NATURAL LIFTS OF DORFMAN BRACKETS

M. JOTZ LEAN AND C. KIRCHHOFF-LUKAT

Abstract. In this note we prove that, for a vector bundle E over a manifold M, a Dorfman bracket on $TM \oplus E^*$ anchored by pr_{TM} and with E a vector bundle over M, is equivalent to a lift from $\Gamma(TM \oplus E^*)$ to linear sections of $TE \oplus T^*E \to E$, that intertwines the given Dorfman bracket with the Courant-Dorfman bracket on sections of $TE \oplus T^*E$.

This shows a universality of the Courant-Dorfman bracket, and allows us to caracterise twistings and symmetries of transitive Dorfman brackets via the corresponding lifts.

Contents

1. Introduction

Theodore Courant and his adviser Alan Weinstein defined[1] the Courant bracket in 1990 [8, 7]: an $\mathbb{R}$-bilinear, skew-symmetric bracket on sections of $TM \oplus T^*M$ that satisfies the Jacobi identity up to an exact one-form. Irene Dorfman independently introduced that structure in her definition and study of Dirac structures in the context of infinite dimensional Hamiltonian structures [9]. Then Liu, Weinstein and Xu discovered in the late nineties that this bracket on sections of $TM \oplus T^*M$ is in fact a particular, "standard" example of a Courant algebroid, when they defined the later notion and proved that the bicrossproduct of any Lie bialgebroid can be understood as a special type of Courant algebroid [21].

Nowadays, for a smooth manifold M, the standard Courant algebroid structure on $TM \oplus T^*M$ is often defined using the Courant-Dorfman bracket on $TM \oplus T^*M$: an $\mathbb{R}$-bilinear bracket on sections of $TM \oplus T^*M$, that is not skew-symmetric but satisfies a Jacobi identity written in Leibniz form (see [25, 26]). The two brackets are equivalent in the sense that the Courant bracket is the skew-symmetrisation of the Courant-Dorfman bracket.

In the context of Courant algebroids and Dirac structures, the Courant-Dorfman bracket plays an important role in the generalised geometry developped first by Nigel Hitchin, Marco Gualtieri (see [12, 11]). It also enters the theoretical physics literature in this context: $TM \oplus T^*M$-generalised geometry turns out to provide a convenient description for the low-energy effective theory of closed string theory referred to as *double field theory* (see for instance [14, 13]).

Subsequently, the low-energy effective theories of the conjectured M-theory were linked to Dorfman brackets on vector bundles of the form $TM \oplus \wedge^{k_1} T^*M \oplus \cdots \oplus \wedge^{k_l} T^*M$ (see [15]).

In all of these applications, Dorfman brackets encode infinitesimal gauge transformations of the physical theory. Gauge transformations or gauge invariances are redundancies in the mathematical description of the theory (not to be confused with physical symmetries) – the physical results are invariant under the application of such transformations. For example, general relativity, a theory of four-dimensional smooth manifolds with Lorentzian metrics, is invariant under diffeomorphisms. The Lie algebra of the diffeomorphism group on a smooth manifold is given by the Lie derivatives $\pounds_X$ for $X \in \mathfrak{X}(M)$, so the Lie bracket (the simplest example of a Dorfman bracket) gives the infinitesimal gauge transformations of general relativity.

Similarly, the theory described by the $TM \oplus T^*M$-generalised geometry, which is a theory of a metric and a 2-form on a smooth manifold M, is invariant under the semi-direct product of the diffeomorphism group with the (additive) group of closed two-forms $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$ – the physics of this theory only depends on the exterior derivative of the two-form. The Lie algebra of this group of *generalised diffeomorphisms* is precisely given by elements $[\![(X,\xi),\cdot]\!], (X,\xi) \in \Gamma(TM \oplus T^*M)$, so the Courant-Dorfman bracket encodes the infinitesimal gauge transformations of this more extended theory. This principle is repeated in the M-theory examples.

Dorfman-type brackets on $TM \oplus \wedge^{k_1} T^*M \oplus \cdots \oplus \wedge^{k_l} T^*M$ and generalisations are studied in great detail in [1] under the name of *closed-form algebroids* as a special case of the general concept of *Leibniz algebroid*. Leibniz algebroids are the

[1]See [19] for a nice exposition of the history of Courant algebroids.

natural generalisation of Lie algebroids, where the bracket is no longer required to be antisymmetric, but still satisfies a form of Jacobi identity.

This paper studies Leibniz algebroids on vector bundles of the form $TM \oplus E^*$, where $E \to M$ is some smooth vector bundle, in the context of double vector bundles; more specifically the standard VB-Courant algebroid $TE \oplus T^*E$ over the vector bundle E. We call Leibniz brackets of this type *Dorfman brackets*, since they constitute the most direct generalisation of the original Courant-Dorfman bracket on $TM \oplus T^*M$.

Section 3 characterises linear sections of $TE \oplus T^*E$ in terms of certain derivations of its core $E \oplus T^*M$. Linear sections of $TE \oplus T^*E$ form a locally free sheaf over M and are thus sections of a vector bundle $\widehat{E} \to M$, the so-called fat vector bundle. $\widehat{E}$ is in fact isomorphic as a vector bundle to the Omni-Lie algebroid $\mathrm{Der}(E^*) \oplus J^1(E^*)$ studied in [5, 6]. Using our results on linear sections, we can show that the E^*-valued Courant algebroid structure on $\mathrm{Der}(E^*) \oplus J^1(E^*)$ is induced from the standard Courant algebroid structure on $TE \oplus T^*E$. Note that according to [17], the VB-Courant algebroid $TE \oplus T^*E \to E$ is equivalent to an E^*-Courant algebroid. In [4] the omni-Lie algebroid associated to E^* is proved to be an E^*-Courant algebroid. To our knowledge, those two E^*- Courant algebroids have never be proved to coincide before.

Furthermore, these results are used in Section 4 to establish the following main result (Theorem 4.2), which shows that all Dorfman brackets on $TM \oplus E^*$ are intimately linked to the Courant-Dorfman bracket on $TE \oplus T^*E$. Therefore, the Courant-Dorfman bracket can be seen as universal in the family of the Dorfman brackets.

Theorem. Let $[\![\cdot, \cdot]\!]$ be any Dorfman bracket on $TM \oplus E^*$ anchored by pr_{TM}. Then there exists an $\mathbb{R}$-linear map $\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$ which satisfies

(1) If $\Phi_E\colon TE \oplus T^*E \to TM \oplus E^*$ is the projection in the double vector bundle $(TE \oplus T^*E; TM \oplus E^*, E; M)$ (see Section 3.5),
$$\Phi_E(\Xi(\nu)(e_m)) = \nu(m)$$
for all $\nu \in \Gamma(TM \oplus E^*)$, $e_m \in E_m$ and $m \in M$.

(2) The lift is *natural* in the sense that for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$, we have:
$$\Xi[\![\nu_1, \nu_2]\!] = [\![\Xi(\nu_1), \Xi(\nu_2)]\!]$$
where the bracket on the right-hand side is the Courant-Dorfman bracket on sections of $TE \oplus T^*E \to E$.

We compare this to results obtained in [16, 5, 6].

Section 5 explores the most important examples of such natural lifts, and sections 6 and 7 describe twistings and internal symmetries of Dorfman brackets in light of the double vector bundle context.

Notation and conventions. We write $p_M\colon TM \to M$, $q_E\colon E \to M$ for vector bundle projections. We write $\langle \cdot\,, \cdot \rangle$ for the canonical pairing of a vector bundle with its dual; i.e. $\langle e_m, \varepsilon_m \rangle = \varepsilon_m(e_m)$ for $e_m \in E$ and $\varepsilon_m \in E^*$. We use several different pairings; in general, which pairing is used is clear from its arguments. Given a section ε of E^*, we write $\ell_\varepsilon\colon E \to \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \varepsilon(m), e_m \rangle$ for all $e_m \in E$. We denote by $\iota_E\colon E \to E \oplus T^*M$ the canonical inclusion.

Let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the sheaves of smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the sheaf of sections of E is written $\Gamma(E)$.

2. Preliminaries on Courant algebroids, Dorfman brackets, dull brackets and Dorfman connections

An anchored vector bundle is a vector bundle $Q \to M$ endowed with a vector bundle morphism $\rho_Q \colon Q \to TM$ over the identity. Consider an anchored vector bundle $(\mathsf{E} \to M, \rho)$ and a vector bundle V over the same base M together with a morphism $\widetilde{\rho} \colon \mathsf{E} \to \operatorname{Der}(V)$, such that the symbol of $\widetilde{\rho}(e)$ is $\rho(e) \in \mathfrak{X}(M)$ for all $e \in \Gamma(\mathsf{E})$. Assume that E is paired with itself via a nondegenerate pairing $\langle \cdot\,, \cdot \rangle \colon \mathsf{E} \times_M \mathsf{E} \to V$ with values in V. Then $\mathsf{E} \to M$ is a **Courant algebroid with pairing in** V if E is in addition equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot\,, \cdot]\!]$ on the smooth sections $\Gamma(\mathsf{E})$ such that the following conditions are satisfied:

(1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
(2) $\widetilde{\rho}(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
(3) $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$,
(4) $\widetilde{\rho}[\![e_1, e_2]\!] = [\widetilde{\rho}(e_1), \widetilde{\rho}(e_2)]$

for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$, where $\mathcal{D} \colon \Gamma(V) \to \Gamma(\mathsf{E})$ is defined by $\langle \mathcal{D}v, e \rangle = \widetilde{\rho}(e)(v)$ for all $v \in \Gamma(V)$. Note that

(5) $[\![e_1, f e_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$

for $e_1, e_2 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$ follows from (2). If $V = \mathbb{R} \times M \to M$ is the trivial bundle, then $\mathcal{D} = \rho^* \circ \mathbf{d} \colon C^\infty(M) \to \Gamma(\mathsf{E})$, where E is identified with E^* via the pairing. The quadruple $(\mathsf{E} \to M, \rho, \langle \cdot\,, \cdot \rangle, [\![\cdot\,, \cdot]\!])$ is then a **Courant algebroid** [21, 25]; then $\widetilde{\rho} = \rho$ and (4) follows from (2) and the nondegeneracy of the pairing (see also [26]). Finally note that Courant algebroids with a pairing in a vector bundle E were defined in [4] and called *E-Courant algebroids*.

Example 2.1. [7] The direct sum $TM \oplus T^*M$ endowed with the projection on TM as anchor map, $\rho = \operatorname{pr}_{TM}$, the symmetric bracket $\langle \cdot\,, \cdot \rangle$ given by

$$\langle (v_m, \theta_m), (w_m, \eta_m) \rangle = \theta_m(w_m) + \eta_m(v_m) \tag{1}$$

for all $m \in M$, $v_m, w_m \in T_m M$ and $\alpha_m, \beta_m \in T_m^* M$ and the **Courant-Dorfman bracket** given by

$$[\![(X, \theta), (Y, \eta)]\!] = ([X, Y], \pounds_X \eta - \mathbf{i}_Y \mathbf{d}\theta) \tag{2}$$

for all $(X, \theta), (Y, \eta) \in \Gamma(TM \oplus T^*M)$, yield the standard example of a Courant algebroid, which is often called the **standard Courant algebroid over** M. The map $\mathcal{D} \colon C^\infty(M) \to \Gamma(TM \oplus T^*M)$ is given by $\mathcal{D}f = (0, \mathbf{d}f)$. We are here particularly interested in the standard Courant algebroid over the total space of a vector bundle.

Next we define dull algebroids and Leibniz algebroids.

Definition 2.2. (1) [16] *A* ***dull algebroid*** *is an anchored vector bundle $(Q \to M, \rho)$ endowed with a bracket $[\![\cdot\,, \cdot]\!]$ on $\Gamma(Q)$ satisfying $\rho[\![q_1, q_2]\!] = [\rho(q_1), \rho(q_2)]$, and the Leibniz identity in both terms*

$$[\![f_1 q_1, f_2 q_2]\!] = f_1 f_2 [\![q_1, q_2]\!] + f_1 \rho(q_1)(f_2) q_2 - f_2 \rho(q_2)(f_1) q_1$$

for all $f_1, f_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$.

(2) [1] *A **Leibniz algebroid** is an anchored vector bundle* $(Q \to M, \rho)$ *endowed with a bracket* $[\![\cdot\,,\cdot]\!]$ *on* $\Gamma(Q)$ *with* $[\![q_1, fq_2]\!] = f[\![q_1,q_2]\!] + \rho(q_1)(f)q_2$ $\forall f \in C^\infty(M), q_1, q_2 \in \Gamma(Q)$*, and satisfying the Jacobi identity in Leibniz form*

$$[\![q_1,[\![q_2,q_3]\!]]\!] = [\![[\![q_1,q_2]\!],q_3]\!] + [\![q_2,[\![q_1,q_3]\!]]\!]$$

for all $q_1, q_2, q_3 \in \Gamma(Q)$.

(3) *A Leibniz algebroid* E' *is **transitive** if the anchor* $\rho\colon E' \to TM$ *is surjective* [1]*. Then the Leibniz algebroid can be written* $E' = TM \oplus E^*$ *with* $\rho = \mathrm{pr}_{TM}$ *and* $E \to M$ *a vector bundle. We call its bracket* $[\![\cdot,\cdot]\!]$ *a **Dorfman bracket***[2].

(4) *A transitive Leibniz algebroid is **split** if there is a section* $\sigma\colon TM \to E'$ *of the anchor map such that* $\sigma(\mathfrak{X}(M))$ *is closed under the Leibniz bracket* [1]*.*

First note that the definition of the Leibniz algebroid implies [1]

$$\rho[\![q_1,q_2]\!] = [\rho(q_1),\rho(q_2)] \quad \text{for all } q_1,q_2 \in \Gamma(Q).$$

Any split transitive Leibniz algebroid E' forms a split short exact sequence of vector bundles:

$$0 \to E^* \hookrightarrow E' \xrightarrow{\rho} TM \to 0 \tag{3}$$

with $E^* = \ker \rho$. The splitting map $\sigma : TM \to E'$ induces an isomorphism $E' \cong TM \oplus E^*$. Since $\sigma(\mathfrak{X}(M))$ is closed under the Leibniz bracket and $\rho \circ \sigma = \mathrm{id}_{TM}$, we have $[\![\sigma(X),\sigma(Y)]\!] = \sigma[X,Y]$. Thus, if we use σ to define the isomorphism $E' \to TM \oplus E^*$, we obtain a Dorfman bracket with the property

$$[\![(X,0),(Y,0)]\!] \stackrel{\text{def}}{=} [\![\sigma(X),\sigma(Y)]\!] = \sigma[X,Y] = ([X,Y],0) \tag{4}$$

Correspondingly, we call a Dorfman bracket *split* precisely if it has this property.

Consider a dull algebroid $(Q,\rho,[\![\cdot\,,\cdot]\!])$. Then the bracket can be dualised to a map

$$\Delta\colon \Gamma(Q)\times\Gamma(Q^*) \to \Gamma(Q^*), \quad \rho(q)\langle q',\tau\rangle = \langle [\![q,q']\!],\tau\rangle + \langle q', \Delta_q\tau\rangle$$

for all $q,q' \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$. The map Δ is then a **Dorfman (Q-)connection on** Q^* [16], i.e. an $\mathbb{R}$-bilinear map with

(1) $\Delta_{fq}\tau = f\Delta_q\tau + \langle q,\tau\rangle \cdot \rho^*\mathbf{d}f$,
(2) $\Delta_q(f\tau) = f\Delta_q\tau + \rho(q)(f)\tau$ and
(3) $\Delta_q(\rho^*\mathbf{d}f) = \rho^*\mathbf{d}(\pounds_{\rho(q)})$

for all $f \in C^\infty(M)$, $q,q' \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$. The curvature of Δ is the map $R_\Delta\colon \Gamma(Q)\times\Gamma(Q) \to \Gamma(Q^*\otimes Q^*)$ defined on $q,q' \in \Gamma(Q)$ by $R_\Delta(q,q') := \Delta_q\Delta_{q'} - \Delta_{q'}\Delta_q - \Delta_{[\![q,q']\!]}$. For all $f \in C^\infty(M)$ and $q_1,q_2,q_3 \in \Gamma(Q)$, $\tau \in \Gamma(Q^*)$, we have

$$\langle R_\Delta(q_1,q_2)\tau, q_3\rangle = \langle [\![[\![q_1,q_2]\!],q_3]\!] + [\![q_2,[\![q_1,q_3]\!]]\!] - [\![q_1,[\![q_2,q_3]\!]]\!], \tau\rangle.$$

Consider a Dorfman bracket $[\![\cdot\,,\cdot]\!]\colon \Gamma(Q)\times\Gamma(Q) \to \Gamma(Q)$. Its dual map is

$$\mathcal{D}\colon \Gamma(Q) \to \mathrm{Der}(Q^*),$$

[2] Occasionally the term "Dorfman bracket" is used for the bracket of arbitrary Leibniz algebroids in the literature, but in this paper it will exclusively refer to the case where the anchor is surjective and the underlying vector bundle is split.

defined by $\rho(q)\langle q', \tau\rangle = \langle q', \mathcal{D}_q\tau\rangle + \langle [\![q, q']\!], \tau\rangle$ for all $q, q' \in \Gamma(Q)$ and $\tau \in \Gamma(Q^*)$. The Jacobi identity in Leibniz form for $[\![\cdot\,,\cdot]\!]$ is equivalent to

$$\mathcal{D}_{q_1} \circ \mathcal{D}_{q_2} - \mathcal{D}_{q_2} \circ \mathcal{D}_{q_1} = \mathcal{D}_{[\![q_1,q_2]\!]} \tag{5}$$

for all $q_1, q_2 \in \Gamma(Q)$.

