $e \in \Gamma(E)$, such that:*

- (1) ∇ is tensorial in the F -argument,
- (2) ∇ is \mathbb{R} -linear in the E -argument,
- (3) ∇ satisfies the Leibniz rule

$$\nabla_X(fe) = X(f)e + f\nabla_X e$$

for all $X \in \Gamma(F)$, $e \in \Gamma(E)$, $f \in C^\infty(M)$.

The connection is **flat** if its curvature tensor vanishes.

³The notation means that σ is first differentiated with respect to u , yielding a curve $v(t) = \frac{\partial \sigma}{\partial u}(t, 0)$ in TM with $\left. \frac{d}{dt} \right|_{t=0} v(t) = \xi$. Thus, $v = \frac{\partial \sigma}{\partial u}(0, 0) = p_{TM}(\xi)$ and $x = \frac{\partial \sigma}{\partial t}(0, 0) = Tp_M(\xi)$.

Example 2.3 (The Bott connection). Let M be a smooth manifold and $F \subseteq TM$ an involutive subbundle. The **Bott connection**

$$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F)$$

defined by

$$\nabla_X^F \bar{Y} = \overline{[X, Y]},$$

where $\bar{Y} \in \Gamma(TM/F)$ is the projection of $Y \in \mathfrak{X}(M)$, is a flat partial F -connection on $TM/F \rightarrow M$.

The class $\bar{Y} \in \Gamma(TM/F)$ of a vector field is ∇^F -flat if and only if $[Y, \Gamma(F)] \subseteq \Gamma(F)$. Since F is involutive, this does not depend on the representative of \bar{Y} . We say by abuse of notation that Y is ∇^F -flat.

The following proposition can be easily shown by using the fact that the parallel transport defined by a flat connection does not depend on the chosen path in simply connected sets (see [19], [21] for similar statements).

Proposition 2.4. *Let $E \rightarrow M$ be a smooth vector bundle of rank k , $F \subseteq TM$ an involutive subbundle and ∇ a flat partial F -connection on E . Then for each point $m \in M$ there exists a frame of local ∇ -flat sections $e_1, \dots, e_k \in \Gamma(E)$ defined on an open neighborhood U of m in M .*

We will also use the following lemma, which is easy to prove.

Lemma 2.5. *Let $E \rightarrow M$ be a smooth vector bundle of rank k , $F \subseteq TM$ an involutive subbundle and ∇ a partial F -connection on E .*

- (1) *Assume that $f \in C^\infty(M)$ is F -invariant, i.e. $X(f) = 0$ for all $X \in \Gamma(F)$. Then $f \cdot e$ is ∇ -flat for any ∇ -flat section $e \in \Gamma(E)$.*
- (2) *Assume that the foliation defined by F on M is simple, i.e. the leaf space has a smooth manifold structure such that the quotient $\pi : M \rightarrow M/F$ is a smooth surjective submersion. Then $X \in \mathfrak{X}(M)$ is ∇^F -flat if and only if there exists $\bar{X} \in \mathfrak{X}(M/F)$ such that $X \sim_\pi \bar{X}$.*

3. FOLIATED GROUPOIDS

3.1. Definition and properties.

Definition 3.1. *Let $G \rightrightarrows M$ be a Lie groupoid. A subbundle $F_G \subseteq TG$ is **multiplicative** if it is a subgroupoid of $TG \rightrightarrows TM$ over $F_G \cap TM =: F_M$. We also say that F_G is a **multiplicative distribution** on G . Also, if F_G is involutive the pair $(G \rightrightarrows M, F_G)$ will be called, by abuse a notation, a **foliated groupoid**.*

Remark 3.2. Multiplicative subbundles were introduced in [35] (see also [14]) as follows. A subbundle $F_G \subseteq TG$ is multiplicative if for all composable $g, h \in G$ and $u \in F_G(g \star h)$, there exist $v \in F_G(g)$, $w \in F_G(h)$ such that $u = v \star w$. It is easy to check that a multiplicative distribution in the sense of Definition 3.1 is multiplicative in the sense of [35], but the converse is not necessarily true, unless for instance if the Lie groupoid is a Lie group (see [17]). The case of involutive wide subgroupoids of $TG \rightrightarrows TM$ has also been studied in [1].

Definition 3.3. *Let F_G be a multiplicative distribution on $G \rightrightarrows M$. The subbundle*

$$K = F_G \cap A = \{v \in F_G \mid p_G(v) \in M \text{ and } T\mathfrak{s}(v) = 0\}$$

*is called the **core** of $(G \rightrightarrows M, F_G)$.*

This name is chosen for the following reason. A multiplicative subbundle $F_G \subseteq TG$ determines a VB-groupoid

$$\begin{array}{ccc} F_G & \xrightarrow{p_G} & G \\ T\mathfrak{t} \downarrow & & \downarrow \mathfrak{s} \\ & T\mathfrak{s} & \\ & \downarrow & \\ F_M & \xrightarrow{p_M} & M \end{array}$$

That is, F_G is a vector bundle over G , which is a Lie groupoid over M , and F_G is also a Lie groupoid over F_M , which is a vector bundle over M , the structure maps of the groupoid structure are vector bundle morphisms and the double source map $(p_G, T\mathfrak{s}) : F_G \rightarrow G \times_M F_M$ is a surjective submersion [25]. The set K of elements of F_G that are sent to units by p_G and to 0 by $T\mathfrak{s}$ is called the **core** of the VB-groupoid. It inherits the structure of a vector bundle over M .

In particular, we have the following lemma [19].

Lemma 3.4. *Let $G \rightrightarrows M$ be a Lie groupoid and $F_G \subseteq TG$ a multiplicative subbundle. Then the intersection $F_M := F_G \cap TM$ has constant rank on M . Since it is the set of units of F_G seen as a subgroupoid of TG , the pair $F_G \rightrightarrows F_M$ is a Lie groupoid.*

The bundle $F_G|_M$ splits as $F_G|_M = F_M \oplus K$, where $K := F_G \cap A$. We have

$$(F_G \cap T^s G)(g) = K(\mathfrak{t}(g)) \star 0_g = T_{\mathfrak{t}(g)} r_g(K(\mathfrak{t}(g)))$$

for all $g \in G$.

In the same manner, if $F^t := (F_G \cap T^t G)|_M$, we have $(F_G \cap T^t G)(g) = 0_g \star F^t(\mathfrak{s}(g))$ for all $g \in G$. As a consequence, the intersections $F_G \cap T^t G$ and $F_G \cap T^s G$ have constant rank on G .

3.2. The connection associated to a foliated groupoid. Our first result on foliated groupoids is easy to prove, by considering right-invariant and \mathfrak{s} -sections of the distribution (see also [20, 16]).

Proposition 3.5. *Let F_G be a multiplicative involutive distribution on a Lie groupoid $G \rightrightarrows M$. Then K is a subalgebroid of A and F_M is an involutive subbundle of TM .*

The main goal of this subsection is the construction of a partial F_M -connection on A/K induced by the Bott F_G -connection on TG/F_G . We will see later how the quadruple (A, F_M, K, ∇) contains the whole information about the foliated groupoid.

We write \bar{a} for the class in A/K of $a \in \Gamma(A)$.

Theorem 3.6. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid. Then there is a partial F_M -connection on A/K*

$$(3.4) \quad \nabla : \Gamma(F_M) \times \Gamma(A/K) \rightarrow \Gamma(A/K).$$

defined by

$$a \in \Gamma(A) \text{ is } \nabla\text{-flat if and only if } a^r \in \mathfrak{X}(G) \text{ is } \nabla^{F_G}\text{-flat.}$$

The triple (F_M, K, ∇) is an infinitesimal ideal system in A .

For the proof of this theorem, we need the following result, which can be shown with the same techniques as its general counterpart on Dirac groupoids in [20, 16].

Lemma 3.7. *Let $(G \rightrightarrows M, F_G)$ be a Lie groupoid endowed with a multiplicative subbundle $F_G \subseteq TG$, X a \mathfrak{t} -section of F_G , i.e. \mathfrak{t} -related to some $\bar{X} \in \Gamma(F_M)$, and consider $a \in \Gamma(A)$. Then the Lie derivative $\mathcal{L}_{a^r} X$ can be written as a sum*

$$\mathcal{L}_{a^r} X = Z_{a,X} + b_{a,X}^r$$

with $b_{a,X} \in \Gamma(A)$, and $Z_{a,X}$ a \mathfrak{t} -section of F_G . In addition, if $X \sim_{\mathfrak{t}} 0$, then $\mathcal{L}_{a^r} X \in \Gamma(F_G \cap T^{\mathfrak{t}}G)$. In particular, its restriction to M is a section of $F^{\mathfrak{t}}$ and $b_{a,X}$ is a section of K .

Assume now that F_G is involutive and define

$$\nabla : \Gamma(F_M) \times \Gamma(A/K) \rightarrow \Gamma(A/K)$$

by

$$\nabla_{\bar{X}} \bar{a} = -\overline{b_{a,X}},$$

with $b_{a,X}$ as in Lemma 3.7, for any choice of \mathfrak{t} -section $X \in \Gamma(F_G)$ such that $X \sim_{\mathfrak{t}} \bar{X}$ and any choice of representative $a \in \Gamma(A)$ for \bar{a} . We will show that this is a well-defined partial F_M -connection and complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Choose $X, X' \in \Gamma(F_G)$ such that $X \sim_{\mathfrak{t}} \bar{X}$ and $X' \sim_{\mathfrak{t}} \bar{X}$. Then $Y := X - X' \sim_{\mathfrak{t}} 0$ and, by Lemma 3.7, we find $b_{a,Y} \in \Gamma(K)$ for any $a \in \Gamma(A)$, i.e. $\overline{b_{a,X}} = \overline{b_{a,X'}}$.

Choose now $a \in \Gamma(K)$ and $X \in \Gamma(F_G)$, $X \sim_{\mathfrak{t}} \bar{X} \in \Gamma(F_M)$. Then we have $a^r \in \Gamma(F_G)$ and since F_G is involutive, $\mathcal{L}_{a^r} X \in \Gamma(F_G)$. Again, since $Z_{a,X} \in \Gamma(F_G)$, we find $b_{a,X} \in \Gamma(K)$. This shows that ∇ is well-defined.

By definition, if $a \in \Gamma(A)$ is such that $\nabla_{\bar{X}} \bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$, then we have $\mathcal{L}_{a^r} X = Z_{a,X} + b_{a,X}^r \in \Gamma(F_G)$ for all \mathfrak{t} -sections $X \in \Gamma(F_G)$. Since $\Gamma(F_G)$ is spanned as a $C^\infty(G)$ -module by its \mathfrak{t} -sections, we get

$$[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G).$$

Conversely, $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$ implies immediately $\nabla_{\bar{X}} \bar{a} = 0$ for all $\bar{X} \in \Gamma(F_M)$. This proves the second claim of the theorem.

We check that ∇ is a flat partial F_M -connection. Choose $a \in \Gamma(A)$, $\bar{X} \in \Gamma(F_M)$, $X \in \Gamma(F_G)$ such that $X \sim_{\mathfrak{t}} \bar{X}$ and $f \in C^\infty(M)$. Then we have $\mathfrak{t}^* f \cdot X \sim_{\mathfrak{t}} f \bar{X}$ and

$$\mathcal{L}_{a^r}(\mathfrak{t}^* f \cdot X) = \mathfrak{t}^*(\rho(a)(f)) \cdot X + \mathfrak{t}^* f \cdot \mathcal{L}_{a^r} X.$$

In particular, we find

$$\begin{aligned} \overline{b_{a,\mathfrak{t}^* f \cdot X}} &= \overline{(1 - T\mathfrak{s})(\mathfrak{t}^*(\rho(a)(f)) \cdot X + \mathfrak{t}^* f \cdot \mathcal{L}_{a^r} X)}|_M \\ &= \overline{\rho(a)(f) \cdot (1 - T\mathfrak{s})X}|_M + \overline{f \cdot (1 - T\mathfrak{s})(\mathcal{L}_{a^r} X)}|_M. \end{aligned}$$

Since $(T\mathfrak{s} - 1)X|_M \in \Gamma(K)$, this leads to $\overline{b_{a,\mathfrak{t}^* f \cdot X}} = f \cdot \overline{b_{a,X}}$ and hence $\nabla_{f\bar{X}} \bar{a} = -f \cdot \overline{b_{a,X}} = f \cdot \nabla_{\bar{X}} \bar{a}$.

Since $(fa)^r = \mathfrak{t}^* f \cdot a^r$, we have in the same manner

$$\mathcal{L}_{(fa)^r} X = -\mathcal{L}_X(\mathfrak{t}^* f \cdot a^r) = -\mathfrak{t}^*(\bar{X}(f)) \cdot a^r + \mathfrak{t}^* f \cdot \mathcal{L}_{a^r} X,$$

which leads to $\nabla_{\bar{X}}(f \cdot \bar{a}) = \bar{X}(f) \cdot \bar{a} + f \cdot \nabla_{\bar{X}} \bar{a}$.