$\mathcal{D}$ allows the extension of the Dorfman bracket to all tensor bundles of Q via the Leibniz rule. In the theoretical physics applications, this operation is called the *generalised Lie derivative* due to its Lie algebra property.

Example 2.3. The bracket of a Courant algebroid E is a Dorfman bracket. Using the nondegenerate pairing to identify E with its dual, we find that $\mathcal{D}$ is in this case the "adjoint action": $\mathcal{D}_e = [\![e, \cdot]\!]$ for $e \in \Gamma(\mathsf{E})$.

Example 2.4. On any vector bundle of the form $TM \oplus E^*$ with $E = \wedge^{k_1}TM \oplus \cdots \oplus \wedge^{k_l}TM$, there is a Dorfman bracket

$$[\![(X,\alpha),(Y,\beta)]\!] = [X,Y] + \pounds_X\beta - i_Y\mathbf{d}\alpha \quad \text{for} \quad (X,\alpha),(Y,\beta) \in \Gamma(TM \oplus E^*) \tag{6}$$

For simplicity of notation, consider the special case $TM \oplus \wedge^k T^*M$ for the rest of this example – the more general case works in the same way. Let $(T,\theta) \in \Gamma(\wedge^k TM \oplus T^*M)$. Then we have

$$\begin{aligned}
\left\langle \mathcal{D}_{(X,\alpha)}(T,\theta),(Y,\beta)\right\rangle &= X\left\langle (T,\theta),(Y,\beta)\right\rangle - \langle [\![(X,\alpha),(Y,\beta)]\!],(T,\theta)\rangle \\
&= \langle \pounds_X\theta, Y\rangle + \langle \pounds_X T, \beta\rangle + \langle \mathbf{i}_Y\mathbf{d}\alpha, T\rangle \\
&= \left\langle (\pounds_X T, \pounds_X\theta + (-1)^k\mathbf{d}\alpha(T,\cdot)), Y+\beta\right\rangle
\end{aligned}$$

which shows $\mathcal{D}_{(X,\alpha)}(T,\theta) = (\pounds_X T, \pounds_X\theta + (-1)^k\mathbf{d}\alpha(T,\cdot))$.

Example 2.5. [1] extensively discusses a generalisation of example 2.4, so-called closed-form Leibniz algebroids. All commonly studied examples of Dorfman brackets belong to this class of Leibniz algebroids.

In addition to the terms in (6), closed form algebroids can for example contain terms that mix different degrees of differential forms:

$$[\![(0;\alpha_k,0,0),(0;0,\beta_j,0)]\!] = (-1)^{(k-1)j}(0;0,0,\mathbf{d}\alpha_k \wedge \beta_j) \tag{7}$$

for the Dorfman bracket on $TM \oplus \wedge^k T^*M \oplus \wedge^j T^*M \oplus \wedge^{k+j+1}T^*M$.

Terms of this type correspond to terms of the following form in $\mathcal{D}$:

$$\begin{aligned}
&\left\langle \mathcal{D}_{(0;\alpha_k,0,0)}(T_k,T_j,T_{k+j+1};\theta),(Y;\beta_k,\beta_j,\beta_{j+k+1})\right\rangle \\
&= -\left\langle [\![(0;\alpha_k,0,0),(Y;\beta_k,\beta_j,\beta_{j+k+1})]\!],(T_k,T_j,T_{k+j+1};\theta)\right\rangle \\
&= \left\langle (0;\mathbf{i}_Y\mathbf{d}\alpha_k,0,(-1)^{(k-1)j+1}\mathbf{d}\alpha_k\wedge\beta_j),(T_k,0,T_{k+j+1};0)\right\rangle \\
&= \left\langle (0,(-1)^{(k-1)j+1}T_{k+j+1}\neg\mathbf{d}\alpha_k,0;(-1)^k\mathbf{i}_{T_k}\mathbf{d}\alpha),(Y;\beta_k,\beta_j,\beta_{k+j+1})\right\rangle
\end{aligned}$$

and therefore

$$\mathcal{D}_{(0;\alpha_k,0,0)}(0,0,T_{k+j+1};0) = (-1)^{(k-1)j+1}(0,T_{k+j+1}\neg\mathbf{d}\alpha_k,0;0), \tag{8}$$

where $\neg$ denotes contraction over the first (in this case) $(k+1)$ indices.

Example 2.6. A more complex example of *closed-form algebroid* underlies the so-called E_7-exceptional generalised geometry (see [1, 15]). The vector bundle

$$TM \oplus \wedge^2T^*M \oplus \wedge^5T^*M \oplus (\wedge^7T^*M \otimes T^*M) \tag{9}$$

carries a natural $E_7 \times \mathbb{R}^*$-structure and the Dorfman bracket (see [1])

$$\begin{aligned}&[\![(X;\alpha_2,\alpha_5,u),(Y;\beta_2,\beta_5,v)]\!]\\&=([X,Y];\pounds_X\beta_2-\mathbf{i}_Y\mathbf{d}\alpha_2,\pounds_X\beta_5-\mathbf{i}_Y\mathbf{d}\alpha_5+\mathbf{d}\alpha_2\wedge\beta_2,\pounds_Xv-\mathbf{d}\alpha_2\diamond\beta_5+\mathbf{d}\alpha_5\diamond\beta_2),\end{aligned}$$

where $(\mathbf{d}\alpha\diamond\beta)(X)=(\mathbf{i}_X\mathbf{d}\alpha)\wedge\beta$ for all $X\in\mathfrak{X}(M)$. The dual map $\mathcal{D}$ is then given as follows: $\mathcal{D}_{(X;\alpha_2,\alpha_5,u)}(T_2,T_5,T_7\otimes Z;\theta)$ is

$$(\pounds_XT_2-T_5\neg\mathbf{d}\alpha_2+T_7\neg\mathbf{i}_Z\mathbf{d}\alpha_5,\pounds_XT_5-T_7\neg\mathbf{i}_Z\mathbf{d}\alpha_2,0;\pounds_X\theta+\mathbf{d}\alpha_2(T_2,\cdot)-\mathbf{d}\alpha_5(T_5,\cdot))$$

Remark 2.7. Note that all examples for Dorfman brackets in this paper are *local*, i.e. their brackets are given in terms of differential operators in both components. There are non-local Leibniz algebroids, for an example see Appendix C.

3. Linear sections of $TE\oplus T^*E\to E$

In this section, we recall some background notions on double vector bundles. Then we describe the double vector bundle structures on TE, on T^*E and on $TE\oplus T^*E$, for a vector bundle $E\to M$. In the last part of this section, we characterise arbitrary linear sections of $TE\oplus T^*E\to E$ via a certain class of derivations.

3.1. **Double vector bundles and linear splittings.** We briefly recall the definitions of double vector bundles and of their **linear** and **core** sections. We refer to [24, 22, 10] for more detailed treatments. A **double vector bundle** is a commutative square

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B \\ {\scriptstyle\pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M \end{array}$$

of vector bundles such that

$$(d_1+_A d_2)+_B(d_3+_A d_4)=(d_1+_B d_3)+_A(d_2+_B d_4) \tag{10}$$

for $d_1,d_2,d_3,d_4\in D$ with $\pi_A(d_1)=\pi_A(d_2)$, $\pi_A(d_3)=\pi_A(d_4)$ and $\pi_B(d_1)=\pi_B(d_3)$, $\pi_B(d_2)=\pi_B(d_4)$. Here, $+_A$ and $+_B$ are the additions in $D\to A$ and $D\to B$, respectively. The vector bundles A and B are called the **side bundles**. The **core** C of a double vector bundle is the intersection of the kernels of π_A and of π_B. From (10) follows easily the existence of a natural vector bundle structure on C over M. The inclusion $C\hookrightarrow D$ is denoted by $C_m\ni c\longmapsto\overline{c}\in\pi_A^{-1}(0_m^A)\cap\pi_B^{-1}(0_m^B)$.

The space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by two special classes of sections (see [23]), the **linear** and the **core sections** which we now describe. For a section $c\colon M\to C$, the corresponding **core section** $c^\dagger\colon B\to D$ is defined as $c^\dagger(b_m)=\widetilde{0}_{b_m}+_A\overline{c(m)}$, $m\in M$, $b_m\in B_m$. We denote the corresponding core section $A\to D$ by $c^\dagger$ also, relying on the argument to distinguish between them. The space of core sections of D over B is written as $\Gamma_B^c(D)$.

A section $\xi\in\Gamma_B(D)$ is called **linear** if $\xi\colon B\to D$ is a bundle morphism from $B\to M$ to $D\to A$ over a section $a\in\Gamma(A)$. The space of linear sections of D over B is denoted by $\Gamma_B^\ell(D)$. Given $\psi\in\Gamma(B^*\otimes C)$, there is a linear section $\widetilde{\psi}\colon B\to D$ over the zero section $0^A\colon M\to A$ given by $\widetilde{\psi}(b_m)=\widetilde{0}_{b_m}+_A\overline{\psi(b_m)}$. We call $\widetilde{\psi}$ a **core-linear section**.

3.2. **The tangent double and the cotangent double of a vector bundle.** Let $q_E\colon E\to M$ be a vector bundle. Then the tangent bundle TE has two vector bundle structures; one as the tangent bundle of the manifold E, and the second as a vector bundle over TM. The structure maps of $TE\to TM$ are the derivatives of the structure maps of $E\to M$.

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ {\scriptstyle Tq_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM & \xrightarrow[p_M]{} & M \end{array}$$

The space TE is a double vector bundle with core bundle $E\to M$. The map$^{-}\colon E\to p_E^{-1}(0^E)\cap(Tq_E)^{-1}(0^{TM})$ sends $e_m\in E_m$ to $\bar e_m=\left.\frac{d}{dt}\right|_{t=0}te_m\in T_{0^E_m}E$. Hence the core vector field corresponding to $e\in\Gamma(E)$ is the vertical lift $e^\uparrow\colon E\to TE$, i.e. the vector field with flow $\phi^{e^\uparrow}\colon E\times\mathbb{R}\to E$, $\phi^{e^\uparrow}_t(e'_m)=e'_m+te(m)$. An element of $\Gamma^\ell_E(TE)=\mathfrak{X}^\ell(E)$ is called a **linear vector field**. It is well-known (see e.g. [22]) that a linear vector field $\xi\in\mathfrak{X}^l(E)$ covering $X\in\mathfrak{X}(M)$ corresponds to a derivation $D\colon\Gamma(E)\to\Gamma(E)$ over $X\in\mathfrak{X}(M)$. The precise correspondence is given by

$$\xi(\ell_\varepsilon)=\ell_{D^*(\varepsilon)}\quad\text{and}\quad\xi(q_E^*f)=q_E^*(X(f)) \tag{11}$$

for all $\varepsilon\in\Gamma(E^*)$ and $f\in C^\infty(M)$, where $D^*\colon\Gamma(E^*)\to\Gamma(E^*)$ is the dual derivation to D. We write $\widehat{D}$ for the linear vector field in $\mathfrak{X}^l(E)$ corresponding in this manner to a derivation D of $\Gamma(E)$. Given a derivation D over $X\in\mathfrak{X}(M)$, the explicit formula for $\widehat{D}$ is

$$\widehat{D}(e_m)=T_meX(m)+_E\left.\frac{d}{dt}\right|_{t=0}(e_m-tD(e)(m)) \tag{12}$$

for $e_m\in E$ and any $e\in\Gamma(E)$ such that $e(m)=e_m$.

Dualizing TE over E, we get the double vector bundle

$$\begin{array}{ccc} T^*E & \xrightarrow{c_E} & E \\ {\scriptstyle r_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ E^* & \xrightarrow[q_{E^*}]{} & M \end{array}\quad.$$

The map r_E is given as follows. For θ_{e_m}, $r_E(\theta_{e_m})\in E^*_m$,

$$\langle r_E(\theta_{e_m}),e'_m\rangle=\left\langle\theta_{e_m},\left.\frac{d}{dt}\right|_{t=0}e_m+te'_m\right\rangle$$

for all $e'_m\in E_m$. The addition in $T^*E\to E^*$ is defined as follows. If θ_{e_m} and $\omega_{e'_m}$ are such that $r_E(\theta_{e_m})=r_E(\omega_{e'_m})=\varepsilon_m\in E^*_m$, then the sum $\theta_{e_m}+_{r_E}\omega_{e'_m}\in T^*_{e_m+e'_m}E$ is given by

$$\langle\theta_{e_m}+_{E^*}\omega_{e'_m},v_{e_m}+_{TM}v_{e'_m}\rangle=\langle\theta_{e_m},v_{e_m}\rangle+\langle\omega_{e'_m},v_{e'_m}\rangle$$

for all $v_{e_m}\in T_{e_m}E$, $v_{e'_m}\in T_{e'm}E$ such that $(q_E)_*(v_{e_m})=(q_E)_*(v_{e'_m})$.

For $\varepsilon\in\Gamma(E^*)$, the one-form $\mathbf{d}\ell_\varepsilon$ is linear over ε: we have $r_E(\mathbf{d}_{e_m}\ell_\varepsilon)=\varepsilon(m)$ for all $m\in M$ and the sum $\mathbf{d}_{e_m}\ell_\varepsilon+_{r_E}\mathbf{d}_{e'_m}\ell_\varepsilon$ equals $\mathbf{d}_{e_m+e'_m}\ell_\varepsilon$. For $\theta\in\Omega^1(M)$, the one-form $q_E^*\theta$ is a core section of $TE\to E$: $r_E((q_E^*\theta)(e_m))=0^{E^*}_m$, and for $\phi\in\Gamma(\mathrm{Hom}(E,T^*M))$ the core-linear section $\widetilde{\phi}\in\Gamma^l_E(T^*E)$ is given by $\widetilde{\phi}(e_m)=$

$(T_{e_m}q_E)^*\phi(e_m)$ for all $e_m \in E$. The vector space $T^*_{e_m}E$ is spanned by $\mathbf{d}_{e_m}\ell_\varepsilon$ and $\mathbf{d}_{e_m}(q_E^*f)$ for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$. Finally note that $\mathbf{d}\ell_{f\varepsilon} = q_E^*\mathbf{d}\ell\varepsilon + \varepsilon \otimes \mathbf{d}f$ for all $\varepsilon \in \Gamma(E^*)$ and $f \in C^\infty(M)$.

By taking the direct sum over E of TE and T^*E, we get a double vector bundle

$$\begin{array}{ccc} TE \oplus T^*E & \xrightarrow{\pi_E} & E \\ {\scriptstyle \Phi_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ TM \oplus E^* & \xrightarrow[q_{TM\oplus E^*}]{} & M \end{array}$$

with side projection $\Phi_E = (q_E)_* \oplus r_E$ and core $E \oplus T^*M$. In the following, for any section (e,θ) of $E \oplus T^*M$, the vertical section $(e,\theta)^\uparrow \in \Gamma_E(T^{q_E}E \oplus (T^{q_E}E)^\circ)$ is the pair defined by

$$(e,\theta)^\uparrow(e'_m) = \left(\left.\frac{d}{dt}\right|_{t=0} e'_m + te(m), (T_{e'_m}q_E)^t\theta(m) \right) \tag{13}$$

for all $e'_m \in E$. Note that by construction the vertical sections $(e,\theta)^\uparrow$ are core sections of $TE \oplus T^*E$ as a vector bundle over E.

The standard Courant algebroid structure over E is linear and

$$\begin{array}{ccc} TE \oplus T^*E & \xrightarrow{\Phi_E := (q_{E*}, r_E)} & TM \oplus E^* \\ {\scriptstyle \pi_E}\downarrow & & \downarrow \\ E & \xrightarrow[q_E]{} & M \end{array}$$

is a VB-Courant algebroid ([20], see also [17]) with base E and side $TM \oplus E^* \to M$, and with core $E \oplus T^*M \to M$.

The anchor $\Theta = \mathrm{pr}_{TE}\colon TE \oplus T^*E \to TE$ restricts to the map $\partial_E = \mathrm{pr}_E\colon E \oplus T^*M \to E$ on the cores, and defines an anchor $\rho_{TM\oplus E^*} = \mathrm{pr}_{TM}\colon TM \oplus E^* \to TM$ on the side. In other words, the anchor of $(e,\theta)^\uparrow$ is $e^\uparrow \in \mathfrak{X}^c(E)$ and if χ is a linear section of $TE \oplus T^*E \to E$ over $(X,\epsilon) \in \Gamma(TM \oplus E^*)$, the anchor $\Theta(\chi) \in \mathfrak{X}^l(E)$ is linear over X.

3.3. **The first jet bundle of a vector bundle.** For convenience of the exposition in the next section and later on in the paper, we recall here some basic facts about the first jet bundle of a vector bundle, and we set some notations.

The first jet bundle J^1E of a vector bundle E over M is the space $\{\eta_m \in \mathrm{Hom}(T_mM, T_{e_m}E) \mid m \in M,\, e_m \in E_m\}$. It has a projection to $\mathrm{pr}_E\colon J^1E \to E$ to E, $\eta_m \in \mathrm{Hom}(T_mM, T_{e_m}E) \mapsto e_m$ and a projection to $\mathrm{pr}\colon J^1E \to M$ to M, $\eta_m \mapsto m$. This second projection is the projection of a vector bundle structure over M; for $\eta_m \in \mathrm{Hom}(T_mM, T_{e_m}E)$ and $\mu_m \in \mathrm{Hom}(T_mM, T_{e'_m}E)$, we have $\alpha\eta_m + \beta\mu_m \in \mathrm{Hom}(T_mM, T_{\alpha e_m + \beta e'_m}E)$,

$$(\alpha\eta_m + \beta\mu_m)(v_m) = \alpha\eta_m(v_m) +_{TM} \beta\mu_m(v_m),$$

where $+_{TM}$ is the addition in the tangent prolongation $TE \to TM$ of the vector bundle $E \to M$. For each $\phi_m \in \mathrm{Hom}(T_mM, E_m)$ we get an element $\iota(\phi_m) \in J^1E_m$ with $\mathrm{pr}_E(\iota(\phi_m)) = 0^E_m$, $\iota(\phi_m)(v_m) = T_m0^E(v_m) + \left.\frac{d}{dt}\right|_{t=0} t\phi_m(v_m)$.

Two elements $\eta_m \in \operatorname{Hom}(T_mM, T_{e_m}E)$ and $\mu_m \in \operatorname{Hom}(T_mM, T_{e_m}E)$ differ by such an element $\phi_m \in \operatorname{Hom}(T_mM, E_m)$ and we have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}(TM, E) \xrightarrow{\iota} J^1E \xrightarrow{\operatorname{pr}_E} E \to 0$$

of vector bundles over M. The corresponding sequence

$$0 \longrightarrow \Gamma(\operatorname{Hom}(TM, E)) \xrightarrow{\iota} \Gamma(J^1E) \xrightarrow{\operatorname{pr}_E} \Gamma(E) \to 0$$

is canonically split by the map $j^1 \colon \Gamma(E) \to \Gamma(J^1E)$, $(j^1e)_m \in \operatorname{Hom}(T_mM, T_{e_m}E)$, $(j^1e)_m(v_m) = T_me(v_m)$. In particular, given $m \in M$ and two sections $e, e' \in \Gamma(E)$ with $e(m) = e'(m)$, we find $(j^1e)_m = (j^1e')_m + \iota(\phi_m)$ for a $\phi_m \in \operatorname{Hom}(T_mM, E_m)$. In other words, there is a canonical isomorphism

$$\Gamma(J^1E) \cong \Gamma(E) \oplus \Gamma(T^*M \otimes E), \quad \mu \mapsto (\operatorname{pr}_E \mu, \mu - j^1(\operatorname{pr}_E \mu)). \tag{14}$$

Furthermore, we have $j^1(e_1 + e_2) = j^1e_1 + j^1e_2$ and $J^1(fe) = fj^1e + \iota(\mathbf{d}f \otimes e)$ for all $e, e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

Note finally that every element $\mu \in J^1_m(E)$ can be written $\mu = (j^1e)_m$ with a local section $e \in \Gamma(E)$. Furthermore, two local sections $e, e' \in \Gamma(E)$ define the same element $(j^1e)_m = (j^1e')_m =: \mu \in J^1_m(E)$ if and only if $T_me = T_me'$ as vector space morphisms $T_mM \to T_{e(m)}E$. That is, $e(m) = e'(m)$ and $T_me(v_m) = T_me'(v_m)$ for all $m \in T_mM$. The later is equivalent to $v_m\langle \epsilon, e\rangle = (T_mev_m)(\ell_\epsilon) = (T_me'v_m)(\ell_\epsilon) = v_m\langle \epsilon, e'\rangle$ for all $v_m \in T_mM$ and all $\epsilon \in \Gamma(E^*)$, and so to

$$\langle \mathbf{d}_{\epsilon(m)}\ell_e, T_m\epsilon v_m\rangle = \langle \mathbf{d}_{\epsilon(m)}\ell_{e'}, T_m\epsilon v_m\rangle$$

for all $v_m \in T_mM$ and all $\epsilon \in \Gamma(E^*)$. Hence, $(j^1e)_m = (j^1e')_m$ if and only if $\mathbf{d}_\epsilon\ell_e = \mathbf{d}_\epsilon\ell_{e'}$ for all $\epsilon \neq 0 \in E^*_m$; by continuity then $\mathbf{d}_\epsilon\ell_e = \mathbf{d}_\epsilon\ell_{e'}$ for all $\epsilon \in E^*_m$.

3.4. **The E^*-valued Courant algebroid structure on the fat bundle $\widehat{E}$.** The space $\Gamma^l_E(TE \oplus T^*E)$ is a $C^\infty(M)$-module: choose $f \in C^\infty(M)$ and $\chi \in \Gamma^l_E(TE \oplus T^*E)$ a linear section over $\nu \in \Gamma(TM \oplus E^*)$. Then $q_E^*f \cdot \chi$ is linear over $f\nu \in \Gamma(TM \oplus E^*)$. The space $\Gamma^l_E(TE \oplus T^*E)$ is a locally free and finitely generated $C^\infty(M)$-module (this follows from the existence of local splittings). Hence, there is a vector bundle $\widehat{E}$ over M such that $\Gamma^l_E(TE \oplus T^*E)$ is isomorphic to $\Gamma(\widehat{E})$ as $C^\infty(M)$-modules. The vector bundle $\widehat{E}$ is called the *fat vector bundle* defined by $\Gamma^l_E(TE \oplus T^*E)$. We prove below that it is isomorphic to $\operatorname{Der}(E^*) \oplus J^1(E^*)$, where $\operatorname{Der}(E^*)$ is the bundle of derivations on E^*, and $J^1(E^*)$ the first jet bundle.

First recall that (11) defines a bijection between the linear vector fields $\mathfrak{X}^l(E)$ and $\Gamma(\operatorname{Der}(E^*))$. It is easy to see from (11) that this bijection is a morphism of $C^\infty(M)$-modules. Hence, the fat bundle defined by $\mathfrak{X}^l(E) = \Gamma^l_E(TE)$ is the vector bundle $\operatorname{Der}(E^*)$.

Next note that $\Gamma^l_E(T^*E)$ fits in the short exact sequence

$$0 \longrightarrow \Gamma(\operatorname{Hom}(E, T^*M)) \xrightarrow{\widetilde{\cdot}} \Gamma^l_E(T^*E) \xrightarrow{r_E} \Gamma(E^*) \longrightarrow 0,$$

of $C^\infty(M)$-modules, where the second map sends $\phi \in \Gamma(\operatorname{Hom}(E, T^*M))$ to the core-linear section $\widetilde{\phi} \in \Gamma^l_E(T^*E)$, $\widetilde{\phi}(e) = (T_eq_E)^*\phi(e)$ for all $e \in E$, and the third map sends $\theta \in \Gamma^l_E(T^*E)$ to its base section $r_E\theta$ in $\Gamma(E^*)$. We define $\Psi \colon \Gamma(J^1E^*) \to \Gamma^l_ET^*E)$ by $\Psi(j^1\epsilon) = \mathbf{d}\ell_\epsilon$ for $\epsilon \in \Gamma(E^*)$ and $\Psi(\iota\phi) = \widetilde{\phi^*} \in \Gamma^l_E(T^*E)$

for $\phi \in \Gamma(\mathrm{Hom}(TM, E^*))$. The map Ψ is $C^\infty(M)$-linear and we get the following commutative diagram of morphisms of $C^\infty(M)$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\mathrm{Hom}(TM, E^*)) & \xrightarrow{\iota} & \Gamma(J^1E^*) & \xrightarrow{\mathrm{pr}_{E^*}} & \Gamma(E^*) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\cdot)^*} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \Gamma(\mathrm{Hom}(E, T^*M)) & \xrightarrow[\widetilde{\cdot}]{} & \Gamma^l_E(T^*E) & \xrightarrow[r_E]{} & \Gamma(E^*) & \longrightarrow & 0 \end{array}$$

with short exact sequences in the top and bottom rows. Since the left and right vertical arrows are isomorphisms, Ψ is an isomorphism by the five lemma. Since Ψ is an isomorphism of $C^\infty(M)$-modules, we obtain a vector bundle isomorphism $\psi\colon J^1E^* \to \widehat{T^*E}$, where $\widehat{T^*E}$ is the fat bundle defined by $\Gamma^l_E(T^*E)$. Finally we obtain a vector bundle isomorphism

$$\Theta\colon \mathrm{Der}(E^*) \oplus J^1(E^*) \to \widehat{E}, \qquad (D_m, (j^1\epsilon)_m) \mapsto \mathrm{ev}_m\left(\widehat{D^*}, \mathbf{d}\ell_\epsilon\right). \tag{15}$$

Recall that for a linear section $\chi \in \Gamma^l_E(TE \oplus T^*E)$, there exists a section $\nu \in \Gamma(TM \oplus E^*)$ such that $\pi_{TM\oplus E^*} \circ \chi = \nu \circ q_E$. The map $\chi \mapsto \nu$ induces a short exact sequence of vector bundles

$$0 \longrightarrow E^* \otimes (E \oplus T^*M) \hookrightarrow \widehat{E} \longrightarrow TM \oplus E^* \longrightarrow 0.$$

Note that the restriction of the pairing on $TE \oplus T^*E$ to linear sections of $TE \oplus T^*E$ defines a nondegenerate pairing on $\widehat{E}$ with values in E^*. Since the Courant bracket of linear sections is again linear, the vector bundle $\widehat{E}$ inherits a Courant algebroid structure with pairing in E^* (see [17]). In particular, the Courant algebroid structure on $TE \oplus T^*E$ defines a Leibniz bracket on sections of $\mathrm{Der}(E^*) \oplus J^1(E^*)$ and a pairing with values in E^* on

$$\left(\mathrm{Der}(E^*) \oplus J^1(E^*)\right) \times_M \left(\mathrm{Der}(E^*) \oplus J^1(E^*)\right).$$

This is called an *Omni-Lie algebroid* in [5], see also [6]. The symmetric bilinear nondegenerate pairing with values in E^* on $\widehat{E}$ is given by $\langle \Theta(D(m)), \Theta((j^1\epsilon)_m + \iota\phi_m)\rangle = \langle \widehat{D^*}, \mathbf{d}\ell_\epsilon + \widetilde{\phi}\rangle(m) = D(\epsilon)(m) + \phi^*(X)(m)$ for D a derivation with symbol $X \in \mathfrak{X}(M)$. Here, the second term is the evaluation at $m \in M$ of the linear function $\ell_{D\epsilon+\phi^*X}$, when identified with $D\epsilon + \phi^*X \in \Gamma(E^*)$. Hence, the corresponding symmetric bilinear nondegenerate pairing with values in E^* on $J^1(E^*) \oplus \mathrm{Der}(E^*)$ is given by $\left\langle D_m, (j^1\epsilon)_m + \iota\phi_m\right\rangle = D_m(\epsilon) + \phi(X)(m)$ for $\epsilon \in \Gamma(E^*)$, $\phi \in \Gamma(\mathrm{Hom}(TM, E^*))$ and $D_m \in \mathcal{D}_m(E^*)$ with symbol $X \in \mathfrak{X}(M)$.

3.5. **Linear sections of $TE \oplus T^*E \to E$.** In this section we build on the techniques summarised in Section 3.2 and we prove original results on linear sections of $TE \oplus T^*E \to E$. Those results will be the basis of our main theorem in Section 4.

We consider a linear section $\chi \in \Gamma^l_E(TE \oplus T^*E)$ over a pair $(X, \varepsilon) \in \Gamma(TM \oplus E^*)$. Given a section $e \in \Gamma(E)$, the difference $\chi(e(m)) - (T_m eX(m), \mathbf{d}_{e(m)}\ell_\varepsilon)$ projects to $e(m)$ in E and to $0_m \in TM \oplus E^*$ and we can define a section $D_\chi(e, 0)\colon M \to E \oplus T^*M$ by

$$\chi(e(m)) - (T_m eX(m), \mathbf{d}_{e(m)}\ell_\varepsilon) = -D_\chi(e, 0)^\uparrow(e(m))$$

for all $m \in M$. By construction and the scalar multiplication in the fibers of $TE \oplus T^*E \to TM \oplus E^*$, we get $D_\chi(re, 0) = rD_\chi(e, 0)$ for a real number $r \in \mathbb{R}$, and

$D_\chi(e_1+e_2,0) = D_\chi(e_1,0)+D_\chi(e_2,0)$ for $e_1, e_2 \in \Gamma(E)$. For a smooth function $f \in C^\infty(M)$, we have $\chi((fe)(m)) = \chi(f(m)e(m))$ and

$$\left(T_m(fe)X(m), \mathbf{d}_{f(m)e(m)}\ell_\varepsilon\right) = \left(T_m(f(m)e)X(m) + (X(f)e)^\uparrow(f(m)e(m)), \mathbf{d}_{f(m)e(m)}\ell_\varepsilon\right).$$

Hence, we find that

$$D_\chi(fe,0) = fD_\chi(e,0) + (X(f)e,0). \tag{16}$$

Now we set $D_\chi\colon \Gamma(E\oplus T^*M) \to \Gamma(E\oplus T^*M)$, $D_\chi(e,\theta) = D_\chi(e,0) + (0, \pounds_X\theta)$. (16) and Theorem 3.3 below shows that D_χ is a smooth derivation. We have found the following result:

Theorem 3.1. *Let χ be a linear section of $TE\oplus T^*E \to E$ over a pair $(X,\varepsilon) \in \Gamma(TM\oplus E^*)$. Then there exists a unique derivation $D_\chi\colon \Gamma(E\oplus T^*M) \to \Gamma(E\oplus T^*M)$ with symbol $X \in \mathfrak{X}(M)$ and which satisfies*

1. *$D_\chi(e,\theta) = D_\chi(e,0) + (0,\pounds_X\theta)$ and*
2. *$\chi(e(m)) = (T_m eX(m), \mathbf{d}_{e(m)}\ell_\varepsilon) - D_\chi(e,0)^\uparrow(e(m))$,*

for all $e \in \Gamma(E)$ and $\theta \in \Omega^1(M)$.

Conversely, given a pair $(X,\varepsilon) \in \Gamma(TM\oplus E^*)$ and a smooth derivation $D\colon \Gamma(E\oplus T^*M) \to \Gamma(E\oplus T^*M)$ over $X \in \mathfrak{X}(M)$, we write $\chi_{\varepsilon,D}$ for the linear section defined by

$$\chi_{\varepsilon,D}(e(m)) = (T_m eX(m), \mathbf{d}_{e(m)}\ell_\varepsilon) - D(e,0)^\uparrow(e(m))$$

for all $e \in \Gamma(E)$. Note that (1) in the last theorem shows that for each $\chi \in \Gamma^l_E(TE\oplus T^*E)$ there exist a derivation $d_\chi \in \Gamma(\mathrm{Der}(E))$ and a tensor $\phi_\chi \in \Gamma(E^*\otimes T^*M)$ with $D_\chi(e,0) = (d_\chi e, \phi_\chi(e))$. More precisely, $d_\chi = \mathrm{pr}_E \circ D_\chi \circ \iota_E\colon \Gamma(E) \to \Gamma(E)$ is a derivation of E with symbol X and the vector bundle morphism is $\phi_\chi = \mathrm{pr}_{T^*M} \circ D_\chi \circ \iota_E\colon E \to T^*M$. The linear section χ can then be written

$$\chi = \left(\widehat{d_\chi}, \mathbf{d}\ell_\varepsilon - \widetilde{\phi_\chi}\right)$$

Remark 3.2. With the results in Section 3.4, we can phrase this correspondence in terms of the bundle isomorphism $\widehat{E} \cong \mathrm{Der}(E^*)\oplus J^1(E^*)$: $\chi = (\widehat{d_\chi}, \mathbf{d}\ell_\varepsilon - \widetilde{\phi_\chi}) \in \Gamma(\widehat{E})$ corresponds to $(d_\chi, j^1\varepsilon - \iota(\phi_\chi))$ in $\Gamma(\mathrm{Der}(E^*)\oplus J^1(E^*))$.

We can use these results on linear sections to prove the following:

Theorem 3.3. *Let χ be a linear section of $TE\oplus T^*E \to E$ over $(X,\varepsilon) \in \Gamma(TM\oplus E^*)$. The Courant-Dorfman bracket on sections of $TE\oplus T^*E \to E$ satisfies*

$$[\![\chi, \tau^\uparrow]\!] = D_\chi \tau^\uparrow$$

and the pairing

$$\langle \chi, \tau^\uparrow\rangle = q_E^*\langle (X,\varepsilon), \tau\rangle.$$

for all $\tau \in \Gamma(E\oplus T^*M)$. The anchor satisfies $\mathrm{pr}_{TE}(\chi) = \widehat{d_\chi}$.