Choose $\bar{X}, \bar{Y} \in \Gamma(F_M)$ and $X, Y \in \Gamma(F_G)$ such that $X \sim_{\mathfrak{t}} \bar{X}$ and $Y \sim_{\mathfrak{t}} \bar{Y}$. Then we have $[X, Y] \sim_{\mathfrak{t}} [\bar{X}, \bar{Y}]$ and $[X, Y] \in \Gamma(F_G)$ since F_G is involutive. For any $a \in \Gamma(A)$, we have by the Jacobi-identity:

$$\begin{aligned} \mathcal{L}_{a^r}[X, Y] &= [\mathcal{L}_{a^r} X, Y] - [\mathcal{L}_{a^r} Y, X] \\ &= [Z_{a,X} + b_{a,X}^r, Y] - [Z_{a,Y} + b_{a,Y}^r, X] \\ &= [Z_{a,X}, Y] - [Z_{a,Y}, X] + \mathcal{L}_{b_{a,X}^r} Y - \mathcal{L}_{b_{a,Y}^r} X \\ &= [Z_{a,X}, Y] - [Z_{a,Y}, X] + Z_{b_{a,X}, Y} + b_{b_{a,X}, Y}^r - Z_{b_{a,Y}, X} - b_{b_{a,Y}, X}^r. \end{aligned}$$

Since $[Z_{a,X}, Y] - [Z_{a,Y}, X] + Z_{b_{a,X}, Y} - Z_{b_{a,Y}, X}$ is a \mathfrak{t} -section of F_G , we find that

$$\nabla_{[\bar{X}, \bar{Y}]} \bar{a} = \overline{b_{b_{a,X}, Y}} - \overline{b_{b_{a,Y}, X}} = \nabla_{\bar{X}} \nabla_{\bar{Y}} \bar{a} - \nabla_{\bar{Y}} \nabla_{\bar{X}} \bar{a}$$

which shows the flatness of ∇ .

Choose now $a \in \Gamma(A)$ such that $\nabla_{\bar{X}}\bar{a} = 0 \in \Gamma(A/K)$ for all $\bar{X} \in \Gamma(F_M)$. If $b \in \Gamma(K)$, then $b^r \in \Gamma(F_G)$, $\rho(b) \in \Gamma(F_M)$ and $b^r \sim_{\mathfrak{t}} \rho(b)$. This leads to

$$\overline{[b, a]} = \nabla_{\rho(b)}\bar{a} = 0 \in \Gamma(A/K)$$

and hence $[a, b] \in \Gamma(K)$. This shows 2. For each $\bar{X} \in \Gamma(F_M)$, there exists $X \in \Gamma(F_G)$ such that $X \sim_{\mathfrak{t}} \bar{X}$. Since $[a^r, X] \in \Gamma(F_G)$, $a^r \sim_{\mathfrak{t}} \rho(a)$ and $\text{Tt}(F_G) = F_M$, we find $[\rho(a), \bar{X}] \in \Gamma(F_M)$, which proves 4.

To show 3., choose two sections $a, b \in \Gamma(A)$ such that \bar{a} and \bar{b} are ∇ -flat. We have then for any \mathfrak{t} -section $X \sim_{\mathfrak{t}} \bar{X}$ of F_G :

$$\begin{aligned} \mathcal{L}_{[a, b]^r} X &= \mathcal{L}_{a^r}(Z_{b, X} + b_{b, X}^r) - \mathcal{L}_{b^r}(Z_{a, X} + b_{a, X}^r) \\ &= \mathcal{L}_{a^r}(Z_{b, X}) + [a, b_{b, X}]^r - \mathcal{L}_{b^r}(Z_{a, X}) - [b, b_{a, X}]^r. \end{aligned}$$

Since \bar{a} and \bar{b} are ∇ -flat, this yields $\nabla_{\bar{X}}\overline{[a, b]} = -\overline{[a, b_{b, X}]} + \overline{[b, b_{a, X}]}$. Since \bar{a} and \bar{b} are flat, we have $b_{b, X}, b_{a, X} \in \Gamma(K)$ and 3. follows using 2. \square

3.3. Involutivity of a multiplicative subbundle of TG . It is natural to ask here how exactly the involutivity of F_G is encoded in the data (F_M, K, ∇) . For an arbitrary (not necessarily involutive) multiplicative subbundle $F_G \subseteq TG$, we can consider the map

$$\begin{aligned} \tilde{\nabla} : \Gamma(F_M) \times \Gamma(A) &\rightarrow \Gamma(A/K), \\ \tilde{\nabla}_{\bar{X}} a &= -\overline{b_{a, X}} \end{aligned}$$

which is well-defined by the proof of Theorem 3.6.

Theorem 3.8. *Let (G, F_G) be a source-connected Lie groupoid endowed with a multiplicative subbundle. Then F_G is involutive if and only if the following holds:*

- (1) $F_M \subseteq TM$ is involutive,
- (2) $\tilde{\nabla}$ vanishes on sections of K ,
- (3) the induced map $\nabla : \Gamma(F_M) \times \Gamma(A/K) \rightarrow \Gamma(A/K)$ is a flat partial F_M -connection on A/K .

The proof of this theorem is a simplified version of the proof of the general criterion for the integrability property of multiplicative Dirac structures (see [16]).

Proof. We have already shown in Proposition 3.5 and Theorem 3.6 that the involutivity of F_G implies (1), (2) and (3).

For the converse implication, note that the \mathfrak{t} -star sections of F_G span F_G outside of the set of units M . Hence, it is sufficient to show involutivity on \mathfrak{t} -star sections and right-invariant sections of F_G . Choose first two right-invariant sections a^r, b^r of F_G , i.e. with $a, b \in \Gamma(K)$. We then have $\rho(b) \in \Gamma(F_M)$, $b^r \sim_{\mathfrak{t}} \rho(b)$ and, since $\tilde{\nabla}_{\rho(b)} a = 0$ by (2), we find that $[a^r, b^r] \in \Gamma(F_G)$. In the same manner, by the definition of $\tilde{\nabla}$ and Lemma 3.7, Condition (2) implies that the bracket of a right-invariant section of F_G and a \mathfrak{t} -star section is always a section of F_G .

We have thus only to show that the bracket of two \mathfrak{t} -star sections of F_G is again a section of F_G . Let K° be the annihilator of K in A^* and consider the dual F_M -connection on $(A/K)^* \simeq K^\circ \subseteq A^*$, i.e. the (by (3)) flat connection $\nabla^* : \Gamma(F_M) \times \Gamma(K^\circ) \rightarrow \Gamma(K^\circ)$ given by $(\nabla_{\bar{X}}^* \alpha)(\bar{a}) = \bar{X}(\alpha(\bar{a})) - \alpha(\nabla_{\bar{X}}\bar{a})$ for all $\bar{X} \in \Gamma(F_M)$, $\alpha \in \Gamma(K^\circ)$ and $a \in \Gamma(A)$.

Choose a \mathfrak{t} -star section $X \in \Gamma(F_G)$, $X \sim_{\mathfrak{t}} \bar{X}$, $X|_M = \bar{X}$ and a \mathfrak{t} -section $\eta \in \Gamma(F_G^\circ)$, $\eta \sim_{\mathfrak{t}} \bar{\eta}$. Then, for any section a of A , we have

$$\begin{aligned} (\mathcal{L}_X \eta)(a^r) &= X(\eta(a^r)) + \eta(\mathcal{L}_{a^r} X) = \mathfrak{t}^*(\bar{X}(\bar{\eta}(\bar{a}))) + \eta(Z_{a, X} + b_{a, X}^r) && \text{by Lemma 3.7} \\ &= \mathfrak{t}^*(\bar{X}(\bar{\eta}(\bar{a})) - \bar{\eta}(\nabla_{\bar{X}}\bar{a})) && \text{since } \eta \in \Gamma(F_G^\circ) \text{ and } Z_{a, X} \in \Gamma(F_G) \\ (3.5) \quad &= \mathfrak{t}^*(\nabla_{\bar{X}}^* \bar{\eta}(\bar{a})). \end{aligned}$$

This shows that $\mathcal{L}_X \eta \sim_{\tilde{\mathfrak{t}}} \nabla_{\bar{X}}^* \bar{\eta} \in \Gamma(K^\circ)$. Note that we have not shown yet that $\mathcal{L}_X \eta$ is a section of F_G° . Choose a second \mathfrak{t} -star section Y of F_G , $Y \sim_{\tilde{\mathfrak{t}}} \bar{Y}$ and $Y|_M = \bar{Y}$. An easy computation using $\eta(X) = \eta(Y) = 0$ yields

$$-2\mathbf{d}(\eta([X, Y])) = \mathcal{L}_X \mathcal{L}_Y \eta - \mathcal{L}_Y \mathcal{L}_X \eta - \mathcal{L}_{[X, Y]} \eta.$$

Using this and (3.5), a straightforward computation yields for $a \in \Gamma(A)$:

$$\begin{aligned} -2 \cdot a^r(\eta([X, Y])) &= (\mathcal{L}_X \mathcal{L}_Y \eta - \mathcal{L}_Y \mathcal{L}_X \eta - \mathcal{L}_{[X, Y]} \eta)(a^r) \\ &= \mathfrak{t}^* \left((\nabla_{\bar{X}}^* \nabla_{\bar{Y}}^* \bar{\eta} - \nabla_{\bar{Y}}^* \nabla_{\bar{X}}^* \bar{\eta})(\bar{a}) - [\bar{X}, \bar{Y}](\bar{\eta}(\bar{a})) \right) \\ &\quad + Y(\eta(Z_{a, X})) - X(\eta(Z_{a, Y})) - \eta \left(Z_{b_{a, X}, Y} + b_{b_{a, X}, Y}^r - Z_{b_{a, Y}, X} - b_{b_{a, Y}, X}^r \right) \\ &\stackrel{(3)}{=} \mathfrak{t}^* \left(\bar{\eta}(-\nabla_{[\bar{X}, \bar{Y}]} \bar{a} - \nabla_{\bar{Y}} \nabla_{\bar{X}} \bar{a} + \nabla_{\bar{X}} \nabla_{\bar{Y}} \bar{a}) \right) \stackrel{(3)}{=} 0. \end{aligned}$$

Hence, $a^r(\eta([X, Y])) = 0$ for all $a \in \Gamma(A)$ and since G is source-connected, this implies that $\eta([X, Y])(g) = \eta([X, Y])(\mathfrak{s}(g))$ for all $g \in G$. But since for $m \in M$, we have

$$[X, Y](m) = [\bar{X}, \bar{Y}](m)$$

and F_M is a subalgebroid of TM by (1), we find that $[X, Y](\mathfrak{s}(g)) \in F_G(\mathfrak{s}(g))$ for all $g \in G$ and hence $\eta([X, Y])(g) = \eta([X, Y])(\mathfrak{s}(g)) = 0$. Since η was a $\tilde{\mathfrak{t}}$ -section of F_G° and $\tilde{\mathfrak{t}}$ -sections of F_G° span F_G° on G , we have shown that $[X, Y] \in \Gamma(F_G)$ and the proof is complete. \square

Remark 3.9. (1) We have seen in this proof that Condition (2) implies the fact that K is a subalgebroid of A .

(2) The same result has been shown independently in [8], using Lie groupoid and Lie algebroid cocycles, in the special case where $F_M = TM$, i.e. where F_G is a wide subgroupoid of TG .

Example 3.10. Assume that G is a Lie group (hence with $M = \{e\}$) with Lie algebra \mathfrak{g} . Let F_G be a multiplicative distribution. In this case, the core $K =: \mathfrak{f}$ is the fiber of F_G over the identity and $F_M = 0$. As a consequence, any partial F_M -connection on $\mathfrak{g}/\mathfrak{f}$ is trivial. We check that all the conditions in Theorem 3.6 are automatically satisfied.

First of all, any element ξ of \mathfrak{g} is ∇ -flat. This implies that

$$[\xi^r, \Gamma(F_G)] \subseteq \Gamma(F_G) \quad \text{for all} \quad \xi \in \mathfrak{g},$$

i.e. F_G is left-invariant, in agreement with [29, 17, 19].

1), 3) and 4) are trivially satisfied and 2) is exactly the fact that \mathfrak{f} is an ideal in \mathfrak{g} . This recovers the results proved in [29, 17, 19]. Note also that all the conditions in Theorem 3.8 are trivially satisfied, hence a multiplicative distribution on a Lie group is always involutive.

Example 3.11. Let $G \rightrightarrows M$ be a Lie groupoid with a smooth, free and proper action of a Lie group H by Lie groupoid automorphisms. Let \mathcal{V}_G be the vertical space of the action, i.e. the smooth subbundle of TG that is generated by the infinitesimal vector fields ξ_G , for all $\xi \in \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H . The involutive subbundle \mathcal{V}_G is easily seen to be multiplicative (see for instance [19]).

The action restricts to a free and proper action of H on M , and it is easy to check that $\mathcal{V}_G \cap TM = \mathcal{V}_M$ is the vertical vector space of the action of H on M . Furthermore, $\mathcal{V}_G \cap T^s G = \mathcal{V}_G \cap T^t G = 0^{TG}$ and we get $K = 0^A$.

The infinitesimal vector fields (ξ_G, ξ_M) , $\xi \in \mathfrak{h}$, are multiplicative (in the sense of [27] for instance). We get hence from [27] that the Lie bracket $[a^r, \xi_G]$ is right-invariant for any $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. We obtain a map (see also [27])

$$\begin{aligned} \mathfrak{h} \times \Gamma(A) &\rightarrow \Gamma(A) \\ (\xi, a) &\mapsto [\xi_G, a^r]|_M \end{aligned} \quad ,$$

and we recover the connection

$$\nabla : \Gamma(\mathcal{V}_M) \times \Gamma(A) \rightarrow \Gamma(A)$$

defined by $\nabla_{\xi_M} a = [\xi_G, a^r]|_M$ for all $\xi \in \mathfrak{h}$ and $a \in \Gamma(A)$. This connection is obviously flat and satisfies all the conditions in Theorem 3.6.