We prove the first identity in Appendix A. The second and third identities follow immediately from (3.5).

We now state our first main theorem.

Theorem 3.4. *Choose two linear sections $\chi_1, \chi_2 \in \Gamma^l_E(TE \oplus T^*E)$, over pairs $(X_1, \varepsilon_1), (X_2, \varepsilon_2) \in \Gamma(TM \oplus E^*)$. Then we have*

$$\begin{aligned}\llbracket \chi_1, \chi_2 \rrbracket &= \left(\widehat{[d_{\chi_1}, d_{\chi_2}]}, \mathbf{d}\ell_{\mathrm{pr}_{E^*} D^*_{\chi_1}(X_2,\varepsilon_2)} - \widetilde{\mathrm{pr}_{T^*M} \circ [D_{\chi_1}, D_{\chi_2}] \circ \iota_E}\right) \\ &= \chi_{\mathrm{pr}_{E^*} D^*_{\chi_1}(X_2,\varepsilon_2), [D_{\chi_1}, D_{\chi_2}]}\end{aligned} \tag{17}$$

and $\langle \chi_1, \chi_2 \rangle = \ell_{\mathrm{pr}_{E^*}(D^*_{\chi_1}(X_2,\varepsilon_2) + D^*_{\chi_2}(X_1,\varepsilon_1))}$.

The Theorem is again proved in Appendix A and gives us an expression for the induced E^*-valued Courant bracket on $\mathrm{Der}(E^*) \oplus J^1(E^*)$:

Corollary 3.5. *Choose $d_1, d_2 \in \Gamma(\mathrm{Der}(E^*))$ with symbols $X_1, X_2 \in \mathfrak{X}(M)$ and choose $\mu_1, \mu_2 \in \Gamma(J^1(E^*))$ corresponding as in* (14) *to $(\varepsilon_1, \phi_1), (\varepsilon_2, \phi_2) \in \Gamma(E^*) \oplus \Gamma(T^*M \otimes E^*)$. Then*

$$\llbracket (d_1, \mu_1), (d_2, \mu_2) \rrbracket = \left([d_1, d_2], \pounds_{d_1}\mu_2 - \pounds_{d_2}\mu_1 + j^1 \langle d_2, \mu_1 \rangle\right), \tag{18}$$

where the $\mathrm{Der}(E^*)$*-Lie derivative on $J^1(E^*)$ is defined in equation (19) of* [5]*:*

$$\pounds_d \mu = \pounds_d(\varepsilon, \phi) = (d\varepsilon, (\pounds_X \circ \phi^* - \phi^* \circ d^*)^*)$$

where d is a derivation of E^* with symbol $X \in \mathfrak{X}(M)$ and $\mu = (\varepsilon, \phi) \in \Gamma(J^1E^*) \simeq \Gamma(E^* \oplus \mathrm{Hom}(TM, E^*))$. Thus, our theorem proves that the E^*-valued Courant algebroid structure on $\mathrm{Der}(E^*) \oplus J^1(E^*)$ given in [5] *is precisely induced from $TE \oplus T^*E$ via the isomorphism Ψ from 3.4.*

Proof. With the correspondence between $\Gamma(\widehat{E})$ and $\Gamma(\mathrm{Der}(E^*) \oplus \Gamma(J^1(E^*)) = \Gamma(\mathrm{Der}(E^*) \oplus E^* \oplus \mathrm{Hom}(TM, E^*))$, $(d_i, (\varepsilon_i, \phi_i))$ corresponds to $\chi_i = \chi_{\varepsilon_i, D_i}$ with $D_i(e, 0) = (d_i(e), -\phi_i^*(e))$. Then we have $\mathrm{pr}_{E^*} D^*_{\chi_1}(X_2, \varepsilon_2) = d_1(\varepsilon_2) + \phi_1(X_2)$ as well as

$$\mathrm{pr}_{T^*M}([D_{\chi_1}, D_{\chi_2}](e, 0)) = -\phi_1^*(d_2^* e) + \phi_2^*(d_1^* e) - \pounds_{X_1}(\phi_2^*(e)) + \pounds_{X_2}(\phi_1^*(e)).$$

By the considerations in Section 3.4, we have further $\langle d_2, \mu_1 \rangle = d_2\varepsilon_1 + \phi_1(X_2)$. We get using the isomorphisms $\Gamma(J^1E^*) \simeq \Gamma(E^* \oplus \mathrm{Hom}(TM, E^*))$ and $\Gamma^l_E(TE \oplus T^*E) \simeq \Gamma(J^1E^* \oplus \mathrm{Der}(E^*))$:

$$\begin{aligned}\llbracket (d_1, \mu_1), (d_2, \mu_2) \rrbracket &= \llbracket (d_1, \varepsilon_1, \phi_1), (d_2, \varepsilon_2, \phi_2) \rrbracket = \left\llbracket \left(\widehat{d_1}, \mathbf{d}\ell_{\varepsilon_1} + \widetilde{\phi_1}\right), \left(\widehat{d_2}, \mathbf{d}\ell_{\varepsilon_2} + \widetilde{\phi_2}\right) \right\rrbracket \\ &= \llbracket \chi_{\epsilon_1, D_1}, \chi_{\epsilon_2, D_2} \rrbracket = \chi_{(d_1(\varepsilon_2) + \phi_1(X_2)), [D_1, D_2]} \\ &= ([d_1, d_2], d_1(\varepsilon_2) + \phi_1(X_2), \pounds_{d_1}\phi_2 - \pounds_{d_2}\phi_1) \\ &= ([d_1, d_2], 0, 0) + (0, d_2\varepsilon_1 + \phi_1(X_2), 0) + (0, \pounds_{d_1}(\varepsilon_2, \phi_2) - \pounds_{d_2}(\varepsilon_1, \phi_1)) \\ &= ([d_1, d_2], j^1\langle d_2, \mu_1 \rangle + \pounds_{d_1}\mu_2 - \pounds_{d_2}\mu_1).\end{aligned}$$

□

Note that the derivation D_χ defines as follows a derivation of $\mathrm{Hom}(E, E \oplus T^*M)$: $(D_\chi \varphi)(e) = D_\chi(\varphi(e)) - \varphi(d_\chi(e))$ for all $e \in \Gamma(E)$.

Corollary 3.6. *In the situation of the preceding theorem, the Courant-Dorfman bracket satisfies $\llbracket \chi, \widetilde{\varphi} \rrbracket = \widetilde{D_\chi \varphi}$ for $\varphi \in \Gamma(\mathrm{Hom}(E, E \oplus T^*M))$.*

Proof. The section $\varphi \in \Gamma(E^* \otimes (E \oplus T^*M))$ can be written as $\varphi = (\phi_1, \phi_2^*)$, with $\phi_1 \in \Gamma(E^* \otimes E)$ and $\phi_2 \in \Gamma(\mathrm{Hom}(TM, E^*))$. Furthermore, ϕ defines a section of $\mathrm{Der}(E^*) \oplus J^1(E^*)$: ϕ_1^* is a derivation of E^* with symbol $0 \in \mathfrak{X}(M)$ and $\phi_2 \simeq \iota\phi_2$ is a section of J^1E^*. Therefore $\widetilde{\phi}$ is simply the corresponding core-linear section under the correspondence outlined above. Choose $\chi = (d, \mu)$ with d a derivation of E^* over $X \in \mathfrak{X}(M)$ and $\mu = j^1\varepsilon + \iota\psi \in \Gamma(J^1E^*)$. Then the results above yield

$$[\![(d,\mu),(\phi_1,\phi_2)]\!] = [\![(d,\varepsilon,\psi),(\phi_1^*,0,\phi_2)]\!] = ([d,\phi_1^*], (\pounds_X \circ \phi_2^* - \phi_2^* \circ d^*)^* + \phi_1^* \circ \psi),$$

which is easily seen to be $D_\chi\varphi$. □

3.6. **Linear closed 3-forms.** Let E be as usual a vector bundle over M. A k-form H on E is **linear** if the induced vector bundle morphism $H^\sharp\colon \oplus^{k-1}TE \to T^*E$ over the identity on E is *also* a vector bundle morphism over a map $h\colon \oplus^{k-1}TM \to E^*$ on the other side of the double vector bundles [2].

According to Proposition 1 in [2], a linear k-form $H \in \Omega^k(E)$ can be written

$$H = \mathbf{d}\Lambda_\mu + \Lambda_\omega$$

with $\mu \in \Omega^{k-1}(M, E^*)$ and $\omega \in \Omega^k(M, E^*)$. Here, given $\omega \in \Omega^k(M, E^*)$, the k-form $\Lambda_\omega \in \Omega^k(E)$ is given by

$$\Lambda_\omega(e_m) = (T_{e_m}q_E)^*(\langle\omega, e\rangle(m)),$$

where $\langle\omega, e\rangle \in \Omega^k(M)$ is the obtained k-form on M. Note that in the equation for H, we have $\mu = (-1)^{k-1}h$.

Example 3.7. For instance, we have seen in §3.2 that for $\varepsilon \in \Gamma(E^*)$, the 1-form $\mathbf{d}\ell_\varepsilon \in \Omega^1(E)$ is linear. Since it projects to $\varepsilon \in \Gamma(E^*)$, we know that any linear 1-form on E can be written $\mathbf{d}\ell_\varepsilon + \widetilde{\phi}$ for $\varepsilon \in \Gamma(E^*) = \Omega^0(M, E^*)$ and $\phi \in \Gamma(\mathrm{Hom}(E, T^*M)) = \Omega^1(M, E^*)$. An easy computation shows $\Lambda_\varepsilon = \ell_\varepsilon \in \Omega^0(E) = C^\infty(E)$ and $\Lambda_\phi = \widetilde{\phi} \in \Omega^1(E)$.

Proposition 3.8. *Consider a linear k-form $H = \mathbf{d}\Lambda_\mu + \Lambda_\omega$, with $\mu \in \Omega^{k-1}(M, E^*)$ and $\omega \in \Omega^k(M, E^*)$. Then H is closed, $\mathbf{d}H = 0$, if and only if $\omega = 0$.*

Proof. H is closed if and only if Λ_ω is closed. It is enough to evaluate $\mathbf{d}\Lambda_\omega$ on linear and core vector fields on E. Take k linear vector fields $\widehat{D_i} \in \mathfrak{X}^l(E)$ over $X_i \in \mathfrak{X}(M)$, $i = 1, \ldots, k$, and one vertical vector field $e^\uparrow \in \mathfrak{X}^c(E)$. Then

$$\begin{aligned}(\mathbf{d}\Lambda_\omega)\left(\widehat{D_1},\ldots,\widehat{D_k},e^\uparrow\right) =& \sum_{i=1}^k(-1)^{i+1}\widehat{D_i}\left(\Lambda_\omega\left(\widehat{D_1},\ldots,\hat{i}\ldots,\widehat{D_k},e^\uparrow\right)\right)\\ &+(-1)^k e^\uparrow\left(\Lambda_\omega\left(\widehat{D_1},\ldots,\widehat{D_k}\right)\right)\\ &+\sum_{1\le i\le j\le k}(-1)^{i+j}\Lambda_\omega\left(\left[\widehat{D_i},\widehat{D_j}\right],\widehat{D_1},\ldots,\hat{i},\ldots,\hat{j},\ldots,\widehat{D_{k-1}},e^\uparrow\right)\\ &+\sum_{i=1}^k(-1)^{i+k}\Lambda_\omega\left(\left[\widehat{D_i},e^\uparrow\right],\widehat{D_1},\ldots,\hat{i},\ldots,\widehat{D_k}\right).\end{aligned}$$

Since $\left[\widehat{D_i}, e^\uparrow\right]$ is again a vertical vector field and Λ_ω vanishes on vertical vector fields, the first, third and fourth terms of this sum all vanish. The remaining term is

$$(-1)^k e^\uparrow\left(\Lambda_\omega\left(\widehat{D_1},\ldots,\widehat{D_k}\right)\right) = (-1)^k q_E^*\langle\omega(X_1,\ldots,X_k),e\rangle.$$

This is 0 for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$ if and only if $\omega = 0$. □

In what follows, we will consider closed linear 3-forms $H = \mathbf{d}\Lambda_\mu$ with $\mu \in \Omega^2(M, E^*)$ the base map of $H^\sharp$. Let us compute the inner product of such a 3-form with two linear vector fields on E.

Recall that any linear vector field can be written $\widehat{D} \in \mathfrak{X}^l(E)$ with a derivation $D\colon \Gamma(E) \to \Gamma(E)$ over $X \in \mathfrak{X}(M)$. The derivation D induces a derivation $D\colon \Omega^1(M, E^*) \to \Omega^1(M, E^*)$ by

$$(D\omega)(Y) = D^*(\omega(Y)) - \omega[X, Y]$$

for all $\omega \in \Omega^1(M, E^*)$ and $Y \in \mathfrak{X}(M)$. In particular, given a Dorfman bracket on sections of $TM \oplus E^*$, the linear vector field $\mathrm{pr}_{TE}\, \Xi(\nu)$ equals $\widehat{\delta_\nu}$, where ν is a section of $TM \oplus E^*$ and δ_ν is the derivation over $\mathrm{pr}_{TM}\, \nu$. We write δ_ν for the induced derivation of $\Omega^1(M, E^*)$.

Proposition 3.9. *Choose $\mu \in \Omega^2(M, E^*)$. Let $\widehat{D_1}, \widehat{D_2}, \widehat{D} \in \mathfrak{X}^l(E)$ be linear vector fields over $X_1, X_2, X \in \mathfrak{X}(M)$ and let e be a section of E. Then*

$$\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = \mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu} + \widetilde{D_1(\mathbf{i}_{X_2}\mu)} - \widetilde{D_2(\mathbf{i}_{X_1}\mu)} - \widetilde{\mathbf{i}_{[X_1,X_2]}\mu} \tag{19}$$

and

$$\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D}}\mathbf{d}\Lambda_\mu = -q_E^*\langle \mathbf{i}_X\mu, e\rangle. \tag{20}$$

Proof. We have for $e \in \Gamma(E)$:

$$\begin{aligned}
\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu &= \widehat{D_1}\left(\Lambda_\mu\left(\widehat{D_2}, e^\uparrow\right)\right) - \widehat{D_2}\left(\Lambda_\mu\left(\widehat{D_1}, e^\uparrow\right)\right) + e^\uparrow\left(\Lambda_\mu\left(\widehat{D_1}, \widehat{D_2}\right)\right)\\
&\quad - \Lambda_\mu\left(\left[\widehat{D_1}, \widehat{D_2}\right], e^\uparrow\right) + \Lambda_\mu\left(\left[\widehat{D_1}, e^\uparrow\right], \widehat{D_2}\right) - \Lambda_\mu\left(\left[\widehat{D_2}, e^\uparrow\right], \widehat{D_1}\right)\\
&= 0 - 0 + e^\uparrow(\ell_{\mu(X_1,X_2)}) - 0 + 0 - 0 = q_E^*\langle \mu(X_1, X_2), e\rangle.
\end{aligned}$$

This shows that $\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = \mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu} + \widetilde{\phi}$ for a section $\phi \in \Gamma(E^* \otimes T^*M)$. We have then

$$\begin{aligned}
\ell_{\langle\phi,X_3\rangle} &= \langle\widetilde{\phi}, \widehat{D_3}\rangle = \mathbf{i}_{\widehat{D_3}}\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu - \widehat{D_3}(\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu})\\
&= \widehat{D_1}\left(\Lambda_\mu\left(\widehat{D_2}, \widehat{D_3}\right)\right) - \widehat{D_2}\left(\Lambda_\mu\left(\widehat{D_1}, \widehat{D_3}\right)\right) + \cancel{\widehat{D_3}\left(\Lambda_\mu\left(\widehat{D_1}, \widehat{D_2}\right)\right)}\\
&\quad - \Lambda_\mu\left(\left[\widehat{D_1}, \widehat{D_2}\right], \widehat{D_3}\right) + \Lambda_\mu\left(\left[\widehat{D_1}, \widehat{D_3}\right], \widehat{D_2}\right) - \Lambda_\mu\left(\left[\widehat{D_2}, \widehat{D_3}\right], \widehat{D_1}\right)\\
&\quad - \cancel{\widehat{D_3}(\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu})}\\
&= \widehat{D_1}(\ell_{\mu(X_2,X_3)}) - \widehat{D_2}(\ell_{\mu(X_1,X_3)}) - \Lambda_\mu\left(\widehat{[D_1, D_2]}, \widehat{D_3}\right)\\
&\quad + \Lambda_\mu\left(\widehat{[D_1, D_3]}, \widehat{D_2}\right) - \Lambda_\mu\left(\widehat{[D_2, D_3]}, \widehat{D_1}\right)\\
&= \ell_{D_1^*(\mu(X_2,X_3))} - \ell_{D_2^*(\mu(X_1,X_3))} - \ell_{\mu([X_1,X_2],X_3)} + \ell_{\mu([X_1,X_3],X_2)} - \ell_{\mu([X_2,X_3],X_1)}\\
&= \ell_{\langle D_1(\mathbf{i}_{X_2}\mu) - D_2(\mathbf{i}_{X_1}\mu) - \mathbf{i}_{[X_1,X_2]}\mu, X_3\rangle}
\end{aligned}$$

and we find (19). In order to prove (20), we use the equation

$$\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = q_E^*\langle \mu(X_1, X_2), e\rangle$$

above to find that

$$\mathbf{i}_{\widehat{D_2}}\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = -\mathbf{i}_{\widehat{D_2}}(q_E^*\langle \mu(X_1), e\rangle)$$

for all linear $\widehat{D_2} \in \mathfrak{X}^l(E)$. Since $\mathbf{i}_{e'^\uparrow}\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = 0 = -\mathbf{i}_{e'^\uparrow}(q_E^*\langle \mu(X_1), e\rangle)$ for all $e' \in \Gamma(E)$, we find that $\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D_1}}\mathbf{d}\Lambda_\mu = -q_E^*\langle \mu(X_1), e\rangle$. □

We will use the following lemma.