Example 3.12. Let $(G \rightrightarrows M, J_G)$ be a complex Lie groupoid, i.e. a Lie groupoid endowed with a complex structure J_G that is multiplicative in the sense that the map

$$\begin{array}{ccc} TG & \xrightarrow{J_G} & TG \\ T\mathfrak{t} \downarrow \downarrow T\mathfrak{s} & & T\mathfrak{t} \downarrow \downarrow T\mathfrak{s} \\ TM & \xrightarrow{J_M} & TM \end{array}$$

is a Lie groupoid morphism over some map J_M . Since $J_G^2 = -\text{Id}_{TG}$, we conclude that $J_M^2 = -\text{Id}_{TM}$ and the Nijenhuis condition for J_M is easy to prove using \mathfrak{s} -related vector fields. The map J_G restricts also to a map j_A on the core A , i.e. a fiberwise complex structure that satisfies also a Nijenhuis condition. (This can be seen by noting that the Nijenhuis tensor of J_G restricts to right-invariant vector fields.)

The subbundles $T^{1,0}G = E_i$ and $T^{0,1}G = E_{-i}$ of $TG \otimes \mathbb{C}$ are multiplicative and involutive with bases $T^{1,0}M$ and $T^{0,1}M$ and cores $A^{1,0}$ and $A^{0,1}$. The quotient $(A \otimes \mathbb{C})/A^{1,0}$ is isomorphic as a vector bundle to $A^{0,1}$ and a straightforward computation shows that the connection that we get from the multiplicative involutive complex distribution $T^{1,0}G$ is exactly the connection $\nabla : \Gamma(T^{1,0}M) \times \Gamma(A^{0,1}) \rightarrow \Gamma(A^{0,1})$ as in Lemma 4.7 of [22].

Since the flat sections of the connection in this Lemma are exactly the holomorphic sections of $A^{0,1}$, one can reconstruct the map $J_A : TA \rightarrow TA$ defined by $J_A = \sigma^{-1} \circ A(J_G) \circ \sigma$ ⁴ as in [23] by requiring that $J_A(Ta) = Ta \circ J_M$ for all flat sections $a \in \Gamma(A)$, and $J_A(\hat{b}) = \widehat{j_A(b)} \circ J_M$ for all sections $b \in \Gamma(A)$.

By Lemma 4.7 in [22] and the integration results in [23], the complex structure J_G is hence equivalent to the datum (J_M, j_A, ∇) with its properties. This is in agreement with the results that we will prove in the next section.

4. FOLIATED ALGEBROIDS

In this section we study Lie algebroids equipped with distributions compatible with both Lie algebroid structures $TA \rightarrow TM$ and $TA \rightarrow A$ on TA . This is the first step towards an infinitesimal description of foliated groupoids.

4.1. Definition and properties.

Definition 4.1. Let $A \rightarrow M$ be a Lie algebroid. A subbundle $F_A \subseteq TA$ over A is called **morphic** if it is a Lie subalgebroid of $TA \rightarrow TM$ over some subbundle $F_M \subseteq TM$.

If F_A is involutive and morphic, then the pair (A, F_A) is called for simplicity a **foliated Lie algebroid**.

Since foliated algebroids have already been shown in [14, 9] to correspond to infinitesimal ideal systems, we only summarize here the approach in [9], see also [18]. (Note that another approach could be found in a former version of this paper.)

⁴ To avoid confusions, we write in this example $\sigma : TA \rightarrow A(TG)$ for the canonical flip map.

4.1.1. *Connections on a vector bundle A , linear splittings of TA and the Lie bracket on $\mathfrak{X}(A)$.* We recall here the relation between a linear connection on a vector bundle A and the Lie bracket of vector fields on A .

Let $q_A: A \rightarrow M$ be a vector bundle. A linear vector field on A is a derivation of $C^\infty(A)$ that sends linear functions to linear functions and pullbacks to pullbacks. More explicitly, $\tilde{X} \in \mathfrak{X}(A)$ is linear over $X \in \mathfrak{X}(M)$ if for all $\xi \in \Gamma(A^*)$, $X(\ell_\xi) = \ell_{D_{\tilde{X}}^* \xi}$ with $D_{\tilde{X}}^* \xi \in \Gamma(A^*)$ and for all $f \in C^\infty(M)$, $\tilde{X}(q_A^* f) = q_A^*(X(f))$. Hence, a linear vector field \tilde{X} which is q_A -related to $X \in \mathfrak{X}(M)$ defines a derivation $D_{\tilde{X}}^*: \Gamma(A^*) \rightarrow \Gamma(A^*)$ with symbol X . The dual derivation $D_{\tilde{X}}: \Gamma(A) \rightarrow \Gamma(A)$ describes then the Lie bracket of \tilde{X} with vertical vector fields:

$$(4.6) \quad [\tilde{X}, a^\dagger] = (D_{\tilde{X}} a)^\dagger$$

for all $a \in \Gamma(A)$. The Lie bracket $[\tilde{X}, \tilde{Y}]$ of two linear vector fields \tilde{X} and $\tilde{Y} \in \mathfrak{X}(A)$ over X and $Y \in \mathfrak{X}(M)$ is again linear over $[X, Y]$ and the derivation $D_{[\tilde{X}, \tilde{Y}]}$ is equal to the commutator of the derivations $D_{\tilde{X}}$ and $D_{\tilde{Y}}$.

Let $\nabla: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ be a connection. For each $X \in \mathfrak{X}(M)$, ∇_X is a derivation of $\Gamma(A)$ and we have a corresponding linear vector field $\widetilde{\nabla_X}$ over X . The set of all vector fields on A defined in this manner spans a linear subbundle H_∇ of $p_A: TA \rightarrow A$ that is in direct sum with the vertical space $V := T^{q_A} A = \{v_{a_m} \in TA \mid T_{a_m} q_A v_{a_m} = 0\}$:

$$TA \cong V \oplus H_\nabla \rightarrow A.$$

Note that V is spanned by the vertical vector fields on A (see (1.1)).

For all functions $\varphi \in C^\infty(M)$ and sections $\xi \in \Gamma(A^*)$, we have

$$(4.7) \quad \widetilde{\nabla_X}(\ell_\xi) = \ell_{\nabla_X^* \xi}, \quad \widetilde{\nabla_X}(q_A^* \varphi) = q_A^*(X(\varphi)), \quad b^\dagger(\ell_\xi) = q_A^*(\xi, b), \quad b^\dagger(q_A^* \varphi) = 0.$$

Conversely, consider a linear splitting $TA \cong V \oplus H$ of $TA \rightarrow A$. Then, since $H \cong TA/V$ is isomorphic to the pullback $q_A^! TM \rightarrow A$, and by the linearity of H , we find for each vector field $X \in \mathfrak{X}(M)$ a unique linear vector field $\tilde{X} \in \Gamma(H)$ such that $\tilde{X} \sim_{q_A} X$. One can then define a connection $\nabla^H: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ by setting

$$\nabla_X^H = D_{\tilde{X}}$$

for all $X \in \mathfrak{X}(M)$.

This shows the correspondence of the two definitions of a connection; the first as the map

$$\nabla: \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A),$$

the second as a linear splitting

$$TA \cong V \oplus H \rightarrow A.$$

Given ∇ or H_∇ , it is easy to see using the equalities in (4.7) that $[\tilde{X}, \tilde{Y}] = \widetilde{[X, Y]} - R_\nabla(X, Y)^\dagger$, $[\tilde{X}, a^\dagger] = (\nabla_X a)^\dagger$, and $[a^\dagger, b^\dagger] = 0$ for all $X, Y \in \mathfrak{X}(M)$ and $a, b \in \Gamma(A)$. Here, $R_\nabla(X, Y)^\dagger \in \mathfrak{X}(A)$ is defined by $R_\nabla(X, Y)^\dagger(a_m) = (R_\nabla(X, Y)a_m)^\dagger$ for all $a_m \in A$. That is, the Lie bracket of vector fields on A can completely be described in terms of the connection.

4.1.2. *The Lie bracket on sections of $TA \rightarrow TM$.* Consider now a connection ∇ on a Lie algebroid A . Then we can define sections $\tilde{a} \in \Gamma_{TM}(H_\nabla)$, for $a \in \Gamma(A)$, by

$$\tilde{a}(v_m) = T_m a v_m - \left. \frac{d}{dt} \right|_{t=0} a_m + t \nabla_{v_m} a$$

for all $v_m \in TM$. Recall that for $a \in \Gamma(A)$, we also have the core section a^\dagger of $\ker(p_A) \rightarrow TM$:

$$a^\dagger(v_m) = T_m 0^A v_m + \left. \frac{d}{dt} \right|_{t=0} ta(m).$$

The vector bundle $TA \rightarrow TM$ is spanned by the sections \tilde{a} and a^\dagger for all $a \in \Gamma(A)$ and the Lie algebroid structure on $TA \rightarrow TM$ can be described as follows:

- (1) $[\tilde{a}, \tilde{b}] = \widehat{[a, b]} - R_{\Delta}^{\text{bas}}(a, b)^\dagger$,
- (2) $[\tilde{a}, b^\dagger] = (\nabla_a^{\text{bas}} b)^\dagger$,
- (3) $[a^\dagger, b^\dagger] = 0$,
- (4) $\rho_{TA}(\tilde{a}) = \widehat{\nabla_a^{\text{bas}}}$ $\in \mathfrak{X}(TM)$,
- (5) $\rho_{TA}(b^\dagger) = (\rho(b))^\dagger \in \mathfrak{X}(TM)$,

where the linear connections $\nabla^{\text{bas}} : \Gamma(A) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ and $\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ are given by

$$\nabla_a^{\text{bas}} X = \rho(\nabla_X a) + [\rho(a), X]$$

for all $a \in \Gamma(A)$ and $X \in \mathfrak{X}(M)$ and

$$\nabla_a^{\text{bas}} b = \nabla_{\rho(b)} a + [a, b]$$

for all $b \in \Gamma(A)$ [10, 11], and $R_{\nabla}^{\text{bas}} \in \Omega^2(A, \text{Hom}(A, TM))$ by

$$R_{\nabla}^{\text{bas}}(a, b)(X) = -\nabla_X[a, b] + [\nabla_X a, b] + [a, \nabla_X b] + \nabla_{\nabla_b^{\text{bas}} X} a - \nabla_{\nabla_a^{\text{bas}} X} b.$$

4.1.3. *Double subbundles of the double vector bundle $(TA; TM, A; M)$.* We consider now a double subbundle $(F_A; F_M, A; M)$ of $(TA; TM, A; M)$. Then, by [31], there exists a *linear* splitting s of the short exact sequence

$$(4.8) \quad 0 \rightarrow q_A^1 K \rightarrow F_A \rightarrow q_A^1 F_M \rightarrow 0$$

of vector bundles over A^5 . That is, the vector bundle morphism $s : q_A^1 F_M \rightarrow F_A$ (over the identity on A) satisfies as well $s(v_m, a_m + b_m) = s(v_m, a_m) +_{F_A \rightarrow F_M} s(v_m, b_m)$ for all $v_m \in F_M(m)$ and $a_m, b_m \in A_m$.⁶ From this follows the existence for each $X \in \Gamma(F_M)$ of a linear vector field \tilde{X} over X with values in F_A :

$$\tilde{X}(a_m) = s(X(m), a_m)$$

for all $a_m \in A$. This defines a linear connection $\tilde{\nabla} : \Gamma(F_M) \times \Gamma(A) \rightarrow \Gamma(A)$ of the anchored vector bundle F_M^7 : $\nabla_X = D_{\tilde{X}}$ for all $X \in \Gamma(F_M)$. Extend $\tilde{\nabla}$ to a TM -connection ∇ on A . The linear connection ∇ is then *adapted* to F_A . That is, F_A splits as

$$(F_A \cap H_{\nabla}) \oplus (F_A \cap V)$$

as a vector bundle over A , and as

$$(F_A \cap H_{\nabla}) \oplus (F_A \cap \ker(p_A))$$

as a vector bundle over F_M . Furthermore, there exists a subbundle $K \subseteq A$ (the *core* of F_A) such that $F_A \cap V$ is spanned by vertical vector fields k^\dagger for all $k \in \Gamma(K)$.

Note that $F_A \cap H_{\nabla} \rightarrow A$ is then spanned by the sections $\tilde{X} = \widehat{\nabla_X}$ for all $X \in \Gamma(F_M)$, $F_A \cap H_{\nabla} \rightarrow F_M$ by the sections $\tilde{a}|_{F_M}$ for all $a \in \Gamma(A)$, $F_A \cap V \rightarrow A$ by the sections k^\dagger for all $k \in \Gamma(K)$ and $F_A \cap \ker(p_A) \rightarrow F_M$ by $a^\dagger|_{F_M}$ for all $a \in \Gamma(K)$.

This leads easily to the following two propositions.

Proposition 4.2. *The subbundle $F_A \rightarrow A$ of $TA \rightarrow A$ is involutive if and only if, for an adapted connection ∇ ,*

- (1) F_M is involutive,
- (2) $\nabla_X a \in \Gamma(K)$ for all $X \in \Gamma(F_M)$ and $a \in \Gamma(K)$,

⁵The first map sends (k_m, a_m) to $\frac{d}{dt} \Big|_{t=0} a_m + tk_m$ and the second map is $q_A^1(Tq_A)|_{F_A}$.

⁶Note that $s(v_m + w_m, a_m) = s(v_m, a_m) +_{F_A \rightarrow A} s(w_m, a_m)$ since s is a vector bundle morphism over A . Here, we denote by $+_{F_A \rightarrow A}$ the addition of F_A as a vector bundle over A and by $+_{F_A \rightarrow F_M}$ the addition of F_M as a vector bundle over F_M .

⁷Note that as long as one does not consider curvature, one does not need a Lie algebroid bracket on $\Gamma(F_M)$ to define an F_M -connection.

(3) and the induced connection $\bar{\nabla} : \Gamma(F_M) \times \Gamma(A/K) \rightarrow \Gamma(A/K)$ is flat.

Proposition 4.3. *The subbundle $F_A \rightarrow F_M$ of $TA \rightarrow TM$ is a subalgebroid if and only if*

- (1) $\rho(K) \subseteq F_M$,
- (2) $\nabla_a^{\text{bas}} X \in \Gamma(F_M)$ for all $a \in \Gamma(A)$ and $X \in \Gamma(F_M)$,
- (3) $\nabla_a^{\text{bas}} b \in \Gamma(K)$ for all $a \in \Gamma(A)$ and $b \in \Gamma(K)$,
- (4) and $R_{\bar{\nabla}}^{\text{bas}}(a, b)(X) \in \Gamma(K)$ for all $a, b \in \Gamma(A)$ and $X \in \Gamma(K)$.

Corollary 4.4. *The subbundle F_A is involutive and a Lie algebroid over F_M if and only if for any adapted connection ∇ , $\nabla_X a \in \Gamma(K)$ for all $X \in \Gamma(F_M)$ and $a \in \Gamma(K)$ and the induced connection $\bar{\nabla} : \Gamma(F_M) \times \Gamma(A/K) \rightarrow \Gamma(A/K)$ defines an infinitesimal ideal system $(F_M, K, \bar{\nabla})$ in A .*

Since for any two connections ∇ and ∇' that are adapted to F_A , the difference $\nabla - \nabla'$ satisfies $(\nabla - \nabla')_{v_m} a \in K(m)$ for all $v_m \in F_M$ and $a \in \Gamma(A)$, the induced connection $\bar{\nabla}$ does not depend on the choice of the adapted connection ∇ . The connection $\bar{\nabla}$ can be defined intrinsically as follows: for any linear vector field $\tilde{X} \in \Gamma(F_A)$ over $X \in \Gamma(F_M)$ and any section $a \in \Gamma(A)$:

$$\nabla_{\tilde{X}}^{F_A} \overline{a^\uparrow} = \overline{[\tilde{X}, a^\uparrow]}$$

is again a vertical section and equals $(\bar{\nabla}_X a)^\uparrow$, where vertical sections of TA/F_A are identified with sections of A/K and ∇^{F_A} is the Bott-connection. We have the following theorem.

Theorem 4.5. *Let A be a Lie algebroid. Morphic involutive subbundles of TA are in one-to-one correspondence with infinitesimal ideal systems in A .*

Remark 4.6. Let (A, F_A) be a foliated algebroid and (F_M, K, ∇) the corresponding infinitesimal ideal system in A . Let X be a linear section of F_A over $\tilde{X} \in \Gamma(F_M)$. Then

$$\nabla_{\tilde{X}} \bar{a} = 0 \quad \Leftrightarrow \quad D_X a \in \Gamma(K)$$

for $a \in \Gamma(A)$.

Example 4.7. Let H be a connected Lie group with Lie algebra \mathfrak{h} . Assume that H acts on a Lie algebroid $A \rightarrow M$ in a free and proper manner, by Lie algebroid automorphisms. Consider the vertical spaces $\mathcal{V}_A, \mathcal{V}_M$ defined as follows

$$\mathcal{V}_A(a) = \{\xi_A(a) \mid \xi \in \mathfrak{h}\}, \quad \mathcal{V}_M(m) = \{\xi_M(m) \mid \xi \in \mathfrak{h}\},$$

for $a \in A$ and $m \in M$. We check that \mathcal{V}_A inherits a Lie algebroid structure over \mathcal{V}_M making the pair (A, \mathcal{V}_A) into a foliated algebroid with core zero. Choose $\xi_A(a_m) \in \mathcal{V}_A(a_m)$ for some $\xi \in \mathfrak{g}$, then $T_{a_m} q_A \xi_A(a_m) = \xi_M(m) \in \mathcal{V}_M(m)$. Hence, if $T_{a_m} q_A \xi_A(a_m) = 0$, then $\xi = 0$ since the action is supposed to be free and we find $K = 0$. Notice that, since the action is by algebroid automorphisms, each infinitesimal generator ξ_A is in fact a morphic vector field covering ξ_M .

If $\rho(a_m) = \dot{c}(0)$, a simple computation, using $\rho \circ \Phi_{\exp(t\xi)} = T\phi_{\exp(t\xi)} \circ \rho$ for all $t \in \mathbb{R}$, yields $\rho_{TA}(\xi_A(a_m)) = \frac{d}{ds} \Big|_{s=0} \xi_M(c(s)) \in T_{\xi_M(m)} \mathcal{V}_M$. If $a \in \Gamma(A)$ is such that $T_m a(\xi_M(m)) \in \mathcal{V}_A(a_m)$ for some $\xi \in \mathfrak{g}$ and $m \in M$, then there exists $\eta \in \mathfrak{g}$ such that $T_m a(\xi_M(m)) = \eta_A(a(m))$. But applying $T_{a_m} q_A$ to both sides of this equality yields then $\xi_M(m) = \eta_M(m)$, which leads to $\xi = \eta$, since the action is free, and hence $T_m a(\xi_M(m)) = \xi_A(a(m))$.

Here, the induced partial \mathcal{V}_M -connection ∇ is defined on A by

$$[\xi_A, a^\uparrow] = (\nabla_{\xi_M} a)^\uparrow$$

for any $a \in \Gamma(A)$. If $a \in \Gamma(A)$ is ∇ -flat, then we find $[\xi_A, a^\uparrow] = 0$ for all $\xi \in \mathfrak{g}$ and hence the flows commute, which leads to

$$\Phi_{\exp(t\xi)}(b_m) + s \cdot a(\phi_{\exp(t\xi)}(m)) = \Phi_{\exp(t\xi)}(b_m) + s \cdot \Phi_{\exp(t\xi)}(a(m))$$

for all $b_m \in A$ and $s, t \in \mathbb{R}$ and hence to

$$a(\phi_{\exp(t\xi)}(m)) = \Phi_{\exp(t\xi)}(a(m)).$$

Since H is assumed to be connected, this yields $a \circ \phi_h = \Phi_h \circ a$ for all $h \in H$.

Because the action is by Lie algebroid morphisms, we find then that the Lie algebroid bracket of ∇ -flat sections $a, b \in \Gamma(A)$ is again ∇ -flat. Since the core sections of \mathcal{V}_A are all trivial, this shows that the Lie bracket of $TA \rightarrow TM$ restricts to $\mathcal{V}_A \rightarrow \mathcal{V}_M$.

4.2. The Lie algebroid of a multiplicative involutive distribution. The following construction can be found in [30] in the more general setting of multiplicative Dirac structures.

Let F_G be a multiplicative subbundle of TG with space of units $F_M \subseteq TM$. Since $F_G \subseteq TG$ is a Lie subgroupoid, we can apply the Lie functor, leading to a Lie subalgebroid $A(F_G) \subseteq A(TG)$ over $F_M \subseteq TM$.

As we have seen in Subsection 2.2, the canonical involution $J_G : TTG \rightarrow TTG$ restricts to an isomorphism of double vector bundles $j_G : TA \rightarrow A(TG)$ inducing the identity map on both the side bundles and the core. Since $j_G : TA \rightarrow A(TG)$ is an isomorphism of Lie algebroids over TM , we conclude that

$$F_A := j_G^{-1}(A(F_G)) \subseteq TA$$

is a Lie algebroid over $F_M \subseteq TM$. Since

$$\begin{array}{ccc} F_G & \xrightarrow{p_G} & G \\ Tt \downarrow \parallel & & \downarrow \parallel Ts \\ F_M & \xrightarrow{p_M} & M \end{array}$$

is a VB-subgroupoid of

$$\begin{array}{ccc} TG & \xrightarrow{p_G} & G \\ Tt \downarrow \parallel & & \downarrow \parallel Ts \\ TM & \xrightarrow{p_M} & M \end{array},$$

the Lie algebroid

$$\begin{array}{ccc} A(F_G) & \xrightarrow{A(p_G)} & A \\ \downarrow & & \downarrow \\ F_M & \xrightarrow{p_M} & M \end{array}$$

is a VB-subalgebroid of

$$\begin{array}{ccc} A(TG) & \xrightarrow{A(p_G)} & A \\ \downarrow & & \downarrow \\ TM & \xrightarrow{p_M} & M \end{array}$$

([4]), and $F_A \rightarrow A$ is also a subbundle of $TA \rightarrow A$. The main theorem of this subsection is the following.

Theorem 4.8. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid with core K . Then $(A, F_A = j_G^{-1}(A(F_G)))$ is a foliated algebroid with core K .*

Conversely, let (A, F_A) be a foliated Lie algebroid. Assume that A integrates to a source simply connected Lie groupoid $G \rightrightarrows M$. Then there is a unique multiplicative distribution F_G on G such that $F_A = j_G^{-1}(A(F_G))$.

We will use a result of [4], which states that a VB-algebroid

$$\begin{array}{ccc} E & \xrightarrow{q_E^h} & A \\ q_E^v \downarrow & & \downarrow q_A \\ B & \xrightarrow{q_B} & M, \end{array}$$

whenever integrable as a VB-algebroid, it integrates to a VB-groupoid

$$\begin{array}{ccc} G(E) & \xrightarrow{q_{G(E)}} & G(A) \\ \Downarrow & & \Downarrow \\ B & \xrightarrow{q_B} & M \end{array}$$

Furthermore, if $E' \hookrightarrow E$, $B' \hookrightarrow B$ is a VB-subalgebroid with the same horizontal base A ,

$$\begin{array}{ccc} E' & \xrightarrow{q_{E'}^h} & A \\ q_{E'}^v \downarrow & & \downarrow q_A \\ B' & \xrightarrow{q_{B'}} & M \end{array},$$

then $E' \rightarrow B'$ integrates to an embedded VB-subgroupoid $G(E') \hookrightarrow G(E)$ over $B' \hookrightarrow B$,

$$\begin{array}{ccc} G(E') & \xrightarrow{q_{G(E')}} & G(A) \\ \Downarrow & & \Downarrow \\ B' & \xrightarrow{q_{B'}} & M \end{array}$$

This is done in [4] using the characterization of vector bundles via homogeneous structures (see [12]). The problem of the integrability of VB-algebroids, that is, the existence of a groupoid integrating a given VB-algebroid, is treated in [3], where explicit obstructions to the integrability of VB-algebroid are shown. Notice that if A is a Lie algebroid which integrates to a source simply connected Lie groupoid G , then the tangent algebroid $TA \rightarrow TM$ is an integrable VB-algebroid. The source simply connected integration of TA is the tangent groupoid $TG \rightrightarrows TM$. In particular, VB-subalgebroids of TA are always integrable.

Proof of Theorem 4.8. The first part has been shown above. Recall from Appendix A the construction of $A(F_G) \rightarrow F_M$ and $j_G(A(F_G)) = F_A \rightarrow F_M$. Since $j_G \circ \tilde{a} = a^\dagger$, we find immediately that the core sections of F_A are $j_G \circ \tilde{a}$ for all $a \in \Gamma(K)$.

Let (A, F_A) be a foliated Lie algebroid with core K . The VB-subgroupoid of $(TG, G; TM, M)$ integrating the subalgebroid $j_G(F_A) \rightarrow F_M$ of $A(TG) \rightarrow TM$ is a multiplicative subbundle $F_G \rightrightarrows F_M$ of $TG \rightrightarrows TM$ with core K . By Theorem 3.8, F_G is involutive. \square

Theorem 4.9. *Let $(G \rightrightarrows M, F_G)$ be a foliated groupoid and (A, F_A) the corresponding foliated algebroid. Then the infinitesimal ideal systems defined by $(G \rightrightarrows M, F_G)$ and (A, F_A) coincide.*

Proof. Since the Lie algebroid of $F_G \rightrightarrows F_M$ is a subalgebroid of $TA \rightrightarrows TM$ over F_M , the two involutive subbundles of TM coincide. We write (F_M, K, ∇^G) for the infinitesimal ideal

system defined by $(G \rightrightarrows M, F_G)$ and (F_M, K, ∇^A) for the infinitesimal ideal system defined by (A, F_A) . We have to show $\nabla^G = \nabla^A$.