Lemma 3.10. *Choose an element* $\beta \in \Omega^1(M, E^*)$ *and a linear vector field* $\widehat{D} \in \mathfrak{X}^l(E)$ *over* $X \in \mathfrak{X}(M)$*. Then* $\mathbf{i}_{\widehat{D}}\mathbf{d}\Lambda_\beta$ *is a linear* 1*-form over* $\beta(X) \in \Gamma(E^*)$*. More precisely,*

$$\mathbf{i}_{\widehat{D}}\mathbf{d}\Lambda_\beta = -\mathbf{d}\ell_\beta + \widetilde{D\beta}.$$

Proof. We have

$$\mathbf{i}_{e^\uparrow}\mathbf{i}_{\widehat{D}}\mathbf{d}\Lambda_\beta = \widehat{D}\langle \Lambda_\beta, e^\uparrow\rangle - e^\uparrow\langle \Lambda_\beta, \widehat{D}\rangle - \langle \Lambda_\beta, (De)^\uparrow\rangle = -e^\uparrow(\ell_{\beta(X)}) = -q_E^*\langle e, \beta(X)\rangle.$$

Therefore $\mathbf{i}_{\widetilde{X}}\mathbf{d}\Lambda_\beta = -\mathbf{d}\ell_{\beta(X)} + \widetilde{\phi}$ with a section $\phi \in \Gamma(\operatorname{Hom}(E, T^*M))$ to be determined. We have

$$\begin{aligned}
\left\langle \widetilde{\phi}, \widehat{D'} \right\rangle &= \langle \mathbf{i}_{\widehat{D}}\mathbf{d}\Lambda_\beta + \mathbf{d}\ell_{\beta(X)}, \widehat{D'}\rangle \\
&= \widehat{D}(\ell_{\beta(Y)}) - \cancel{\widehat{D'}(\ell_{\beta(X)})} - \ell_{\beta[X,Y]} + \cancel{\widehat{D'}(\ell_{\beta(X)})} \\
&= \ell_{D^*(\beta(Y)) - \beta[X,Y]} = \ell_{(D\beta)(Y)} = \left\langle \widetilde{D\beta}, \widehat{D'} \right\rangle
\end{aligned}$$

for any linear vector field $\widehat{D'} \in \mathfrak{X}^l(E)$ over $Y \in \mathfrak{X}(M)$. This shows that $\phi = D\beta$. □

4. Dorfman brackets and natural lifts.

Consider now an $\mathbb{R}$-linear lift

$$\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma_E^l(TE \oplus T^*E),$$

sending each section (X, ε) of $TM \oplus E^*$ to a linear section over (X, ε). Then the lift defines an $\mathbb{R}$-linear map

$$\mathcal{D}\colon \Gamma(TM \oplus E^*) \to \operatorname{Der}(E \oplus T^*M), \quad \Xi(X, \varepsilon) = \chi_{\varepsilon, \mathcal{D}_{(X,\varepsilon)}}.$$

Consider the dual

$$[\![\cdot\,,\cdot]\!]\colon \Gamma(TM \oplus E^*) \times \Gamma(TM \oplus E^*) \to \Gamma(TM \oplus E^*)$$

of $\mathcal{D}$, written in bracket form and defined by

$$X\langle \nu, \tau\rangle = \langle \nu, \mathcal{D}_{(X,\varepsilon)}\tau\rangle + \langle [\![(X,\varepsilon), \nu]\!], \tau\rangle$$

for all $(X, \varepsilon), \nu \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$. Any bracket defined in this manner is $\mathbb{R}$-bilinear, anchored by pr_{TM} and satisfies a Leibniz identity in its second component. We easily get the following result.

Proposition 4.1. *Lifts*

$$\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma_E^l(TE \oplus T^*E),$$

sending each section (X, ε) *of* $TM \oplus E^*$ *to a linear section over* (X, ε) *are equivalent to* $\mathbb{R}$*-bilinear brackets* $[\![\cdot\,,\cdot]\!]\colon \Gamma(TM \oplus E^*) \times \Gamma(TM \oplus E^*) \to \Gamma(TM \oplus E^*)$*, that are anchored by* pr_{TM} *and satisfy a Leibniz identity in the second component.*

Define further the map $\delta\colon \Gamma(TM \oplus E^*) \to \operatorname{Der}(E)$ by

$$\delta_\nu = \operatorname{pr}_E \circ \mathcal{D}_\nu \circ \iota_E.$$

As we have seen before, the lift $\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma_E^l(TE \oplus T^*E)$ can be written

$$\Xi(X, \varepsilon)(e_m) = (T_m e X(m), \mathbf{d}_{e_m}\ell_\varepsilon) - (\mathcal{D}_{(X,\varepsilon)}(e, 0))^\uparrow(e_m), \tag{21}$$

for any $e \in \Gamma(E)$ with $e(m) = e_m$, or

$$\Xi(X, \varepsilon) = (\widehat{\delta_{(X,\varepsilon)}}, \mathbf{d}\ell_\varepsilon - \widetilde{\mathrm{pr}_{T^*M}\,\mathcal{D}_{(X,\varepsilon)} \circ \iota_E}).$$

In terms of sections of the Omni-Lie algebroid $\mathrm{Der}(E^*) \oplus J^1(E^*)$, this says that anchored $\mathbb{R}$-bilinear brackets on $TM \oplus E^*$ with Leibniz rule in the second component are in one-to-one correspondence with splittings

$$\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma(\mathrm{Der}(E^*) \oplus J^1(E^*)),$$

of the short exact sequence

$$0 \to \Gamma(E^* \otimes (E \oplus T^*M)) \to \Gamma(\mathrm{Der}(E^*) \oplus J^1(E^*)) \to \Gamma(TM \oplus E^*) \to 0$$

Note that in either description the map Ξ is a map of sections only, so its image will in general not span a sub-vector bundle of $\widehat{E} \cong \mathrm{Der}(E^*) \oplus J^1(E^*)$.

We prove the following theorem, which shows that a chosen lift as above is *natural*, if and only if the bracket $[\![\cdot\,,\cdot]\!]$ is a Dorfman bracket.

Theorem 4.2. *Let E be a smooth vector bundle over a manifold M. Consider an $\mathbb{R}$-bilinear bracket $[\![\cdot\,,\cdot]\!]$ on sections of $TM \oplus E^*$, that is anchored by pr_{TM} and satisfies the Leibniz identity in its second component. Then $[\![\cdot\,,\cdot]\!]$ is a Dorfman bracket if and only if the corresponding lift as in Proposition 4.1 or* (21) *is natural, i.e. if and only if*

$$[\![\Xi(\nu_1), \Xi(\nu_2)]\!] = \Xi[\![\nu_1, \nu_2]\!]$$

for all $\nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$, where the bracket on the left-hand side is the Courant-Dorfman bracket.

The proof of this theorem follows from the general results in 3.5 and is given in Appendix B. Note that the proof of this theorem can also be adapted in a straightforward manner from the proof of the main theorem in [16] (see the following remark); the only difference being that $\mathcal{D}$ is not $C^\infty(M)$-linear in its lower entry. The proof in [16] is however independent of this property.

Remark 4.3. Note that horizontal lifts $\sigma\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$ satisfying $\sigma(\nu_1 + \nu_2) = \sigma(\nu_1) + \sigma(\nu_2)$ and $\sigma(f \cdot \nu) = q_E^* f \sigma(\nu)$ are called *linear*. The horizontal lifts above are in general not linear; they are additive, but in general they are not $C^\infty(M)$-homogeneous.

Linear horizontal lifts $\sigma\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$ were proved in [16] to be equivalent to dull brackets on sections of $TM \oplus E^*$, or equivalently to Dorfman connections $\Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$.

Let $\Delta\colon \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \to \Gamma(E \oplus T^*M)$ be a Dorfman connection and consider the dual dull bracket $[\![\cdot\,,\cdot]\!]_\Delta$. Note that the map

$$\nabla\colon \Gamma(TM \oplus E^*) \times \Gamma(E) \to \Gamma(E), \qquad \nabla_\nu e = \mathrm{pr}_E(\Delta_\nu(e, 0))$$

is a linear connection. Choose $\nu, \nu_1, \nu_2 \in \Gamma(TM \oplus E^*)$ and $\tau \in \Gamma(E \oplus T^*M)$. [16] proves the following identities

(1) $\left\langle \sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2) \right\rangle = \ell_{[\![\nu_1,\nu_2]\!]_\Delta + [\![\nu_2,\nu_1]\!]_\Delta}$,
(2) $\left\langle \sigma^\Delta(\nu), \tau^\uparrow \right\rangle = q_E^* \langle \nu, \tau \rangle$,
(3) $\mathrm{pr}_{TE}\left(\sigma^\Delta(\nu)\right) = \widehat{\nabla_\nu}$ and $\mathrm{pr}_{TE}(\tau^\uparrow) = (\mathrm{pr}_E \tau)^\uparrow$,
(4) $\left[\!\left[\sigma^\Delta(\nu), \tau^\uparrow\right]\!\right] = (\Delta_\nu \tau)^\uparrow$,
(5) $\left[\!\left[\sigma^\Delta(\nu_1), \sigma^\Delta(\nu_2)\right]\!\right] = \sigma^\Delta([\![\nu_1, \nu_2]\!]_\Delta) - R_\Delta(\widetilde{\nu_1, \nu_2}) \circ \iota_E$.

Those results could now be easily deduced from Theorems 3.3 and 3.4, as we deduce our main result Theorem 4.2 from those.

Here, we have the following result, a counterpart for Dorfman brackets of the results described in Remark 4.3. Note that since $[\![\cdot\,,\cdot]\!]$ is anchored by pr_{TM}, the sum $[\![\nu_1,\nu_2]\!]+[\![\nu_2,\nu_1]\!]$ is a section of E^* for all $\nu_1,\nu_2\in\Gamma(TM\oplus E^*)$.

Theorem 4.4. *Let $[\![\cdot\,,\cdot]\!]$ be a Dorfman bracket on sections of $TM\oplus E^*$. For all $\nu,\nu_1,\nu_2\in\Gamma(TM\oplus E^*)$ and $\tau\in\Gamma(E\oplus T^*M)$, we have*

(1) $\langle\Xi(\nu_1),\Xi(\nu_2)\rangle=\ell_{[\![\nu_1,\nu_2]\!]+[\![\nu_2,\nu_1]\!]}$ *and* $\left\langle\Xi(\nu),\tau^\uparrow\right\rangle=q_E^*\langle\nu,\tau\rangle$,
(2) $\mathrm{pr}_{TE}(\Xi(\nu))=\widehat{\delta_\nu}$ *and* $\mathrm{pr}_{TE}(\tau^\uparrow)=(\mathrm{pr}_E\,\tau)^\uparrow$,
(3) $\left[\!\left[\Xi(\nu),\tau^\dagger\right]\!\right]=\mathcal{D}_\nu\tau^\dagger$.

Proof. Those identities are all given by Theorems 3.3 and 3.4. □

4.1. **Links to known results on Omni-Lie algebroids, on Dorfman connections and on the standard VB-Courant algebroid.** [5, 6] prove the following result on Lie algebroid structures on subbundles of $TM\oplus E^*$ versus Dirac structures inside the E^*-valued Courant-algebroid $\mathcal{E}:=\mathrm{Der}(E^*)\oplus J^1(E^*)$. Note that such a Dirac structure is called *reducible* if its projection to $TM\oplus E^*$ is surjective.

Theorem 4.5. (Theorem 3.7 in [6]) *There is a one-to-one correspondence between reducible Dirac structures $L\subset\mathcal{E}$ and projective Lie algebroids $A\subset TM\oplus E^*$ such that A is the quotient Lie algebroid of L. (As a Dirac structure, L carries a Lie bracket induced by the Courant-Dorfman bracket.)*

A *projective Lie algebroid* is a subbundle $A\subset TM\oplus E^*$ with a Lie algebroid structure $(A,[\cdot,\cdot]_A,\rho_A)$, with anchor given by $\rho_A=\mathrm{pr}_{TM}|_A$. A *reducible* Dirac structure $L\subset\mathcal{E}$ is a Dirac structure the image of which in $TM\oplus E^*$ under $\mathbf{b}\colon\mathcal{E}\to TM\oplus E^*$ is a regular subbundle. The correspondence in the theorem is such that $A=\mathbf{b}(L)$, and the Lie bracket is the quotient Lie bracket on A induced by the short exact sequence

$$0\to A^0\to L\xrightarrow{\mathbf{b}}A\to 0$$

For details, see [6].

This result, our results from Section 4 and the results from [16] as outlined in Remark 4.3, suggest the following relationships between subspaces of $\Gamma(\widehat{E})\cong\Gamma(\mathcal{E})$ that are closed under $[\![\cdot,\cdot]\!]$ and project to locally-free subsheaves of $\Gamma(TM\oplus E^*)$, and $\mathbb{R}$-bilinear brackets on subbundles of $TM\oplus E^*$:

Let $\mathcal{V}\subset\Gamma(\widehat{E})$ be a subspace that is closed under $[\![\cdot,\cdot]\!]$ and such that $\mathcal{V}$ maps to $\Gamma(F), F$ a subbundle of $TM\oplus E^*$. Then, collectively, we have the following results:

4.1.1. *Setting 1:* $F=TM\oplus E^*,\mathcal{V}=\Im(\Xi)$, where $\Xi\colon\Gamma(TM\oplus E^*)\to\Gamma(\widehat{E})$ is a splitting of $p:\Gamma(\widehat{E})\to\Gamma(TM\oplus E^*)$. This is just the setting of Proposition 4.1, i.e. such lifts precisely correspond to brackets on $TM\oplus E^*$ that satisfy a Leibniz identity in the second component.
Now, if $\mathcal{V}$ is additionally a sub-vector bundle of $\widehat{E}$ and Ξ a morphism of vector bundles, we are in the setting of dull brackets and Dorfman connections, as studied in [16], i.e. the resulting bracket satisfies the Leibniz identity also in its first component.
If instead (or additionally) the lift Ξ is *natural*, i.e. $[\![\Xi\cdot,\Xi\cdot]\!]=\Xi[\![\cdot,\cdot]\!]$, the bracket on $TM\oplus E^*$ satisfies the Dorfman condition (the Jacobi identity in Leibniz form).

If $\mathcal{V}$ is such that $\langle \nu, \nu' \rangle = 0$ for all $\nu, \nu' \in \mathcal{V}$, the bracket $[\![\cdot,\cdot]\!]$ on $TM \oplus E^*$ is antisymmetric (see Theorem 3.4).

4.1.2. *Setting 2:* $\mathcal{V} = \Gamma(L), L \subset \widehat{E}$ a Dirac structure. This is the case studied by [5, 6] as described above. The parallels to the first setting are obvious: $\mathcal{V}$ is closed under $[\![\cdot,\cdot]\!]$, which is necessary to induce an $\mathbb{R}$-bilinear bracket on its projection to $TM \oplus E^*$ at all, $\mathcal{V}$ is isotropic under $\langle \cdot,\cdot \rangle$, so the resulting bracket is antisymmetric, and $\mathcal{V}$ is given by the sections of a vector bundle, i.e. the Leibniz rule in the first component is satisfied.
However, in this case there is not necessarily a splitting $\Xi\colon F \to L$.

4.1.3. *Setting 3:* Of course the first two settings are not mutually exclusive: According to our results, Dirac structures $L \subset \widehat{E}$ which project surjectively to $TM \oplus E^*$ allow a lift $\Xi : TM \oplus E^* \to L$, which is natural – a projective Lie bracket on $TM \oplus E^*$ is in particular a Dorfman bracket.

5. Standard examples

We illustrate the result in Theorem 4.2 with the examples of standard Dorfman brackets on $TM \oplus E^*$, by giving explicitly the lifts.

5.1. **Lift of the Courant-Dorfman bracket.** We consider here the case where $E = TM$ and the Dorfman bracket on $TM \oplus T^*M$ is the Courant-Dorfman bracket

$$[\![(X_1,\theta_1),(X_2,\theta_2)]\!] = ([X_1,X_2], \pounds_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1)$$

for $X_1, X_2 \in \mathfrak{X}(M)$ and $\theta_1,\theta_2 \in \Omega^1(M)$. First, recall that the derivation $\mathcal{D}\colon \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ is just $\mathcal{D}_{\nu_1}\nu_2 = [\![\nu_1,\nu_2]\!]$. Hence, by definition, the value $\Xi(X,\theta)(Y(m))$ is

$$\left(T_mYX(m) - \left.\frac{d}{dt}\right|_{t=0} (Y + t[X,Y])(m), \mathbf{d}\ell_\theta - (T_{Y(m)}p_M)^*(-\mathbf{i}_Y\mathbf{d}\theta) \right).$$

Using (12), we get $\Xi\colon \Gamma(TM \oplus T^*M) \to \Gamma^l_{TM}(TTM \oplus T^*TM)$,

$$\Xi(X,\theta) = \left(\widehat{[X,\cdot]}, \mathbf{d}\ell_\theta - \widetilde{\mathbf{d}\theta} \right), \tag{22}$$

where $\widetilde{\mathbf{d}\theta}$ is the one-form on TM defined by $\widetilde{\mathbf{d}\theta}(v) = (T_v p_M)^*(-\mathbf{i}_v\mathbf{d}\theta) \in T^*_v(TM)$ for all $v \in TM$. This choice of sign is for consistency with the notations in the next section for the general case E, e.g. in the proof of (19). We have indeed $\langle \widetilde{\mathbf{d}\theta}, \widehat{D} \rangle = \ell_{\mathbf{i}_X\mathbf{d}\theta}$ for any derivation D of TM over $X \in \mathfrak{X}(M)$, since evaluated at $Y(m) \in TM$, $\langle \widetilde{\mathbf{d}\theta}, \widehat{D} \rangle (Y(m))$ is $\langle (T_{Y(m)}p_M)^*(-\mathbf{i}_{Y(m)}\mathbf{d}\theta), T_mY(X(m)) \rangle = \langle -\mathbf{i}_{Y(m)}\mathbf{d}\theta, X(m) \rangle = \ell_{\mathbf{i}_X\mathbf{d}\theta}(Y(m))$.

For the convenience of the reader, let us compute here explicitly the Courant-Dorfman bracket $[\![\Xi(X_1,\theta_1),\Xi(X_2,\theta_2)]\!]$ of two images of Ξ. The Lie bracket of two linear vector fields $\widehat{D_1}, \widehat{D_2} \in \mathfrak{X}^l(E)$ is $[\widehat{D_1},\widehat{D_2}] = \widehat{[D_1,D_2]} = \widehat{D_1 \circ D_2 - D_2 \circ D_1}$. To see this, one only needs to apply $[\widehat{D_1},\widehat{D_2}]$ on linear and pullback functions. Since $[[X_1,\cdot],[X_2,\cdot]]$ is $[[X_1,X_2],\cdot]$ by the Jacobi identity, we find that the Lie bracket of the vector fields $\widehat{[X_1,\cdot]}$ and $\widehat{[X_2,\cdot]}$ is $\widehat{[[X_1,X_2],\cdot]}$. Let us compute $\pounds_{\widehat{[X_1,\cdot]}}(\mathbf{d}\ell_{\theta_2} - \widetilde{\mathbf{d}\theta_2}) - \mathbf{i}_{\widehat{[X_2,\cdot]}}\mathbf{d}(\mathbf{d}\ell_{\theta_1} - \widetilde{\mathbf{d}\theta_1})$. We have $\pounds_{\widehat{[X_1,\cdot]}}\mathbf{d}\ell_{\theta_2} = \mathbf{d}\left(\widehat{[X_1,\cdot]}(\ell_{\theta_2})\right) = \mathbf{d}\ell_{\pounds_{X_1}\theta_2}$ and

$$\pounds_{\widehat{[X_1,\cdot]}}(\widetilde{\mathbf{d}\theta_2}) = \widetilde{\mathbf{d}(\pounds_{X_1}\theta_2)}.$$

The second equation is more difficult to see and requires some explanations. Take $Y \in \mathfrak{X}(M)$. Then

$$\begin{aligned}\left\langle \pounds_{\widehat{[X,\cdot]}}\widetilde{\mathbf{d}\theta}, \widehat{[Y,\cdot]}\right\rangle &= \widehat{[X,\cdot]}\left\langle \widetilde{\mathbf{d}\theta}, \widehat{[Y,\cdot]}\right\rangle - \left\langle \widetilde{\mathbf{d}\theta}, \widehat{[[X,Y],\cdot]}\right\rangle\\ &= \widehat{[X,\cdot]}\ell_{\mathbf{i}_Y\mathbf{d}\theta} - \ell_{\mathbf{i}_{[X,Y]}\mathbf{d}\theta} = \ell_{\pounds_X\mathbf{i}_Y\mathbf{d}\theta - \mathbf{i}_{[X,Y]}\mathbf{d}\theta}\\ &= \ell_{\mathbf{i}_Y\mathbf{d}\pounds_X\theta} = \left\langle \widetilde{\mathbf{d}\pounds_X\theta}, \widehat{[Y,\cdot]}\right\rangle.\end{aligned}$$

Since $\left\langle \pounds_{\widehat{[X,\cdot]}}\widetilde{\mathbf{d}\theta}, Y^\uparrow\right\rangle$ is easily seen to vanish, as $\left\langle \widetilde{\mathbf{d}\pounds_X\theta}, Y^\uparrow\right\rangle$ does, we find that $\pounds_{\widehat{[X,\cdot]}}\widetilde{\mathbf{d}\theta} = \widetilde{\mathbf{d}\pounds_X\theta}$. Therefore we get

$$\begin{aligned}&\pounds_{\widehat{[X_1,\cdot]}}(\mathbf{d}\ell_{\theta_2} - \widetilde{\mathbf{d}\theta_2}) - \mathbf{i}_{\widehat{[X_2,\cdot]}}\mathbf{d}(\mathbf{d}\ell_{\theta_1} - \widetilde{\mathbf{d}\theta_1})\\ =&\pounds_{\widehat{[X_1,\cdot]}}(\mathbf{d}\ell_{\theta_2} - \widetilde{\mathbf{d}\theta_2}) + \pounds_{\widehat{[X_2,\cdot]}}\widetilde{\mathbf{d}\theta_1} - \mathbf{d}(\mathbf{i}_{\widehat{[X_2,\cdot]}}\widetilde{\mathbf{d}\theta_1})\\ =&\mathbf{d}\ell_{\pounds_{X_1}\theta_2} - \mathbf{d}(\widetilde{\pounds_{X_1}\theta_2}) + \mathbf{d}(\widetilde{\pounds_{X_2}\theta_1}) - \mathbf{d}\langle\widetilde{\mathbf{d}\theta_1}, \widehat{[X_2,\cdot]}\rangle\\ =&\mathbf{d}\ell_{\pounds_{X_1}\theta_2} - \mathbf{d}(\widetilde{\pounds_{X_1}\theta_2}) + \mathbf{d}(\widetilde{\pounds_{X_2}\theta_1}) - \mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{d}\theta_1}.\end{aligned}$$

Since $\mathbf{d}(\pounds_{X_2}\theta_1) = \mathbf{d}(\mathbf{i}_{X_2}\mathbf{d}\theta_1)$, this shows

$$\begin{aligned}[\![\Xi(X_1,\theta_1), \Xi(X_2,\theta_2)]\!] &= \left(\widehat{[[X_1,X_2],\cdot]}, \mathbf{d}\ell_{\pounds_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1} - \widetilde{\mathbf{d}(\pounds_{X_1}\theta_2 - \mathbf{i}_{X_2}\mathbf{d}\theta_1)}\right)\\ &= \Xi[\![(X_1,\theta_1),(X_2,\theta_2)]\!].\end{aligned}$$

Remark 5.1. There is a canonical isomorphism of double vector bundles

$$\Sigma : T(TM \oplus T^*M) \to TTM \oplus T^*TM,$$

which maps the natural VB-Courant algebroid structure on $T(TM \oplus T^*M)$, the tangent prolongation of the standard Courant algebroid on $TM \oplus T^*M$, to the standard VB-Courant algebroid structure on $T(TM) \oplus T^*(TM)$. The lift Ξ is then precisely $\Xi = \Sigma \circ T$, where T denotes the tangent prolongation of a section,

$$(s\colon M \to TM \oplus T^*M) \mapsto (Ts : TM \to T(TM \oplus T^*M)).$$

A precise description and proof can be found in [18].

5.2. **Another lift to $TTM \oplus T^*TM$.** Consider this time the natural lift $\Xi\colon \Gamma(TM \oplus T^*M) \to \Gamma^l_{TM}(T(TM) \oplus T^*(TM))$, $\Xi(X,\theta) = \left(\widehat{[X,\cdot]}, \mathbf{d}\ell_\theta\right)$. This is equivalent to the Dorfman bracket $[\![\cdot\,,\cdot]\!]\colon \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$,

$$[\![(X_1,\theta_1),(X_2,\theta_2)]\!] = ([X_1,X_2], \pounds_{X_1}\theta_2).$$

To see this, let us compute the Courant-Dorfman bracket of $\Xi(X_1,\theta_1)$ with $\Xi(X_2,\theta_2)$. We have

$$[\![\Xi(X_1,\theta_1), \Xi(X_2,\theta_2)]\!] = \left(\left[\widehat{[X_1,\cdot]}, \widehat{[X_2,\cdot]}\right], \pounds_{\widehat{[X_1,\cdot]}}\mathbf{d}\ell_{\theta_2} - \mathbf{i}_{\widehat{[X_2,\cdot]}}\mathbf{d}^2\ell_{\theta_1}\right).$$

By the formulas found in the preceding example, we get

$$[\![\Xi(X_1,\theta_1), \Xi(X_2,\theta_2)]\!] = \left(\widehat{[[X_1,X_2],\cdot]}, \mathbf{d}\ell_{\pounds_{X_1}\theta_2}\right) = \Xi([X_1,X_2], \pounds_{X_1}\theta_2). \tag{23}$$

In fact, we call the lifts associated to Dorfman brackets "natural" because they generalise the properties of this lift.

5.3. **More general examples.** More generally, according to (21) the lift corresponding to the Dorfman bracket in Example 2.4 has the same form:

$$\begin{aligned}\Xi((X,\alpha))(e_m) &= \Big((T_m e)(X(m)) - (\pounds_X e)^{\uparrow}(e_m), \mathbf{d}\ell_\alpha(e_m) - \widetilde{\mathbf{d}\alpha}\Big)\\ &= \Big(\widehat{\pounds_X\cdot}, \mathbf{d}\ell_\alpha - \widetilde{\mathbf{d}\alpha}\Big)(e_m) \qquad (24)\end{aligned}$$

for all $e_m \in \wedge^k TM$, where, in the second equality, we have used the definition of the derivation $\widehat{D}$ in (12). Here in order to be consistent with the next section, as well as the previous example, $\widetilde{\mathbf{d}\alpha}$ is defined by:

$$\widetilde{\mathbf{d}\alpha}(e_m) = (T_{e_m}\operatorname{pr}_{TM})^*((-1)^k \mathbf{i}_{e_m}\mathbf{d}\alpha) = (T_{e_m}\operatorname{pr}_{TM})^*(\mathbf{d}\alpha(\cdot, e_m)).$$

In all examples so far, the lift $\Xi\colon \Gamma(TM\oplus E^*) \to \Gamma^l_E(TE\oplus T^*E)$ is really a direct sum of two lifts $\Xi_{TM}\colon \mathfrak{X}(M) \to \Gamma^l_E(TE)$ and $\Xi_{E^*}\colon \Gamma(E^*) \to \Gamma^l_E(T^*E)$. All the examples discussed so far are split Dorfman brackets. For these, we always have:

Proposition 5.2. *For all split Dorfman brackets on $TM\oplus E^*$, $\Xi(X,0) \in \mathfrak{X}^l(E)$.*

Proof. We show that $\mathcal{D}_{(X,0)}(e,0) = (\delta_{(X,0)}e, 0)$:

$$\begin{aligned}\big\langle \mathcal{D}_{(X,0)}(e,0),(Y,0)\big\rangle &= X\langle (Y,0),(e,0)\rangle - \langle [\![(X,0),(Y,0)]\!],(e,0)\rangle\\ &= -\langle ([X,Y],0),(e,0)\rangle = 0\end{aligned}$$

for all $Y \in \mathfrak{X}(M)$. □

However, for general split Dorfman brackets Ξ_{E^*} is a map $\Gamma(E^*) \to \Gamma^l_e(TE\oplus T^*E)$. For example the term in (8) gives rise to a term $(e\neg\mathbf{d}\alpha_k)^{\uparrow}(e_m) \in \Gamma(TE)$ in $\Xi_{E^*}(\alpha_k)(e_m)$. If the Dorfman bracket is *not* split, mixing can also occur in the TM-part of the lift: $\Xi_{TM}\colon TM \to \Gamma^l_E(TE\oplus T^*E)$, as illustrated by the following example:

Example 5.3. Let $H \in \Omega^3_{\mathrm{cl}}(M)$ be a closed 3-form. Then $[\![(X_1,\theta_1),(X_2,\theta_2)]\!]_H = [\![(X_1,\theta_1),(X_2,\theta_2)]\!] + (0, \mathbf{i}_{X_2}\mathbf{i}_{X_1}H)$ (with $[\![\cdot,\cdot]\!]$ the Courant-Dorfman bracket) is also a Dorfman bracket on $TM\oplus T^*M$. This Dorfman bracket is not split, and we have $\mathcal{D}^H_{(X,0)}(Y,0) = ([X,Y], \mathbf{i}_Y\mathbf{i}_X H)$ by Example 2.3, which shows

$$\Xi^H(X,0)(Y(m)) = (\Xi_{TM}(X), -p_M^*(\mathbf{i}_Y\mathbf{i}_X H))(Y(m)).$$

The following section studies in detail such *twistings* of Dorfman brackets in relation to their lifts.

6. Twisted Courant-Dorfman bracket over vector bundles.

Here we consider the standard Courant-Dorfman bracket on $TE\oplus T^*E$ over a vector bundle E, twisted by a linear closed 3-form $H \in \Omega^3(E)$. That is, we have

$$[\![(X_1,\alpha_1),(X_2,\alpha_2)]\!]_H = [\![(X_1,\alpha_1),(X_2,\alpha_2)]\!] + (0, \mathbf{i}_{X_2}\mathbf{i}_{X_1}H).$$

Given a form $\mu \in \Omega^2(M,E^*)$ and a Dorfman bracket $[\![\cdot\,,\cdot]\!]$ on sections of $TM\oplus E^*$, we can define a twisted bracket $[\![\cdot\,,\cdot]\!]_\mu\colon \Gamma(TM\oplus E^*)\times\Gamma(TM\oplus E^*) \to \Gamma(TM\oplus E^*)$ by

$$[\![(X_1,\epsilon_1),(X_2,\epsilon_2)]\!]_\mu = [\![(X_1,\epsilon_1),(X_2,\epsilon_2)]\!] + (0, \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu).$$

This satisfies a Leiniz equality in the second term (as always, with anchor pr_{TM}) and is compatible with the anchor. We make the following definition.

Definition 6.1. *Let* $[\![\cdot\,,\cdot]\!]\colon \Gamma(TM\oplus E^*)\times\Gamma(TM\oplus E^*)\to\Gamma(TM\oplus E^*)$ *be a Dorfman bracket and* $\mu\in\Omega^2(M,E^*)$ *a form. Then we say that* μ ***twists*** $[\![\cdot\,,\cdot]\!]$ *if* $[\![\cdot\,,\cdot]\!]_\mu$ *satisfies the Jacobi identity in Leibniz form, i.e. if* $[\![\cdot\,,\cdot]\!]_\mu$ *is a new Dorfman bracket.*

In this section we will describe in terms of the lift associated to $[\![\cdot\,,\cdot]\!]$ a necessary and sufficient condition for μ to twist $[\![\cdot\,,\cdot]\!]$.

Example 6.2. The standard Dorfman bracket on $TM\oplus\wedge^kT^*M$ (Example 2.4) is twisted by $\mu\in\Omega^2(M,\wedge^kT^*M)$ if and only if $\mu\in\Omega^{k+2}_{\mathrm{cl}}(M)$, i.e. actually antisymmetric in all components and closed.

We define the dual derivation $\mathcal{D}^\mu\colon\Gamma(TM\oplus E^*)\times\Gamma(E\oplus T^*M)\to\Gamma(E\oplus T^*M)$ to $[\![\cdot\,,\cdot]\!]_\mu$ and find

$$\mathcal{D}^\mu_{(X,\epsilon)}(e,\theta)=\mathcal{D}_{(X,\epsilon)}(e,\theta)-(0,\langle\mathbf{i}_X\mu,e\rangle). \tag{25}$$

The corresponding lift $\Xi^\mu\colon\Gamma(TM\oplus E^*)\to\Gamma^l_E(TE\oplus T^*E)$ as in (21) is then just

$$\Xi^\mu(X,\epsilon)=\Xi(X,\epsilon)+\widetilde{(0,\mathbf{i}_X\mu)}.$$

Recall that it is natural if and only if $[\![\cdot\,,\cdot]\!]_\mu$ satisfies the Jacobi identity.