Choose a ∇^G -flat section $a \in \Gamma(A)$. Then $a^r \in \mathfrak{X}(G)$ is ∇^{F_G} -flat, or, in other words, a^r preserves F_G . For $v \in F_M$, we have hence $T \text{Exp}(ta)v \in F_G$ for all $t \in \mathbb{R}$ where this makes sense, and so $\beta_A(v) \in A(F_G)$. This shows that $\beta_a|_{F_M}$ is a section of $A(F_G)$, and so that $Ta|_{F_M}$ is a section of $F_A \rightarrow F_M$ (see Appendix A). As a consequence, we find $\tilde{\nabla}_X a \in \Gamma(K)$ for any connection $\tilde{\nabla}$ adapted to F_A and any $X \in \Gamma(F_M)$, and this finally leads to $\nabla_X^A a = 0$.

We have thus shown

$$\{a \in \Gamma(A) \mid a \text{ } \nabla^G\text{-flat}\} \subseteq \{a \in \Gamma(A) \mid a \text{ } \nabla^A\text{-flat}\}.$$

Since both connections are flat, it is easy to conclude from this that they have the same sets of flat sections. Again by the flatness of the connections, one finds then that $\nabla^G = \nabla^A$. \square

As a corollary of this and Theorem 4.5, we get the following result.

Corollary 4.10. *Let $G \rightrightarrows M$ be a source-simply connected Lie groupoid with Lie algebroid $A \rightarrow M$. Then multiplicative involutive distributions on G are in one-to-one correspondence with infinitesimal ideal systems in A .*

Example 4.11. Assume that H acts on a Lie groupoid G over M by groupoid automorphisms. Assume also that the action is free and proper. Starting from the data $(A, \mathcal{V}_M, 0, \nabla)$ where ∇ is the partial \mathcal{V}_M -connection on A determined by

$$[\xi_G, a^r] = (\nabla_{\xi_M} a)^r$$

for all $\xi \in \mathfrak{g}$ and $a \in \Gamma(A)$, the last theorem states that we recover exactly the foliated Lie algebroid $\mathcal{V}_A \rightarrow \mathcal{V}_M$ obtained by applying the Lie functor to the foliated groupoid $\mathcal{V}_G \rightrightarrows \mathcal{V}_M$.

Example 4.12. Assume that \mathfrak{g} is a Lie algebra, i.e. a Lie algebroid over a point. In this case, the tangent Lie algebroid $T\mathfrak{g}$ is also a Lie algebroid over a point, that is, $T\mathfrak{g}$ is a Lie algebra. It is easy to see that the Lie algebra structure on $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ is the semi-direct product Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation of \mathfrak{g} on itself. Note also that the fact that a triple $(0, \mathfrak{f}, \nabla = 0)$ is an (infinitesimal) ideal system on \mathfrak{g} is equivalent to saying that $\mathfrak{f} \subseteq \mathfrak{g}$ is an ideal.

The morphic involutive distribution $F_{\mathfrak{g}}$ associated to the infinitesimal ideal system $(0, \mathfrak{f}, 0)$ is given by $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$. The property that $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$ is a morphic involutive distribution is equivalent to saying that $\mathfrak{g} \times \mathfrak{f}$ is a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{g}$. In particular, if G is the connected and simply connected Lie group integrating \mathfrak{g} , we conclude that the foliated algebroid $F_{\mathfrak{g}} = \mathfrak{g} \times \mathfrak{f}$ integrates to a Lie subgroup $G \times \mathfrak{f}$ of the semi-direct Lie group $G \ltimes \mathfrak{g}$ determined by the adjoint action of G on its Lie algebra \mathfrak{g} . Using right (or left) translations, we get a subbundle $F_G \subseteq TG$ which is involutive and multiplicative. Thus, in the case of Lie groups and Lie algebras, this recovers the results in [29, 17, 19].

5. EXAMPLES OF INFINITESIMAL IDEAL SYSTEMS

In this section, we present several natural examples of ideal systems. These examples show that the infinitesimal ideal systems are the object that should be considered as the right notion of ideals in Lie algebroids.

5.1. Regular Dirac structures and the kernel of the associated presymplectic groupoids. Let (M, \mathbf{D}) be a Dirac manifold. Recall that $(\mathbf{D} \rightarrow M, \text{pr}_{TM}, [\cdot, \cdot])$ is then a Lie algebroid, where $[\cdot, \cdot]$ is the Courant Dorfman bracket on sections of $TM \oplus T^*M$.

Assume that the characteristic distribution $F_M \subseteq TM$, defined by

$$F_M(m) = \{v_m \in T_m M \mid (v_m, 0_m) \in \mathbf{D}(m)\}$$

for all $m \in M$, is a subbundle of TM . The involutivity of F_M follows from the properties of the Dirac structure. Set $K := F_M \oplus \{0\} \subseteq \mathbf{D}$. It is easy to check that K is a subalgebroid of \mathbf{D} . Define

$$\begin{aligned} \nabla : \Gamma(F_M) \times \Gamma(\mathbf{D}/K) &\rightarrow \Gamma(\mathbf{D}/K) \\ \nabla_X \bar{d} &= \overline{[(X, 0), d]}. \end{aligned}$$

This map is easily seen to be a well-defined, flat, partial F_M -connection on \mathbf{D}/K , and the verification of the fact that $(\mathbf{D}, F_M, K, \nabla)$ is an infinitesimal ideal system on the Lie algebroid $\mathbf{D} \rightarrow M$ is straightforward.

We show that if $\mathbf{D} \rightarrow M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$ [6, 5], then $(\mathbf{D}, F_M, K, \nabla)$ integrates to the involutive subbundle $F_G = \ker \omega_G \subseteq TG$.

The map $\rho := \text{pr}_{TM} : \mathbf{D} \rightarrow TM$ is the anchor of the Dirac structure \mathbf{D} viewed as a Lie algebroid over M , and the map $\sigma := \text{pr}_{T^*M} : \mathbf{D} \rightarrow T^*M$ defines an IM-2-form on the Lie algebroid \mathbf{D} (see [6], [5]). Note that K is the kernel of σ and F_M is the kernel of $\sigma^t : TM \rightarrow \mathbf{D}^*$. The two-form $\Lambda := \sigma^* \omega_{\text{can}} \in \Omega^2(\mathbf{D})$ is morpich in the sense that

$$\begin{array}{ccc} TD & \xrightarrow{\Lambda^\sharp} & T^*\mathbf{D} \\ \downarrow & & \downarrow \\ TM & \xrightarrow{-\sigma^t} & \mathbf{D}^* \end{array}$$

is a Lie algebroid morphism ([5]). See, for instance [25], for the Lie algebroid structure on $T^*\mathbf{D} \rightarrow \mathbf{D}^*$. If $\mathbf{D} \rightarrow M$ integrates to a presymplectic groupoid $(G \rightrightarrows M, \omega_G)$, the Lie algebroid $T^*\mathbf{D} \rightarrow \mathbf{D}^*$ is isomorphic to the Lie algebroid of the cotangent groupoid $T^*G \rightarrow \mathbf{D}^*$ and the map Λ^\sharp integrates via the identifications $TD \simeq A(TG)$ and $T^*\mathbf{D} \simeq A(T^*G)$ to the vector bundle map ω_G^\sharp , that is a Lie groupoid morphism. See [5] for more details.

We show that the morpich involutive distribution $F_{\mathbf{D}} \subseteq TD$ corresponding to $(\mathbf{D}, F_M, K, \nabla)$ is equal to the kernel of Λ^\sharp .

Let n be the dimension of M and k the rank of F_M . Then \mathbf{D} is spanned locally by frames of n flat sections, the first k of them spanning K . If d is a flat section of \mathbf{D} , we have $\mathcal{L}_X d \in \Gamma(K)$ for all $X \in \Gamma(F_M)$, that is, $\mathcal{L}_X(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Since $\mathbf{i}_X \sigma(d) = 0$ for all $X \in \Gamma(F_M)$, this yields $\mathbf{i}_X \mathbf{d}(\sigma(d)) = 0$ for all $X \in \Gamma(F_M)$. Hence, using this type of frames, we find with formulas (4.57) and (4.58) in [5], that the kernel of Λ^\sharp is spanned by the restriction to F_M of the linear sections defined by flat sections of \mathbf{D} , and by the restrictions to F_M of the core sections defined by sections of K . Hence, by construction, the distribution $F_{\mathbf{D}}$ is the kernel of Λ^\sharp .

Since the kernel of ω_G^\sharp is multiplicative with Lie algebroid equal to the kernel of Λ^\sharp , this yields $F_G = \ker \omega_G^\sharp$.

Note that if $F_M \subseteq TM$ is simple, then the leaf space M/F_M has a natural Poisson structure such that the projection $(M, \mathbf{D}) \rightarrow (M/F_M, \pi)$ is a forward Dirac map. Under a completeness condition and if $F_G \subseteq TG$ is also simple, we get a Lie groupoid $G/F_G \rightrightarrows M/F_M$, with a natural symplectic structure ω such that the projection $\pi_G : G \rightarrow G/F_G$ satisfies $\pi_G^* \omega = \omega_G$. It would be interesting to study the relation between the integrability of the Poisson manifold $(M/F_M, \pi)$ and the completeness conditions on F_G (see [19]) so that the quotient $(G/F_G \rightrightarrows M/F_M, \omega)$ is a symplectic groupoid.

Note finally that, under the obvious smoothness and trivial holonomy conditions, the quotient of the Lie algebroid D by the infinitesimal ideal system (F_M, K, ∇) is the Lie algebroid $D_\pi = \text{graph}(\pi^\sharp : T^*(M/F_M) \rightarrow T(M/F_M))$.

5.2. Foliated algebroids in the sense of Vaisman. In [37], foliated Lie algebroids are defined as follows. A foliated Lie algebroid is a Lie algebroid $A \rightarrow M$ together with a subalgebroid B of A and an involutive subbundle $F_M \subseteq TM$ such that

- (1) $\rho(B) \subseteq F_M$,
- (2) A is locally spanned over $C^\infty(M)$ by B -foliated cross sections, i.e. sections a of A such that $[a, b] \in \Gamma(B)$ for all $b \in B$.

Recall our definition of infinitesimal ideal system on a Lie algebroid (Definition 1.1). Since the F_M -partial connection is flat, we get by Proposition 2.4 the existence of frames of flat sections for A . By the properties of the connection, these are K -foliated cross sections. Since (1) is also satisfied by hypothesis, our infinitesimal ideal systems are foliated Lie algebroids in the sense of Vaisman if we set $B := K$.

Conversely, take a Lie algebroid A over a smooth manifold M such that there exists an ideal I (in the sense of the next subsection) in A and an involutive subbundle $F_M \subseteq TM$. Then the triple (A, I, F_M) is a foliated algebroid in the sense of Vaisman, but does not (in general) define an infinitesimal ideal system.

The object that integrates the foliated algebroid in the sense of Vaisman is the right invariant image of B , which defines a subbundle of TG that is tangent to the \mathfrak{s} -fibers and invariant under left multiplication (and sent by Tt to a subset of F_M). This is exactly the intersection of our multiplicative subbundle $F_G \subseteq TG$, integrating (F_M, K, ∇) , with T^sG .

5.3. The usual notion of ideals in Lie algebroids. An ideal I in a Lie algebroid $A \rightarrow M$ is a subbundle over M such that $[a, i] \in \Gamma(I)$ for all $i \in \Gamma(I)$ and all $a \in \Gamma(A)$. The inclusion $I \subseteq \ker(\rho)$ follows immediately and shows that this definition of an ideal is very restrictive. In the other hand, usual ideals correspond obviously to the infinitesimal ideal systems $(F_M = 0, K = I, \nabla = 0)$ in A . Note that in this case, the quotient Lie algebroid A/I over $M/F_M = M$ is always defined. This is a trivial class of example for the results in the next section.

5.4. The Bott connection and reduction by simple foliations. The second standard example of a Lie algebroid is the tangent bundle TM of a smooth manifold M , endowed with the usual Lie bracket of vector fields and the identity Id_{TM} as anchor. Consider an involutive subbundle $F_M \subseteq TM$ and the Bott connection $\nabla^{F_M} : \Gamma(F_M) \times \Gamma(TM/F_M) \rightarrow \Gamma(TM/F_M)$ associated to it. Then it is straightforward to check that the triple (F_M, F_M, ∇^{F_M}) is an infinitesimal ideal system in TM .

This infinitesimal ideal system corresponds to the subbundle of TM given by the tangent lift of F_M . The foliated groupoid associated to this infinitesimal ideal system is $(M \times M \rightrightarrows M, F_M \times F_M \rightrightarrows F_M)$ (we assume here for simplicity that M is simply connected).

5.5. The infinitesimal ideal system associated to a fibration of Lie algebroids. Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ q_A \downarrow & & \downarrow q_{A'} \\ M & \xrightarrow{f} & M' \end{array}$$

be a fibration of Lie algebroids, i.e. the map f is a surjective submersion and $f^!\varphi : A \rightarrow \varphi^!A'$ is a surjective vector bundle morphism over the identity on A .

Then $K := \ker(\varphi) \subseteq A$, i.e. $K(m) = \left\{ a_m \in A_m \mid \varphi(a_m) = 0_{f(m)}^{A'} \right\}$ is a subalgebroid of A and $T^f M \subseteq TM$ is an involutive subbundle. The equality $Tf \circ \rho = \rho' \circ \varphi$ yields immediately $\rho(K) \subseteq F_M$.