Theorem 6.3. *With the notations above, we have*

$$[\![\Xi^\mu(\nu_1),\Xi^\mu(\nu_2)]\!]_{-\mathbf{d}\Lambda_\mu}=\Xi^\mu[\![\nu_1,\nu_2]\!]$$

for all $\nu_1,\nu_2\in\Gamma(TM\oplus E^*)$.

Proof. We just compute

$$\begin{aligned}
[\![\Xi^\mu(\nu_1),\Xi^\mu(\nu_2)]\!]_{-\mathbf{d}\Lambda_\mu}&=\left[\!\!\left[\Xi(\nu_1)+\widetilde{(0,\mathbf{i}_{X_1}\mu)},\Xi(\nu_2)+\widetilde{(0,\mathbf{i}_{X_2}\mu)}\right]\!\!\right]_{-\mathbf{d}\Lambda_\mu}\\
&\overset{(19)}{=}[\![\Xi(\nu_1),\Xi(\nu_2)]\!]-\left(0,\mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu}+\widetilde{\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu)}-\widetilde{\mathcal{D}_{\nu_2}(\mathbf{i}_{X_1}\mu)}-\widetilde{\mathbf{i}_{[X_1,X_2]}\mu}\right)\\
&\quad+\left[\!\!\left[\Xi(\nu_1),\widetilde{(0,\mathbf{i}_{X_2}\mu)}\right]\!\!\right]+\left[\!\!\left[\widetilde{(0,\mathbf{i}_{X_1}\mu)},\Xi(\nu_2)\right]\!\!\right]\\
&=\Xi[\![\nu_1,\nu_2]\!]-\left(0,\mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu}+\widetilde{\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu)}-\widetilde{\mathcal{D}_{\nu_2}(\mathbf{i}_{X_1}\mu)}-\widetilde{\mathbf{i}_{[X_1,X_2]}\mu}\right)\\
&\quad+\left(0,\widetilde{\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu)}\right)+\left(0,\mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu}\right)-\left(0,\widetilde{\mathcal{D}_{\nu_2}(\mathbf{i}_{X_1}\mu)}\right)\\
&=\Xi^\mu[\![\nu_1,\nu_2]\!]
\end{aligned}$$

□

In the third equality, we have used Lemma 3.6. We are now ready to prove our main theorem.

Theorem 6.4. *Consider a Dorfman bracket*

$$[\![\cdot\,,\cdot]\!]\colon\Gamma(TM\oplus E^*)\times\Gamma(TM\oplus E^*)\to\Gamma(TM\oplus E^*)$$

and the corresponding lift $\Xi\colon\Gamma(TM\oplus T^*M)\to\Gamma^l_{TM}(TE\oplus T^*E)$.

Then a form $\mu\in\Omega^2(M,E^*)$ *twists* $[\![\cdot\,,\cdot]\!]$ *if and only if*

$$[\![\Xi(\nu_1),\Xi(\nu_2)]\!]_{\mathbf{d}\Lambda_\mu}=\Xi[\![\nu_1,\nu_2]\!]_\mu$$

for all $\nu_1,\nu_2\in\Gamma(TM\oplus TM^*)$.

In other words, μ *twists a Dorfman bracket if and only its natural lift lifts the twisted bracket to the twist by* $\mathbf{d}\Lambda_\mu$ *of the Courant-Dorfman bracket.*

Note that we also have the following result, which follows from (20) and (25).

Proposition 6.5. *In the situation of the previous theorem, we have*

$$\left[\!\left[\Xi(\nu), \tau^\uparrow\right]\!\right]_{\mathbf{d}\Lambda_\mu} = \mathcal{D}^\mu_\nu \tau^\uparrow,$$

for $\nu \in \Gamma(TM \oplus E^*)$ *and* $\tau \in \Gamma(E \oplus T^*M)$*, no matter if* μ *twists the Dorfman bracket or not.*

Proof of Theorem 6.4. Assume that $[\![\cdot\,,\cdot]\!]_\mu$ is a Dorfman bracket. Then by Theorem 4.2, we have

$$[\![\Xi^\mu(\nu_1), \Xi^\mu(\nu_2)]\!] = \Xi^\mu[\![\nu_1,\nu_2]\!]_\mu = \Xi^\mu[\![\nu_1,\nu_2]\!] + \Xi^\mu(0,\mu(X_1,X_2)). \tag{26}$$

Since $\Xi^\mu(\nu) = \Xi(\nu) + \widetilde{(0, \mathbf{i}_X\mu)}$, we find that

$$\Xi^\mu(0,\mu(X_1,X_2)) = \Xi(0,\mu(X_1,X_2)) \tag{27}$$

and also that $\operatorname{pr}_{TE}\Xi^\mu(\nu) = \operatorname{pr}_{TE}\Xi(\nu) = \widehat{\delta_\nu}$ for all $\nu \in \Gamma(TM \oplus E^*)$. By Theorem 6.3, we have

$$[\![\Xi^\mu(\nu_1), \Xi^\mu(\nu_2)]\!] = \Xi^\mu[\![\nu_1,\nu_2]\!] + \left(0, \mathbf{i}_{\widehat{\delta_{\nu_2}}}\mathbf{i}_{\widehat{\delta_{\nu_1}}}\mathbf{d}\Lambda_\mu\right). \tag{28}$$

(26), (27) and (28) yield together $\Xi(0,\mu(X_1,X_2)) = \left(0, \mathbf{i}_{\widehat{\delta_{\nu_2}}}\mathbf{i}_{\widehat{\delta_{\nu_1}}}\mathbf{d}\Lambda_\mu\right)$, and so

$$\begin{aligned}[\![\Xi(\nu_1), \Xi(\nu_2)]\!]_{\mathbf{d}\Lambda_\mu} &= [\![\Xi(\nu_1), \Xi(\nu_2)]\!] + \left(0, \mathbf{i}_{\widehat{\delta_{\nu_2}}}\mathbf{i}_{\widehat{\delta_{\nu_1}}}\mathbf{d}\Lambda_\mu\right) \\ &= \Xi[\![\nu_1,\nu_2]\!] + \Xi(0,\mu(X_1,X_2)) = \Xi[\![\nu_1,\nu_2]\!]_\mu. \end{aligned}$$ □

Example 6.6. Consider $E = TM$ and choose the Courant-Dorfman bracket on $TM \oplus T^*M$. Recall from §5.1 the corresponding natural lift. Then if $\nu_1 = (X_1, \theta_1)$, we get $\delta_\nu X_2 = [X_1, X_2]$ and $\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu) = \pounds_{X_1}\mathbf{i}_{X_2}\mu$. As a consequence,

$$\begin{aligned}\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu) - \mathcal{D}_{\nu_2}(\mathbf{i}_{X_1}\mu) - \mathbf{i}_{[X_1,X_2]}\mu &= \mathbf{i}_{X_2}\pounds_{X_1}\mu - \pounds_{X_2}\mathbf{i}_{X_1}\mu \\ &= \mathbf{i}_{X_2}\mathbf{i}_{X_1}\mathbf{d}\mu - \mathbf{d}(\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu)\end{aligned} \tag{29}$$

and $\mathcal{D}_{\nu_1}(\mathbf{i}_{X_2}\mu) - \mathcal{D}_{\nu_2}(\mathbf{i}_{X_1}\mu) - \mathbf{i}_{[X_1,X_2]}\mu = -\mathbf{d}(\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu)$ if and only if μ is closed. We get then using (19)

$$\begin{aligned}&[\![\Xi(X_1,\theta_1), \Xi(X_2,\theta_2)]\!]_{\mathbf{d}\Lambda_\mu} \\ &= \Xi[\![(X_1,\theta_1),(X_2,\theta_2)]\!] + \left(0, \mathbf{i}_{\widehat{[X_2,\cdot]}}\mathbf{i}_{\widehat{[X_1,\cdot]}}\mathbf{d}\Lambda_\mu\right) \\ &= \Xi[\![(X_1,\theta_1),(X_2,\theta_2)]\!] + \left(0, \mathbf{d}\ell_{\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu} - \mathbf{d}\widetilde{(\mathbf{i}_{X_2}\mathbf{i}_{X_1}\mu)}\right) = \Xi[\![(X_1,\theta_1),(X_2,\theta_2)]\!]_\mu.\end{aligned}$$

7. Symmetries of Dorfman brackets

In this section we use the known symmetries of the standard Courant algebroid over E to study a similar class of symmetries for Dorfman brackets on $TM \oplus E^*$.

Consider $B \in \Omega^2_{\mathrm{cl}}(E)$. We denote by $\Phi_B\colon TE \oplus T^*E \to TE \oplus T^*E$ the vector bundle morphism over the identity on E that is defined by

$$\Phi_B(X,\theta) = (X, \theta + \mathbf{i}_X B)$$

for all $X \in \mathfrak{X}(E)$ and $\theta \in \Omega^1(E)$. Then Φ_B is a symmetry of the Courant-Dorfman bracket on $TE \oplus T^*E$ [3]:

$$[\![\Phi_B(\chi_1), \Phi_B(\chi_2)]\!] = \Phi_B[\![\chi_1, \chi_2]\!]$$

for all $\chi_1, \chi_2 \in \Gamma(TE \oplus T^*E)$.

According to [2] (see Section 6), given a form $\beta \in \Omega^1(M, E^*)$, the closed form $B = -\mathbf{d}\Lambda_\beta$ is linear. In particular, if $[\![\cdot\,,\cdot]\!]$ is a Dorfman bracket on $TM \oplus E^*$ and $\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$ the associated lift, $\Phi_B(\Xi(\nu)) = \Xi(\nu) + \mathbf{i}_{\widehat{\delta_\nu}} B$ is a linear section of $TE \oplus T^*E$ over $\Phi_\beta(\nu) = \nu + (0, \mathbf{i}_X\beta)$ (see Lemma 3.10), where $\Phi_\beta\colon TM \oplus E^* \to TM \oplus E^*$ is the vector bundle morphism over the identity on M:

$$\Phi_\beta(X, \epsilon) = (X, \epsilon + \mathbf{i}_X\beta).$$

In this section we aim to understand when this map defines a symmetry of a Dorfman bracket on $TM \oplus E^*$. We prove the following result.

Theorem 7.1. *A form $\beta \in \Omega^1(M, E^*)$ defines a symmetry of a Dorfman bracket $[\![\cdot,\cdot]\!]$ via $(X, \epsilon) \mapsto (X, \epsilon + \mathbf{i}_X\beta)$ if and only if*

$$\Phi_{-\mathbf{d}\Lambda_\beta} \circ \Xi = \Xi \circ \Phi_\beta$$

for the corresponding lift $\Xi\colon \Gamma(TM \oplus E^*) \to \Gamma^l_E(TE \oplus T^*E)$.

The proof relies on the following lemma. We set $B := -\mathbf{d}\Lambda_\beta$ for $\beta \in \Omega^1(M, E^*)$.

Lemma 7.2. *Choose $\phi \in \Gamma(\mathrm{Hom}(E, E \oplus T^*M))$. Then $\mathbf{d}\langle \Phi_B(\Xi(\nu)), \widetilde{\phi}\rangle$ is a core linear section of $T^*E \to E$ for all $\nu \in \Gamma(TM \oplus E^*)$ if and only if $\phi = 0$.*

Proof. Since $\langle \tilde{\phi}, \Phi_B(\Xi(\nu))\rangle$ is linear, $\mathbf{d}\langle \tilde{\phi}, \Phi_B(\Xi(\nu))\rangle$ is a core linear section if and only if $\Phi_E\left(\mathbf{d}\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle\right) = 0$. We have

$$\begin{aligned}\left\langle \Phi_E\left(\mathbf{d}\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle\right), e\right\rangle &= \left\langle \mathbf{d}\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle, e^\uparrow\right\rangle \\ &= e^\uparrow\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle.\end{aligned}$$

Write $\nu = (X, \epsilon) \in \Gamma(TM \oplus E^*)$. Since $\Phi_B(\Xi(X, \epsilon)) = \Xi(X, \epsilon) + (0, \mathbf{d}\ell_{\beta(X)} - \widetilde{\delta_{(X,\epsilon)}\beta})$, we find

$$\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle = \ell_{\phi^*(X, \epsilon + \beta(X))}$$

and so $e^\uparrow\left\langle \Phi_B(\Xi(\nu)), \tilde{\phi}\right\rangle = q_E^*\langle \phi^*(X, \epsilon + \beta(X)), e\rangle$. This vanishes for all $e \in \Gamma(E)$ and all $(X, \epsilon) \in \Gamma(TM \oplus E^*)$ if and only if $\phi^*(X, \epsilon + \beta(X)) = 0$ for all $(X, \epsilon) \in \Gamma(TM \oplus E^*)$. In particular, $\phi^*(0, \epsilon)$ must be 0 for all $\epsilon \in \Gamma(E^*)$ or, in other words, ϕ must have image in T^*M. Using this, we find $\phi^*(X, 0) = \phi^*(X, \beta(X))$ for $X \in \mathfrak{X}(M)$. Since this must vanish for all $X \in \mathfrak{X}(M)$, we have shown that ϕ must be 0. □

Proof of Theorem 7.1. We define $\phi_{(X,\epsilon)} \in \Gamma(\mathrm{Hom}(E, E \oplus T^*M))$ by

$$\widetilde{\phi_{(X,\epsilon)}} = \Xi(0, \mathbf{i}_X\beta) - (0, \mathbf{i}_{\widehat{\delta_{(X,\epsilon)}}} B) = \Xi(0, \mathbf{i}_X\beta) - \left(0, \mathbf{d}\ell_{\beta(X)} - \widetilde{\delta_{(X,\epsilon)}\beta}\right). \tag{30}$$

We have used Lemma 3.10. Note that this difference is a core-linear section of $TE \oplus T^*E$ because the linear sections $\Xi(0, \mathbf{i}_X\beta)$ and $\left(0, \mathbf{d}\ell_{\beta(X)} - \widetilde{\delta_{(X,\epsilon)}\beta}\right)$ both project to $(0, \mathbf{i}_X\beta)$ in $\Gamma(TM \oplus E^*)$.

Consider

$$[\![\Phi_\beta(X_1,\epsilon_1),\Phi_\beta(X_2,\epsilon_2)]\!] = [\![(X_1,\epsilon_1+\mathbf{i}_{X_1}\beta),(X_2,\epsilon_2+\mathbf{i}_{X_2}\beta)]\!]$$

in $\Gamma(TM\oplus E^*)$. This lifts to $\Xi[\![(X_1,\epsilon_1+\mathbf{i}_{X_1}\beta),(X_2,\epsilon_2+\mathbf{i}_{X_2}\beta)]\!]$, which equals

$$[\![\Xi(X_1,\epsilon_1)+\Xi(0,\mathbf{i}_{X_1}\beta),\Xi(X_2,\epsilon_2)+\Xi(0,\mathbf{i}_{X_2}\beta)]\!]$$

But this is

$$\left[\!\!\left[\Phi_B(\Xi(X_1,\epsilon_1))+\widetilde{\phi_{(X_1,\epsilon_1)}},\Phi_B(\Xi(X_2,\epsilon_2)))+\widetilde{\phi_{(X_2,\epsilon_2)}}\right]\!\!\right],$$

which can be expanded to

$$\begin{aligned}&\Phi_B(\Xi[\![(X_1,\epsilon_1),(X_2,\epsilon_2)]\!])+\left[\!\!\left[\Phi_B(\Xi(X_1,\epsilon_1)),\widetilde{\phi_{(X_2,\epsilon_2)}}\right]\!\!\right]\\ &+\left[\!\!\left[\widetilde{\phi_{(X_1,\epsilon_1)}},\Phi_B(\Xi(X_2,\epsilon_2)))\right]\!\!\right]+\left[\!\!\left[\widetilde{\phi_{(X_1,\epsilon_1)}},\widetilde{\phi_{(X_2,\epsilon_2)}}\right]\!\!\right]\end{aligned} \tag{31}$$

The second and fourth terms are again core-linear (see Lemma 3.6 and Lemma 4.5 in [17], respectively) so project to 0, but the third is

$$-\left[\!\!\left[\Phi_B(\Xi(X_2,\epsilon_2)),\widetilde{\phi_{(X_1,\epsilon_1)}}\right]\!\!\right]+\left(0,\mathbf{d}\langle\Phi_B(\Xi(X_2,\epsilon_2)),\widetilde{\phi_{(X_1,\epsilon_1)}}\rangle\right).$$

The left-hand term is core-linear, so projects to 0. By Lemma 7.2, the right-hand term also has values in the core for arbitrary $(X_2,\epsilon_2)\in\Gamma(TM\oplus E^*)$ if and only if $\phi_{(X_1,\epsilon_1)}=0$. This happens exactly when (31) projects to $[\![(X_1,\epsilon_1),(X_2,\epsilon_2)]\!]+(0,\mathbf{i}_{[X_1,X_2]}\beta)$ on $TM\oplus T^*M$, so when

$$[\![(X_1,\epsilon_1+\mathbf{i}_{X_1}\beta),(X_2,\epsilon_2+\mathbf{i}_{X_2}\beta)]\!] = [\![(X_1,\epsilon_1),(X_2,\epsilon_2)]\!]+(0,\mathbf{i}_{[X_1,X_2]}\beta).$$

Now $\widetilde{\phi}_{(X,\epsilon)}=0$ is equivalent to $(\Xi\circ\Phi_\beta)(X,\epsilon)=(\Phi_B\circ\Xi)(X,\epsilon)$ because

$$\begin{aligned}(\Xi\circ\Phi_\beta)(X,\epsilon)&=\Xi(X,\epsilon+\beta(X))=\Xi(X,\epsilon)+\Xi(0,\mathbf{i}_X\beta)\\ &=\Xi(X,\epsilon)+(0,\mathbf{i}_{\widehat{\delta_{(X,\epsilon)}}}B)+\widetilde{\phi}_{(X,\epsilon)}=(\Phi_B\circ\Xi)(X,\epsilon)+\widetilde{\phi}_{(X,\epsilon)}.\end{aligned}$$ □

Note that so far, we have not made any statement as to the existence of forms like in Theorem 7.1. The theorem rather provides a simple reformulation of the condition for being a symmetry.