Define a connection $\nabla^\varphi : \Gamma(T^f M) \times \Gamma(A/K) \rightarrow \Gamma(A/K)$ by setting $\nabla_X^\varphi \bar{a} = 0$ for all sections $a \in \Gamma(A)$ that are (φ, f) -related to some section $a' \in \Gamma(A')$, i.e. such that $\varphi \circ a = a' \circ f$. Then the properties of the Lie algebroid morphism (φ, f) imply that $(T^f M, K, \nabla^\varphi)$ is an infinitesimal ideal system in A .

By the results in the next section, we can roughly say that any infinitesimal ideal system can be constructed this way.

6. THE LEAF SPACE OF A FOLIATED ALGEBROID

Assume that (F_M, K, ∇) is an infinitesimal ideal system in A . Then there is an induced involutive subbundle $F_A \subseteq TA$ as in Corollary 4.4. Assume that

- (1) F_M is **simple**, i.e. the leaf space M/F_M is a smooth manifold such that the quotient map $\pi_M : M \rightarrow M/F_M$ is a surjective submersion,
- (2) ∇ has trivial holonomy.

We will show in this section that (F_M, K, ∇) “integrates” under these two conditions to an ideal system $(R(\pi_M), K, \theta)$ in the sense of Mackenzie [25], and that A/F_A is the same as the quotient of A/K by the action θ .

As a consequence, we will find an induced Lie algebroid structure on A/F_A over M/F_M such that the projection

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/F_A \\ q_A \downarrow & & \downarrow [q_A] \\ M & \xrightarrow{\pi_M} & M/F_M \end{array}$$

is a fibration of Lie algebroids.

Assume that A is the Lie algebroid of a source-simply connected Lie groupoid $G \rightrightarrows M$ and take the multiplicative involutive distribution F_G on G that integrates the infinitesimal ideal system (F_M, K, ∇) in A . One of the authors has found in [19] some completeness and regularity conditions for the leaf space $G/F_G \rightrightarrows M/F_M$ to be a Lie groupoid, such that the quotient map $G \rightarrow G/F_G$ is a Lie groupoid morphism over $M \rightarrow M/F_M$. We prove that in that case, the Lie algebroid $A/F_A \rightarrow M/F_M$ found above is isomorphic to the Lie algebroid of $G/F_G \rightrightarrows M/F_M$.

For the convenience of the reader, we recall here Mackenzie and Higgins’ definition of an ideal system.

Definition 6.1. [25] *Let A be a Lie algebroid on M with anchor $\rho : A \rightarrow TM$. An ideal system of A is a triple $(K, R(\pi), \theta)$, where K is a wide Lie subalgebroid of A , where $\pi : M \rightarrow N$ is a surjective submersion and $R(\pi)$ the closed, embedded, wide subgroupoid $\{(m, m') \mid \pi(m) = \pi(m')\} \subseteq M \times M$, and where θ is a linear action of $R(\pi)$ on $A/K \rightarrow M$ such that:*

- (1) *if $a, b \in \Gamma(A)$ are θ -stable, then $[a, b]$ is θ -stable,*
- (2) *if $a \in \Gamma(A)$ is θ -stable, then for all $k \in \Gamma(K)$: $[a, k] \in \Gamma(K)$,*
- (3) *the anchor ρ maps K into $T^\pi M$ and*
- (4) *the induced map $A/K \rightarrow TM/T^\pi M$ is $R(\pi)$ -equivariant with respect to θ and the canonical action of $R(\pi)$ on $TM/T^\pi M$.*

Here, the canonical action θ_0 of $R(\pi)$ on $TM/T^\pi M$ is defined as follows. For $(m, m') \in R(\pi)$ and $v_{m'} \in T_{m'}M$, the element $\theta_0(m, n)(v_{m'} + T_M^\pi)$ is $v_m + T^\pi M$, where v_m is chosen

such that $T_m\pi(v_m) = T_{m'}\pi(v_{m'})$. To see that θ_0 is well-defined, note that the map $T\pi : TM \rightarrow TN$ is fiberwise surjective and factors to a map $T\pi : TM/T^\pi M \rightarrow TN$, which is consequently fiberwise an isomorphism.

In the following, the class in A/F_A of $a_m \in A_m$ will be written $[a_m] \in A/F_A$, and the class in M/F_M of $m \in M$ will be denoted by $[m] \in M/F_M$. The class in A/K of $a_m \in A_m$ will be written \bar{a}_m .

Proposition 6.2. *Let (F_M, K, ∇) be an infinitesimal ideal system in $A \rightarrow M$ and $F_A \subseteq TA$ the corresponding morphic involutive distribution as in Corollary 4.4.*

- (1) *The map $\pi : A \rightarrow A/F_A$ factors as a composition*

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow \pi & \\ A/K & \xrightarrow{\bar{\pi}} & A/F_A \end{array}$$

That is, we have $\pi(a_m + k_m) = \pi(a_m)$ for all $a_m \in A$ and $k_m \in K(m)$.

- (2) *The equivalence relation $\sim := \sim_{F_A}$ on A can be described as follows.*

$$(6.9) \quad a_m \sim a_n \Leftrightarrow \begin{array}{l} \text{There exist linear sections } (X_1, \bar{X}_1), \dots, (X_r, \bar{X}_r) \text{ of } F_A \rightarrow A \\ \text{with flows } \phi^1, \dots, \phi^r \text{ such that} \\ a_m \in \phi_{t_1}^1 \circ \dots \circ \phi_{t_r}^r(a_n) + K(m) \\ \text{for some } t_1, \dots, t_r \in \mathbb{R}. \end{array}$$

- (3) *The map q_A induces a map $[q_A] : A/F_A \rightarrow M/F_M$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/F_A \\ q_A \downarrow & & \downarrow [q_A] \\ M & \xrightarrow{\pi_M} & M/F_M \end{array}$$

commutes.

Proof. (1) Recall that all the core sections $k^\dagger \in \mathfrak{X}(A)$ with $k \in \Gamma(K)$ are sections of F_A . Choose $a_m \in A$ and $k_m \in K(m)$. Then there exists a section $k \in \Gamma(K)$ with $k(m) = k_m$. The flow ϕ^{k^\dagger} of k^\dagger is given by $\phi_t^{k^\dagger}(a) = a + tk(q_A(a))$ for all $a \in A$ and $t \in \mathbb{R}$. Hence, we have $a_m \sim a_m + tk(m) = a_m + tk_m$ for all $t \in \mathbb{R}$, and in particular $a_m \sim a_m + k_m$. The map $\bar{\pi} : A/K \rightarrow A/F_A$, $\bar{\pi}(\bar{a}_m) = [a_m]$ is hence well-defined and the diagram commutes.

- (2) Since the family of linear sections of F_A and the family of core sections of F_A span together F_A , its leaves are the accessible sets of these two families of vector fields ([32, 33, 34], see [28] for a review of these results). Hence, two points a_m and a_n in A are in the same leaf of F_A if they can be joined by finitely many curves along flow lines of core sections k^\dagger for $k \in \Gamma(K)$ and linear vector fields $X \in \Gamma(F_A)$. By the involutivity of F_A , we have $D_X k \in \Gamma(K)$ for all $k \in \Gamma(K)$ and linear vector fields $X \in \Gamma(F_A)$. Hence, by Lemma B.3, we get that K is invariant under the flow lines of linear vector fields with values in F_A . That is, using the fact that ϕ_t^X is a vector bundle morphism, we have

$$\left(\phi_t^X \circ \phi_s^{k^\dagger} \right) (a_m) \in \phi_t^X(a_m + K(m)) = \phi_t^X(a_m) + K\left(\phi_t^{\bar{X}}(m)\right)$$

for all $a_m \in A$, $t \in \mathbb{R}$ where this makes sense and $s \in \mathbb{R}$. Since

$$\phi_s^{k^\dagger} \circ \phi_t^X(a_m) \in \phi_t^X(a_m) + K\left(\phi_t^{\bar{X}}(m)\right),$$

the proof is finished.

- (3) Assume that $a_m \sim a_n$ for some elements $a_m, a_n \in A$. Then there exists, without loss of generality, one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, an element $k_m \in K(m)$ and $t \in \mathbb{R}$ such that $a_m = \phi_t^X(a_n) + k_m$. We have then immediately

$$m = q_A(a_m) = (q_A \circ \phi_t^X)(a_n) = \phi_t^{\bar{X}}(n),$$

which shows $m \sim_{F_M} n$. □

Corollary 6.3. *Let (F_M, K, ∇) be an infinitesimal ideal system in a Lie algebroid $A \rightarrow M$. Choose \bar{a}_m and \bar{a}_n in A/K .*

- (1) $\bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}_n)$ if and only if $\bar{a}_m \in A/K$ is the ∇ -parallel transport of \bar{a}_n over a piecewise smooth path along the foliation defined by F_M on M .
- (2) If ∇ has trivial holonomy, then $\bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}'_m)$ if and only if $\bar{a}_m = \bar{a}'_m$.
- (3) If ∇ has trivial holonomy, the map

$$\theta_\nabla : R(\pi) \times_M (A/K) \rightarrow A/K,$$

$$\theta_\nabla(m, m')(\bar{a}_m) = \bar{a}_m \iff \bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}_{m'})$$

is well-defined and a linear action of $R(\pi)$ on $A/K \rightarrow M$.

Proof. (1) Assume first that $\bar{\pi}(\bar{a}_m) = \bar{\pi}(\bar{a}_n)$. Then there exists without loss of generality one linear vector field $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$ and $t \in \mathbb{R}$ such that $\bar{a}_m = \overline{\phi_t^X(a_n)}$.

Consider the curve $a : [0, t] \rightarrow A$ over $c := \phi_t^{\bar{X}}(n)$ defined by

$$a(\tau) = \phi_\tau^X(a_n)$$

for $\tau \in [0, t]$. For each $\tau \in [0, t]$, we find a flat section a^τ of A and $\varepsilon_\tau > 0$ such that $\phi_s^X(a_n)$ is defined on $(-\varepsilon_\tau, \tau + \varepsilon_\tau)$, $\phi_s^X(\tau - \varepsilon_\tau, \tau + \varepsilon_\tau) \subseteq \text{Dom}(a^\tau)$ and $a^\tau(c(\tau)) = \overline{a(\tau)}$. Since ϕ_s^X preserves K for all s where defined, we get then

$$\begin{aligned} \overline{a^\tau(c(s))} &= \overline{a^\tau(\phi_s^{\bar{X}}(n))} = \overline{a^\tau(\phi_{s-\tau}^{\bar{X}}\phi_\tau^{\bar{X}}(n))} \stackrel{(6.11)}{=} \overline{\phi_{s-\tau}^X a^\tau(c(\tau))} \\ &= \overline{\phi_{s-\tau}^X a(\tau)} = \overline{\phi_{s-\tau}^X \phi_\tau^X(a_n)} = \overline{\phi_s^X(a_n)} = \overline{a(s)} \end{aligned}$$

for $s \in (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$. This yields $\nabla_{\bar{X}(c(\tau))}\bar{a} = 0$ for all τ .

Conversely, assume that $\bar{a}_m \in A/K$ is the ∇ -parallel transport of \bar{a}_n over a piecewise smooth path along a path lying in the leaf of F_M through n . Without loss of generality, we may assume that this path is a segment of a flow curve of a vector field $\bar{X} \in \Gamma(F_M)$, $m = \phi_t^{\bar{X}}(n)$ for some $t \in \mathbb{R}$, and there exists a ∇ -flat section a of A such that $\overline{a(m)} = \bar{a}_m$ and $\overline{a(n)} = \bar{a}_n$. Choose any linear vector field $X \in \Gamma(F_A)$ over \bar{X} . Then we get as in the proof of Proposition 6.2, 4) that

$$\overline{a(m)} = \overline{a(\phi_t^{\bar{X}}(n))} = \overline{\phi_t^X(a(n))}$$

and hence $a_m \sim a_n$ by Proposition 6.2, 2).

- (2) This is immediate since here, parallel transport does not depend on the path along the leaf of F_M through m .
- (3) This follows easily from the two previous statements. □

Remark 6.4. Note that corollary 6.3 implies that the quotient space $\bar{\pi} : A/K \rightarrow A/F_A$ is the quotient by the equivalence relation given by parallel transport, or equivalently by the action θ_∇ .

Proposition 6.5. *Let (F_M, K, ∇) be an infinitesimal ideal system in A and assume that F_M is simple and ∇ has trivial holonomy. Then a section $a \in \Gamma(A)$ is θ_∇ -stable if and only if it is ∇ -flat.*

Proof. Choose a ∇ -flat section $a \in \Gamma(A)$. Assume first that \bar{a} does not vanish on its domain of definition. Since \bar{a} is ∇ -flat, we have $D_X a \in \Gamma(K)$ for all linear vector fields $X \in \Gamma(F_A)$, $X \sim_{q_A} \bar{X} \in \Gamma(F_M)$. By Lemma B.3, this yields

$$(6.10) \quad \phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + K\left(\phi_t^{\bar{X}}(m)\right)$$

for all $t \in \mathbb{R}$ where this makes sense. As a consequence, $\pi(a(m)) = \pi\left(a\left(\phi_t^{\bar{X}}(m)\right)\right)$, or equivalently

$$\bar{\pi}\left(\overline{a(m)}\right) = \bar{\pi}\left(\overline{a\left(\phi_t^{\bar{X}}(m)\right)}\right),$$

In general, we have $a = \sum_{i=1}^n f_i a_i$ on an open set U with non-vanishing ∇ -flat sections a_1, \dots, a_n of A such that $a_1, \dots, a_r \in \Gamma(K)$ for some $r \leq n$ and functions $f_1, \dots, f_n \in C^\infty(U)$ such that f_{r+1}, \dots, f_n are F_M -invariant. This yields using (6.10):

$$(6.11) \quad \begin{aligned} \phi_t^X(a(m)) &= \phi_t^X\left(\sum_{i=1}^n f_i(m)a_i(m)\right) \\ &\in \sum_{i=r+1}^n f_i\left(\phi_t^{\bar{X}}(m)\right)\phi_t^X(a_i(m)) + K\left(\phi_t^{\bar{X}}(m)\right) = a\left(\phi_t^{\bar{X}}(m)\right) + K\left(\phi_t^{\bar{X}}(m)\right) \end{aligned}$$

and we get the statement in the same manner as above.

Conversely, choose a θ_∇ -stable section $a \in \Gamma(A)$. Then we have $\theta_\nabla(m, m')(\overline{a(m')}) = \overline{a(m)}$ for all $(m, m') \in R(\pi) \cap \text{Dom}(a)$, which is by definition equivalent to $\bar{\pi}(\overline{a(m')}) = \bar{\pi}(\overline{a(m)})$. By Corollary 6.3, we can conclude. \square

Proposition 6.6. *Let (F_M, K, ∇) be an infinitesimal ideal system in A and assume that F_M is simple and ∇ has trivial holonomy. Then the induced map $\bar{\rho} : A/K \rightarrow TM/F_M$ is equivariant with respect to the actions θ_∇ and θ_0 of $R(\pi_M)$ on A/K and TM/F_M , respectively.*

Proof. If $\theta_\nabla(m, m')(\overline{a(m')}) = \overline{a(m)}$, then $a_m = k_m + \phi_t^X(a_{m'})$ for some linear section $X \in \Gamma(F_A)$ over $\bar{X} \in \Gamma(F_M)$, $t \in \mathbb{R}$ and $k_m \in K(m)$. As in the proof of Corollary 6.3, consider the curve $a : [0, t] \rightarrow A$ over $c := \phi_t^{\bar{X}}(m)$ defined by

$$a(\tau) = \phi_\tau^X(a_{m'})$$

for $\tau \in [0, t]$. Then $\nabla_{\bar{X}(c(\tau))}\bar{a} = 0$ for all τ . For each $\tau \in [0, t]$, we find $\varepsilon_\tau > 0$ and a flat section a^τ of A such that $\overline{a^\tau(c(s))} = \overline{a(s)}$ for $s \in [\tau - \varepsilon_\tau, \tau + \varepsilon_\tau]$. Then, $\rho \circ a^\tau$ is ∇^{F_M} -flat, and we get by Lemma 2.5 that $(\rho \circ a^\tau) \sim_{\pi_M} Y^\tau$ for some $Y^\tau \in \mathfrak{X}(\bar{M})$.

Since $[c(\tau - \varepsilon_\tau)] = [c(s)]$ for all $s \in [\tau - \varepsilon_\tau, \tau + \varepsilon_\tau]$, we have then

$$\begin{aligned} T_{c(\tau - \varepsilon_\tau)}\pi_M(\rho(a(\tau - \varepsilon_\tau))) &= T_{c(\tau - \varepsilon_\tau)}\pi_M(\rho(a^\tau(c(\tau - \varepsilon_\tau)))) \\ &= Y^\tau([c(\tau - \varepsilon_\tau)]) = Y^\tau([c(s)]) = T_{c(s)}\pi_M(\rho(a(s))) \end{aligned}$$

for all $s \in (\tau - \varepsilon_\tau, \tau + \varepsilon_\tau)$. Since $[0, t]$ is covered by (finitely many) intervals like this, we get $T_m\pi_M(\rho(a_m)) = T_m\pi_M(\rho(a(0))) = T_n\pi_M(\rho(a_{m'}))$, which proves the equality

$$\bar{\rho}(\theta_\nabla(m, m')(\overline{a(m')})) = \bar{\rho}(\overline{a(m)}) = \theta_0(m, m')(\bar{\rho}(\overline{a(m')})).$$

\square

Using Proposition 6.6, Proposition 6.5 and the properties of the infinitesimal ideal system (F_M, K, ∇) , we get the following result.

Corollary 6.7. *Let (F_M, K, ∇) be an infinitesimal ideal system in a Lie algebroid A . Assume that F_M is simple and ∇ has trivial holonomy. Then the triple $(K, R(\pi), \theta_\nabla)$ constructed as above is an ideal system in A .*

This, [25, Theorem 4.4.3] and Remark 6.4 imply the following result.

Corollary 6.8. *Let (F_M, K, ∇) be an infinitesimal ideal system in a Lie algebroid A . Assume that F_M is simple and ∇ has trivial holonomy. Then there is an induced Lie algebroid structure on A/F_A over M/F_M such that the projection (π, π_M) is a Lie algebroid morphism.*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/F_A \\ q_A \downarrow & & \downarrow [q_A] \\ M & \xrightarrow{\pi_M} & M/F_M \end{array}$$

Example 6.9. (1) In the situation of Example 5.1, assume that the leaf space of the foliation defined by F_M is a smooth manifold. Then the reduced Lie algebroid constructed as above is the graph of the Poisson structure that is induced by the Dirac structure D on M/F_M .

- (2) In the case of an ideal in the usual sense as in Example 5.3, the reduced algebroid is just the induced structure on $A/I \rightarrow M$.
- (3) As already mentioned, the reduced algebroid in Example 5.4 is the tangent space $T(M/F_M)$ of the leaf space of the foliation defined by F_M .
- (4) In the case of the kernel of a fibration as in Example 5.5, the reduced algebroid is the Lie algebroid structure on $A' \rightarrow M'$.
- (5) Assume that $F_M = TM$ (this is the special case of infinitesimal ideal systems found in [8]). Then the quotient Lie algebroid is a Lie algebra.

Take now an involutive multiplicative distribution $F_G \subseteq TG$ on a Lie groupoid $G \rightrightarrows M$, such that F_G and F_M are both simple. Assume that the leaf space G/F_G is a Lie groupoid over the leaf space M/F_M , such that the quotient map $G \rightarrow G/F_G$ is a Lie groupoid morphism (recall that conditions for this to be true have been found in [19]). The multiplicative involutive distribution F_G determines an infinitesimal ideal system (F_M, K, ∇) in the Lie algebroid A of $G \rightrightarrows M$ and, under the trivial holonomy condition on ∇ , a Lie algebroid structure on $A/F_A \rightarrow M/F_M$ as in the preceding corollary. We conclude this subsection with the comparison of this Lie algebroid with the Lie algebroid of the quotient groupoid $G/F_G \rightrightarrows M/F_M$.

Theorem 6.10. *Let (F_M, K, ∇) be an infinitesimal ideal system in A . Assume that A integrates to a Lie groupoid $G \rightrightarrows M$, and F_A to a multiplicative involutive distribution F_G on G . Assume furthermore that F_G and F_M are simple, ∇ has trivial holonomy and $G/F_G \rightrightarrows M/F_M$ is a Lie groupoid such that the quotient maps define a Lie groupoid morphism. Then we have*

$$A(G/F_G) \simeq A/F_A,$$

where A/F_A is equipped with the Lie algebroid structure in the previous corollary.

Remark 6.11. It would be interesting to study the relation between the trivial holonomy property of ∇ and the conditions on F_G for $G/F_G \rightrightarrows M/F_M$ to have a quotient Lie groupoid structure.

Proof of Theorem 6.10. Let $\pi_G : G \rightarrow G/F_G$ be the quotient map, and $[s], [t]$ the source and target maps of $G/F_G \rightrightarrows M/F_M$. Recall from Theorem 3.6 that a section $a \in \Gamma(A)$ is ∇ -flat

if and only if $[a^r, \Gamma(F_G)] \subseteq \Gamma(F_G)$ and the vector field a^r is then π_G -related to a vector field $\bar{a}^r \in \mathfrak{X}(G/F_G)$. We have

$$T[s] \circ \bar{a}^r \circ \pi_G = T[s] \circ T\pi_G \circ a^r = T\pi_G(Ts \circ a^r) = 0,$$

which shows that \bar{a}^r is tangent to the $[s]$ -fibers. By Lemma 3.18 in [19], we get

$$\bar{a}^r([g]) = T_g\pi_G(a^r(g)) = T_g\pi_G(a(\mathfrak{t}(g)) \star 0_g) = T_{\mathfrak{t}(g)}\pi_G(a(\mathfrak{t}(g))) \star 0_{[g]} = \bar{a}^r([\mathfrak{t}][g]) \star 0_{[g]},$$

which shows that $\bar{a}^r = \tilde{a}^r$ for $\tilde{a} := \bar{a}^r|_{M/F_M} \in \Gamma(A(G/F_G))$.

Since (π_G, π_M) is a Lie groupoid morphism

$$\begin{array}{ccc} G & \xrightarrow{\pi_G} & G/F_G \\ \mathfrak{t} \downarrow & \mathfrak{s} & \downarrow [\mathfrak{t}] \\ M & \xrightarrow{\pi_M} & M/F_M \end{array}$$

the map $A(\pi_G) = T\pi_G|_A$

$$\begin{array}{ccc} A & \xrightarrow{A(\pi_G)} & A(G/F_G) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi_M} & M/F_M \end{array}$$

is a Lie algebroid morphism and $K = \ker(A(\pi_G))$. For any ∇ -flat section $a \in \Gamma(A)$ we have $a^r \sim_{\pi_G} \tilde{a}^r$ and hence

$$(6.12) \quad A(\pi_G) \circ a = T\pi_G a = \tilde{a} \circ \pi_M.$$

Define the map

$$\begin{array}{ccc} A/F_A & \xrightarrow{\Psi} & A(G/F_G) \\ \downarrow & & \downarrow \\ \bar{M} & \xrightarrow{\text{Id}_M} & M/F_M \end{array}$$

by

$$\Psi([a_m]) = A(\pi_G)(a_m)$$

for all $a_m \in A$. To see that this does not depend on the representative, use $K = \ker(A(\pi_G))$ and recall that $a_m \sim a_n$ if and only if \bar{a}_m is the ∇ -parallel transport of \bar{a}_n along a path lying in the leaf through m of F_M (Corollary 6.3). Without loss of generality, there exists a ∇ -flat section $a \in \Gamma(A)$ such that $a(m) = a_m$ and $a(n) = a_n + k_n$ for some $k_n \in K(n)$. Then, using (6.12), we get

$$\begin{aligned} \Psi([a_m]) &= A(\pi_G)(a_m) = (A(\pi_G) \circ a)(m) = (\tilde{a} \circ \pi_M)(m) \\ &= (\tilde{a} \circ \pi_M)(n) = A(\pi_G)(a_n) = \Psi([a_n]). \end{aligned}$$

Hence, Ψ is a well-defined vector bundle morphism over the identity on M/F_M . Furthermore, the considerations above show that for any ∇ -flat section a of A and corresponding section $[a]$ of A/F_A , we get $\Psi \circ [a] = \tilde{a}$. The compatibility of the Lie algebroid brackets and anchors is then immediate by the definition of the Lie algebroid structure on A/F_A , and the fact that $A(\pi_G)$ is a Lie algebroid morphism. \square

Example 6.12. In the situation of Example 5.4 with M simply connected, the foliated Lie groupoid integrating the infinitesimal ideal system was $(M \times M \rightrightarrows M, F_M \times F_M)$. It is easy to check that the leaf space of the foliation defined by $F_M \times F_M$ is the groupoid $M/F_M \times M/F_M \rightrightarrows M/F_M$ (see also [19]), hence a Lie groupoid if M/F_M is a smooth

manifold. As we have seen above, the reduced Lie algebroid $TM \rightarrow M$ by the infinitesimal ideal system is equal to $T(M/F_M)$. This is the Lie algebroid of $M/F_M \times M/F_M \rightrightarrows M/F_M$.

APPENDIX A. THE LIE ALGEBROID OF THE TANGENT GROUPOID

If $G \rightrightarrows M$ is a Lie groupoid with Lie algebroid A , then we can consider the Lie algebroid $q_{A(TG)} : A(TG) \rightarrow TM$ of the tangent Lie groupoid $TG \rightrightarrows TM$. Since the projection $p_G : TG \rightarrow G$ is a Lie groupoid morphism, we have a Lie algebroid morphism $A(p_G) : A(TG) \rightarrow A$ over $p_M : TM \rightarrow M$:

$$(1.13) \quad \begin{array}{ccc} A(TG) & \xrightarrow{A(p_G)} & A \\ q_{A(TG)} \downarrow & & \downarrow q_A \\ TM & \xrightarrow{p_M} & M \end{array}$$

Let a be a section of A , choose $v \in TM$ and consider the curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow TG$ defined by

$$\gamma(t) = T \operatorname{Exp}(ta)v$$

for ε small enough. Then we have $\gamma(0) = v$ and $Ts(\gamma(t)) = v$ for all $t \in (-\varepsilon, \varepsilon)$. Hence, $\dot{\gamma}(0) \in A_v(TG)$ and we can define a **linear** section $\beta_a : TM \rightarrow A(TG)$ by

$$(1.14) \quad \beta_a(v) = \left. \frac{d}{dt} \right|_{t=0} T \operatorname{Exp}(ta)v$$

for all $v \in TM$. It is easy to check that $\beta_a^r \in \mathfrak{X}(TG)^r$ is the complete lift of a^r (see [25]). In particular the flow of β_a^r is $TL_{\operatorname{Exp}(\cdot a)}$, and (β_a, a) is a morphism of vector bundles.

In the same manner, we can consider $v_m \in TM$, $a \in A_m$ and the curve $\gamma : \mathbb{R} \rightarrow TG$ defined by

$$\gamma(t) = v + ta,$$

where TM and A are seen as subsets of TG , $TM_G = TM \oplus A$. We have again $\gamma(0) = v$ and $Ts(\gamma(t)) = v$ for all t , which yields $\dot{\gamma}(0) \in A_v(TG)$. Given $a \in \Gamma_M(A)$, we define a **core** section \tilde{a} of $A(TG)$ by

$$(1.15) \quad \tilde{a}(v) = \left. \frac{d}{dt} \right|_{t=0} v + ta(p_M(v))$$

for all $v \in TM$. We have for $v_g \in T_g G$ with $T_g t(v_g) = v_m$:

$$\tilde{a}^r(v_g) = \tilde{a}(v_m) \star 0_{v_g} = \left. \frac{d}{dt} \right|_{t=0} v_g + ta^r(g).$$

The vector bundle $A(TG)$ is spanned by the two types of sections β_a and \tilde{a} , for $a \in \Gamma_M(A)$, and, using the flows of β_a^r and $\tilde{b}^r \in \mathfrak{X}^r(TG)$, it is easy to check that the equalities

$$[\beta_a, \beta_b]_{A(TG)} = \beta_{[a, b]}, \quad [\beta_a, \tilde{b}]_{A(TG)} = [\widetilde{a, b}], \quad [\tilde{a}, \tilde{b}]_{A(TG)} = 0$$

hold for all $a, b \in \Gamma_M(A)$.

There exists a natural injective bundle map $\iota_A : A \rightarrow TG$ over $\epsilon : M \rightarrow G$. The canonical involution $J_G : TTG \rightarrow TTG$ restricts to an isomorphism of Lie algebroids

$j_G : TA \longrightarrow A(TG)$. More precisely, there exists a commutative diagram

$$(1.16) \quad \begin{array}{ccc} TA & \xrightarrow{j_G} & A(TG) \\ T\iota_A \downarrow & & \downarrow \iota_{A(TG)} \\ TTG & \xrightarrow{J_G} & TTG \end{array}$$

We check the following identities:

- (1) $j_G \circ Ta = \beta_a$ and
- (2) $j_G \circ a^\dagger = \tilde{a}$,

where a^\dagger is defined as in (2.3). First, we have for $v_m = \dot{c}(0) \in TM$:

$$j_G(T_m a v_m) = j_G \left(\frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \text{Exp}(sa)c(t) \right) = \frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} \text{Exp}(sa)c(t) = \beta_a(v_m).$$

In the same manner, we compute

$$\begin{aligned} j_G \left(T_m 0^A v_m + \frac{d}{ds} \Big|_{s=0} sa_m \right) &= j_G \left(\frac{d}{ds} \Big|_{s=0} sa(c(s)) \right) = j_G \left(\frac{d}{ds} \Big|_{s=0} \frac{d}{dt} \Big|_{t=0} \text{Exp}(tsa)c(s) \right) \\ &= \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} \text{Exp}(tsa)c(s) = \frac{d}{dt} \Big|_{t=0} v_m + ta(m), \end{aligned}$$

which proves the second equality.

The identity

$$\rho_{A(TG)} \circ j_G = J_M \circ T\rho_A = \rho_{TA}$$

is verified easily on these linear and core sections. This shows that the Lie algebroid $A(TG) \rightarrow TM$ of the tangent groupoid is canonically isomorphic to the tangent Lie algebroid $TA \rightarrow TM$ of A .

APPENDIX B. INVARIANCE OF BUNDLES UNDER FLOWS

We prove here a result that is standard, but the proof of which is difficult to find in the literature.

Theorem B.1. *Let M be a smooth manifold and E be a subbundle of the direct sum vector bundle $\mathbb{T}M := TM \oplus T^*M$. Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t . If $\mathcal{L}_Z e \in \Gamma(E)$ for all $e \in \Gamma(E)$, then*

$$\phi_t^* e \in \Gamma(E) \quad \text{for all } e \in \Gamma(E) \quad \text{and} \quad t \in \mathbb{R} \text{ where this makes sense.}$$

Corollary B.2. *Let F be a subbundle of the tangent bundle TM of a smooth manifold M . Let $Z \in \mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t . If $[Z, \Gamma(F)] \subseteq \Gamma(F)$, then $T_m \phi_t F(m) = F(\phi_t(m))$ for all $m \in M$ and t where this makes sense.*

Proof. Choose $X \in \Gamma(F)$ and $m \in M$. Then, by Theorem B.1, we have $T\phi_t \circ X \circ \phi_{-t} = \phi_{-t}^* X \in \Gamma(F)$ for all t where this makes sense, and hence:

$$T_m \phi_t X(m) = (\phi_{-t}^* X)(\phi_t(m)) \in F(\phi_t(m)).$$

□

Proof of Theorem B.1. The subbundle E of $\mathbb{T}M$ is an embedded submanifold of $\mathbb{T}M$. For each section σ of $\mathbb{T}M$, the smooth function $l_\sigma : \mathbb{T}M \rightarrow \mathbb{R}$ is defined by

$$l_\sigma(v, \alpha) = \langle \sigma(p(v, \alpha)), (v, \alpha) \rangle$$

for all $(v, \alpha) \in \mathbb{T}M$, where $p : \mathbb{T}M \rightarrow M$ is the projection. For all $e \in E$, the tangent space $T_e E$ of the submanifold E of $\mathbb{T}M$ is equal to

$$\ker \{ \mathbf{d}_e l_\sigma \mid \sigma \in \Gamma(E^\perp) \},$$

where E^\perp is the orthogonal space to E relative to the canonical symmetric fiberwise pairing on $TM \oplus T^*M$:

$$\langle (v_m, \alpha_m), (w_m, \beta_m) \rangle = \alpha_m(w_m) + \beta_m(v_m)$$

for all $v_m, w_m \in T_m M$, $\alpha_m, \beta_m \in T_m^* M$, $m \in M$.

Consider the complete lift \tilde{Z} to $\mathbb{T}M$ of Z , i.e. the vector field $\tilde{Z} \in \mathfrak{X}(\mathbb{T}M)$ defined by

$$\tilde{Z}(l_\sigma) = l_{\mathcal{L}_Z \sigma} \quad \text{and} \quad \tilde{Z}(p^* f) = p^*(Z(f))$$

for all $\sigma \in \Gamma(\mathbb{T}M)$ and $f \in C^\infty(M)$ (see [25]). Choose $e \in E$ and $\sigma \in \Gamma(E^\perp)$. Then we have $\mathcal{L}_Z \sigma \in \Gamma(E^\perp)$ since for all $\tau \in \Gamma(E)$:

$$\langle \mathcal{L}_Z \sigma, \tau \rangle = Z(\langle \sigma, \tau \rangle) - \langle \sigma, \mathcal{L}_Z \tau \rangle = 0.$$

This leads to

$$(\mathbf{d}_e l_\sigma)(\tilde{Z}(e)) = \left(\tilde{Z}(l_\sigma) \right)(e) = l_{\mathcal{L}_Z \sigma}(e) = 0.$$

Hence, the vector field \tilde{Z} is tangent to E on E . As a consequence, its flow curves starting at points of e remain in the submanifold E .

It is easy to check that the flow Φ_t of the vector field \tilde{Z} is equal to $(T\phi_t, (\phi_{-t})^*)$, i.e.,

$$\Phi_t(v_m, \alpha_m) = (T_m \phi_t(v_m), \alpha_m \circ T_{\phi_t(m)} \phi_{-t})$$

for all $(v_m, \alpha_m) \in \mathbb{T}_m M$. Choose a section $(X, \alpha) \in \Gamma(E)$ and a point $m \in M$. We find

$$(\phi_t^*(X, \alpha))(m) = (T_{\phi_t(m)} \phi_{-t} X(\phi_t(m)), \alpha_{\phi_t(m)} \circ T_m \phi_t) = \Phi_{-t}((X, \alpha)(\phi_t(m))) \in E(m)$$

since $(X, \alpha)(\phi_t(m)) \in E(\phi_t(m))$. Thus, we have shown that $\phi_t^*(X, \alpha)$ is a section of E . \square

Assume now that $q_A : A \rightarrow M$ is a vector bundle, and consider a linear vector field X on A , i.e. the map $X : A \rightarrow TA$ is a vector bundle morphism over $\bar{X} : M \rightarrow TM$ such that $X \sim_{q_A} \bar{X}$. Let ϕ_t^X be the flow of X and $\phi_t^{\bar{X}}$ the flow of \bar{X} . Then $\phi_t^X : A \rightarrow A$ is a vector bundle morphism over $\phi_t^{\bar{X}}$ for all $t \in \mathbb{R}$ where this is defined.

Note that for any $a \in \Gamma(A)$, the section $D_X a \in \Gamma(A)$ is defined by

$$(D_X a)(m) = \left. \frac{d}{dt} \right|_{t=0} \phi_{-t}^X(a(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. In the same manner, if $\varphi \in \Gamma(A^*)$, we can define

$$(D_X \varphi)(m) = \left. \frac{d}{dt} \right|_{t=0} (\phi_t^X)^*(\varphi(\phi_t^{\bar{X}}(m)))$$

for all $m \in M$. We have then $\varphi(a) \in C^\infty(M)$, and

$$(2.17) \quad \bar{X}(m)(\varphi(a)) = \varphi(D_X a)(m) + (D_X \varphi)(a)(m).$$

We can now show the following lemma.

Lemma B.3. *Let A be a vector bundle and $B \subseteq A$ a subbundle.*

(1) *If (X, \bar{X}) is a linear vector field on A such that*

$$D_X b \in \Gamma(B)$$

for all $b \in \Gamma(B)$, then $\phi_t^X(b_m) \in B(\phi_t^{\bar{X}}(m))$ for all $b_m \in B_m$.

- (2) Assume furthermore that $a \in \Gamma(A)$ is such that $a(m)$ is linearly independent to $B(m)$ for all m in $\text{Dom}(a)$ and

$$D_X a \in \Gamma(B).$$

Then

$$\phi_t^X(a(m)) \in a\left(\phi_t^{\bar{X}}(m)\right) + B\left(\phi_t^{\bar{X}}(m)\right)$$

for all $m \in U$ and $t \in \mathbb{R}$ where this makes sense.

Proof. (1) We check that the vector field X is tangent to B on points in B . Let $\varphi \in \Gamma(A^*)$ be a section of B° , i.e. $\varphi_m(b_m) = 0$ for all $b_m \in B$. Let $l_\varphi \in C^\infty(A)$ be the linear function defined by φ . By (2.17), we have then $D_X \varphi \in \Gamma(B^\circ)$. Choose $b_m \in B$. We have then

$$\begin{aligned} \mathbf{d}_{b_m} l_\varphi(X(b_m)) &= \frac{d}{dt} \Big|_{t=0} l_\varphi(\phi_t^X(b_m)) = \frac{d}{dt} \Big|_{t=0} \varphi_{\phi_t^{\bar{X}}(m)}(\phi_t^X(b_m)) \\ &= (D_X \varphi)(b_m) = 0. \end{aligned}$$

Thus, X is tangent to B on B and the flow of X preserves B .

- (2) Assume now that (b_1, \dots, b_k) is a local frame for B on an open set $U \subseteq M$. Complete this frame to a local frame (b_1, \dots, b_n) for A defined on an open U such that $b_{k+1} := a \in \Gamma(A)$. Let $\varphi_1, \dots, \varphi_n$ be a frame for A^* that is dual to (b_1, \dots, b_n) , i.e. such that $(\varphi_{k+1}, \dots, \varphi_n)$ is a frame for B° and $\varphi_{k+1}(a) = 1$. Then, the closed submanifold C of $A|_U$ defined by $C \cap A_m = a(m) + B_m$ is the level set with value $(1, 0, \dots, 0)$ of the function

$$(l_{\varphi_{k+1}}, \dots, l_{\varphi_n}) : A|_U \rightarrow \mathbb{R}^{n-k}.$$

Since $D_X a \in \Gamma(B)$ for the linear vector field (X, \bar{X}) on A , we get

$$0 = \bar{X}(\varphi_i(a)) = \varphi_i(D_X a) + D_X \varphi_i(a) = 0 + D_X \varphi_i(a)$$

for $i = k+1, \dots, n$ and this yields as before for all $b_m \in B$:

$$\begin{aligned} \mathbf{d}_{a(m)+b_m} l_{\varphi_i}(X(a(m) + b_m)) &= \frac{d}{dt} \Big|_{t=0} l_{\varphi_i}(\phi_t^X(a(m) + b_m)) \\ &= (D_X \varphi_i)(a(m) + b_m) = 0. \end{aligned}$$

Hence, X is tangent to C on points of C . That is, the flow of X preserves C . \square

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