Example 7.3. Consider $E=\wedge^kTM$ and the standard Dorfman bracket on $TM\oplus\wedge^kT^*M$ already studied earlier. Choose a morphism $\beta\colon TM\to\wedge^kT^*M$ and consider $-\mathbf{d}\Lambda_\mu$ the associated linear 2-form on $E=\wedge^kTM$.

For β to define a symmetry of the Dorfman bracket on $TM\oplus\wedge^kT^*M$, we need

$$\Xi(0,\mathbf{i}_X\beta)(e_m)=-\mathbf{d}\Lambda_\beta(\widehat{\delta_{(X,\alpha_k)}})(e_m)$$

for all $e_m\in E$, which is equivalent to $(0,\mathbf{d}\ell_{\mathbf{i}_X\beta}-\widetilde{\mathbf{di}_X\beta})=-\mathbf{d}\Lambda_\beta(\widehat{\pounds_X})$.

Both sides of this equation are sections of T^*E, and they are equal if and only if they map all linear and all core vector fields in the same way. On core vector fields $T^\uparrow$, for $T\in\Gamma(\wedge^kTM)$, we have

$$\begin{aligned}\mathbf{d}\Lambda_\beta(\widehat{\pounds_X},T^\uparrow)&=\widehat{\pounds_X}(\Lambda_\beta(T^\uparrow))-T^\uparrow(\Lambda_\beta(\widehat{\pounds_X}))-\Lambda_\beta([\widehat{\pounds_X},T^\uparrow])\\ &=0-q_E^*\langle T,\mathbf{i}_X\beta\rangle-0=0,\end{aligned}$$

$\mathbf{d}\ell_{\mathbf{i}_X\beta}(T^\uparrow) = q_E^*\langle T, \mathbf{i}_X\beta\rangle$ and $\widetilde{\mathbf{di}_X\beta}(T^\uparrow)(e_m) = 0$. On a linear vector field $\widehat{D} \in \mathfrak{X}^l(E)$ over $Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}\mathbf{d}\Lambda_\beta(\widehat{\pounds_X}, \widehat{D}) &= \widehat{\pounds_X}(\Lambda_\beta(\widehat{D})) - \widehat{D}(\Lambda_\beta(\widehat{\pounds_X})) - \Lambda_\beta([\widehat{\pounds_X}, \widehat{D}]) \\ &= \ell_{\pounds_X(\mathbf{i}_Y\beta) - D^*(\mathbf{i}_X\beta) - \mathbf{i}_{[X,Y]}\beta},\end{aligned}$$

$\widetilde{\mathbf{di}_X\beta}(\widehat{D}) = \ell_{\mathbf{i}_Y\mathbf{di}_X\beta}$ and $\widehat{D}(\ell_{\mathbf{i}_X\beta}) = \ell_{D^*(\mathbf{i}_X\beta)}$. Thus we are left with the following condition on β:

$$\pounds_X(\mathbf{i}_Y\beta) - \mathbf{i}_{[X,Y]}\beta - D^*(\mathbf{i}_X\beta) = \mathbf{i}_Y\mathbf{di}_X\beta - D^*(\mathbf{i}_X\beta)$$

for all $X, Y \in \mathfrak{X}(M)$, which is equivalent to $\beta \in \Omega^{k+1}(M)$ and $\mathbf{d}\beta = 0$.

Appendix A. On the proofs of Theorems 3.3 and 3.4

Choose a linear section χ of $TE \oplus T^*E \to E$ over a pair $(X, \varepsilon) \in \Gamma(TM \oplus E^*)$. Then $\chi = \left(\widehat{d_\chi}, \mathbf{d}\ell_\varepsilon - \widetilde{\phi_\chi}\right)$, following the notations set after Theorem 3.1. For simplicity, we write θ_χ for $\mathbf{d}\ell_\varepsilon - \widetilde{\phi_\chi} \in \Omega^1(E)$.

Lemma A.1. *Choose linear sections χ, χ' of $TE \oplus T^*E \to E$ over $(X, \varepsilon), (X', \varepsilon') \in \Gamma(TM \oplus E^*)$, a section $e \in \Gamma(E)$ and a derivation D of E with symbol Y. Then*

(1) $\langle \theta_\chi, e^\uparrow\rangle = q_E^*\langle \varepsilon, e\rangle$,
(2) $\langle \theta_\chi, \widehat{D}\rangle = \ell_{D^*\varepsilon - \phi_\chi^*(Y)}$,
(3) $\pounds_{e^\uparrow}\theta_\chi = q_E^*\left(\mathbf{d}\langle \varepsilon, e\rangle - \phi_\chi(e)\right)$,
(4) $\pounds_{\widehat{d_{\chi'}}}\theta_\chi = \mathbf{d}\ell_{d^*_{\chi'}\varepsilon} - (\widetilde{d_{\chi'}(\phi_\chi^*)})^*$.

Note that in the last equation, ϕ_χ^* is an element of $\Omega^1(M, E^*)$. For a derivation D of E over $X \in \mathfrak{X}(M)$, the derivation $D\colon \Omega^1(M, E^*) \to \Omega^1(M, E^*)$ over X is defined by $(D\omega)(Y) = D^*(\omega(Y)) - \omega[X, Y]$ for all $Y \in \mathfrak{X}(M)$.

Proof. The first identity is immediate. For the second, we recall (12). The pairing of $\hat{D}$ with θ_χ at e_m is

$$Y\langle \varepsilon, e\rangle - \langle \phi_\chi(e), Y\rangle - \langle \varepsilon, De\rangle = \langle D^*\varepsilon, e\rangle - \langle \phi_\chi(e), Y\rangle$$

at m. Hence we have found (2). Next we prove (3). We have

$$\langle \pounds_{e^\uparrow}\theta_\chi, {e'}^\uparrow\rangle = e^\uparrow\langle \theta_\chi, {e'}^\uparrow\rangle - \langle \theta_\chi, \left[e^\uparrow, {e'}^\uparrow\right]\rangle = e^\uparrow(q_E^*\langle \varepsilon, e'\rangle) = 0$$

for $e' \in \Gamma(E)$ and

$$\begin{aligned}\langle \pounds_{e^\uparrow}\theta_\chi, \widehat{D}\rangle &= e^\uparrow\langle \theta_\chi, \widehat{D}\rangle - \langle \theta_\chi, \left[e^\uparrow, \widehat{D}\right]\rangle = q_E^*\langle D^*\varepsilon - \phi_\chi^*(Y), e\rangle + \langle \theta_\chi, (De)^\uparrow\rangle \\ &= q_E^*\langle D^*\varepsilon - \phi_\chi^*(Y), e\rangle + q_E^*\langle \varepsilon, De\rangle = q_E^*(Y\langle \varepsilon, e\rangle - \langle Y, \phi_\chi(e)\rangle)\end{aligned}$$

for a derivation D of E over $Y \in \mathfrak{X}(M)$. Since $q_E^*\left(\mathbf{d}\langle \varepsilon, e\rangle - \phi_\chi(e)\right)$ takes the same values on ${e'}^\uparrow$ and $\hat{D}$, we are done.

Finally, we compute using the first identity

$$\begin{aligned}\langle \pounds_{\widehat{d_{\chi'}}}\theta_\chi, e^\uparrow\rangle &= \widehat{d_{\chi'}}\langle \theta_\chi, e^\uparrow\rangle - \langle \theta_\chi, \left[\widehat{d_{\chi'}}, e^\uparrow\right]\rangle = q_E^*(X'\langle \varepsilon, e\rangle - \langle \varepsilon, d_{\chi'}e\rangle) \\ &= q_E^*\langle d^*_{\chi'}\varepsilon, e\rangle = \langle \mathbf{d}\ell_{d^*_{\chi'}\varepsilon}, e^\uparrow\rangle = \langle \mathbf{d}\ell_{d^*_{\chi'}\varepsilon} - (\widetilde{d_{\chi'}(\phi_\chi^*)})^*, e^\uparrow\rangle\end{aligned}$$

for $e \in \Gamma(E)$. Similarly, using (2) above

$$\begin{aligned}\langle \pounds_{\widehat{d_{\chi'}}}\theta_\chi, \widehat{D}\rangle &= \widehat{d_{\chi'}}\langle \theta_\chi, \widehat{D}\rangle - \langle \theta_\chi, \left[\widehat{d_{\chi'}}, \widehat{D}\right]\rangle = \ell_{d^*_{\chi'}(D^*\varepsilon - \phi^*_\chi(Y))} - \langle \theta_\chi, \widehat{[d_{\chi'}, D]}\rangle \\ &= \ell_{d^*_{\chi'}(D^*\varepsilon - \phi^*_\chi(Y)) - [d_{\chi'}, D]^*\varepsilon + \phi^*_\chi[X', Y]}\end{aligned}$$

for a derivation D of E over $Y \in \mathfrak{X}(M)$. An easy calculation shows $[d_{\chi'}, D]^* = [d^*_{\chi'}, D^*]$, which leads to

$$\langle \pounds_{\widehat{d_{\chi'}}}\theta_\chi, \widehat{D}\rangle = \ell_{D^* d^*_{\chi'}\varepsilon - (d_{\chi'}\phi^*_\chi)(Y)} = \langle \mathbf{d}\ell_{d^*_{\chi'}\varepsilon} - \widetilde{(d_{\chi'}\phi^*_\chi)^*}, \hat{D}\rangle. \qquad \square$$

Proof of Theorem 3.3. We write $\tau = (e, \theta) \in \Gamma(E \oplus T^*M)$. First we find that $\langle \pounds_{\widehat{d_\chi}} q_E^*\theta, e'^\uparrow\rangle$ equals $\widehat{d_\chi}\langle q_E^*\theta, e'^\uparrow\rangle - \langle q_E^*\theta, [\widehat{d_\chi}, e'^\uparrow]\rangle = 0 - 0 = 0$ and $\langle \pounds_{\widehat{d_\chi}} q_E^*\theta, \widehat{D}\rangle = \widehat{d_\chi}(q_E^*\langle \theta, Y\rangle) - \langle q_E^*\theta, [\widehat{d_\chi}, \widehat{D}]\rangle = q_E^*(X\langle \theta, Y\rangle - \langle \theta, [X, Y]\rangle) = q_E^*\langle \pounds_X\theta, Y\rangle$ for all $e' \in \Gamma(E)$ and any derivation D of E over $Y \in \mathfrak{X}(M)$. This shows $\pounds_{\widehat{d_\chi}} q_E^*\theta = q_E^*(\pounds_X\theta)$. In the same manner, we have $\mathbf{i}_{e^\uparrow}\mathbf{d}\theta_\chi = \pounds_{e^\uparrow}\theta_\chi - \mathbf{d}\langle \theta_\chi, e^\uparrow\rangle = q_E^*(-\phi_\chi(e))$ by (1) and (3) in Lemma A.1. We get

$$\begin{aligned}[\![\chi, \tau^\uparrow]\!] &= \left(\left[\widehat{d_\chi}, e^\uparrow\right], \pounds_{\widehat{d_\chi}} q_E^*\theta - \mathbf{i}_{e^\uparrow}\mathbf{d}\theta_\chi\right) = \left((d_\chi e)^\uparrow, q_E^*(\pounds_X\theta + \operatorname{pr}_{T^*M} D_\chi(e, 0))\right) \\ &= \left((d_\chi e)^\uparrow, q_E^*(\operatorname{pr}_{T^*M} D_\chi(e, \theta))\right) = D_\chi\tau^\uparrow,\end{aligned}$$

which proves Theorem 3.3. $\square$

Proof of Theorem 3.4. We simply compute

$$[\![\chi_1, \chi_2]\!] = \left(\left[\widehat{d_{\chi_1}}, \widehat{d_{\chi_2}}\right], \pounds_{\widehat{d_{\chi_1}}}\theta_{\chi_2} - \mathbf{i}_{\widehat{d_{\chi_2}}}\mathbf{d}\theta_{\chi_1}\right). \tag{32}$$

The TE-part is $\widehat{[d_{\chi_1}, d_{\chi_2}]}$. By definition of D_χ, we have $\operatorname{pr}_E \circ D_\chi \circ \iota_E \circ \operatorname{pr}_E = \operatorname{pr}_E \circ D_\chi$ and so $[d_{\chi_1}, d_{\chi_2}] = \operatorname{pr}_E \circ [D_{\chi_1}, D_{\chi_2}] \circ \iota_E$.

The T^*E-component of (32) is

$$\mathbf{d}\ell_{d^*_{\chi_1}\varepsilon_2} - \widetilde{(d_{\chi_1}(\phi^*_{\chi_2}))^*} \cancel{-\mathbf{d}\ell_{d^*_{\chi_2}\varepsilon_1}} + \widetilde{(d_{\chi_2}(\phi^*_{\chi_1}))^*} + \mathbf{d}\ell_{\cancel{d^*_{\chi_2}\varepsilon_1} - \phi^*_{\chi_1}(X_2)}$$

by Lemma A.1. First we find that $\langle d^*_{\chi_1}\varepsilon_2 - \phi^*_{\chi_1}(X_2), e\rangle$ equals

$$\begin{aligned}&X_1\langle \varepsilon_2, e\rangle - \langle \varepsilon_2, d_{\chi_1}e\rangle - \langle X_2, \phi_{\chi_1}(e)\rangle \\ &= X_1\langle \varepsilon_2, e\rangle - \langle (X_2, \varepsilon_2), D_{\chi_1}(e, 0)\rangle = \langle D^*_{\chi_1}(X_2, \varepsilon_2), (e, 0)\rangle\end{aligned}$$

for any $e \in \Gamma(E)$. Then we find that $\langle (d_{\chi_1}\phi^*_{\chi_2} - d_{\chi_2}\phi^*_{\chi_1})^*(e), X\rangle$ equals

$$\begin{aligned}&\left(d^*_{\chi_1}(\phi^*_{\chi_2}(X)) - \phi^*_{\chi_2}[X_1, X] - d^*_{\chi_2}(\phi^*_{\chi_1}(X)) + \phi^*_{\chi_1}[X_2, X]\right)(e) \\ =&X_1\langle X, \phi_{\chi_2}(e)\rangle - \langle X, \phi_{\chi_2}(d_{\chi_1}(e))\rangle - \langle [X_1, X], \phi_{\chi_2}(e)\rangle \\ &- X_2\langle X, \phi_{\chi_1}(e)\rangle + \langle X, \phi_{\chi_1}(d_{\chi_2}(e))\rangle + \langle [X_2, X], \phi_{\chi_1}(e)\rangle \\ =&\langle X, \pounds_{X_1}(\operatorname{pr}_{T^*M} D_{\chi_2}(e, 0)) + \operatorname{pr}_{T^*M} \circ D_{\chi_2} \circ \iota_E \circ \operatorname{pr}_E \circ D_{\chi_1}(e, 0)\rangle \\ &- \langle X, \pounds_{X_2}(\operatorname{pr}_{T^*M} D_{\chi_1}(e, 0)) + \operatorname{pr}_{T^*M} \circ D_{\chi_1} \circ \iota_E \circ \operatorname{pr}_E \circ D_{\chi_2}(e, 0)\rangle\end{aligned}$$

for $X \in \mathfrak{X}(M)$ and $e \in \Gamma(E)$. Since $\pounds_{X_1}(\operatorname{pr}_{T^*M} D_{\chi_2}(e, 0))$ equals $\operatorname{pr}_{T^*M} D_{\chi_1}(0, \operatorname{pr}_{T^*M} D_{\chi_2}(e, 0))$ and $\operatorname{pr}_{T^*M} \circ D_{\chi_1} \circ \iota_E \circ \operatorname{pr}_E \circ D_{\chi_2}(e, 0)$ equals $\operatorname{pr}_{T^*M} D_{\chi_1}(\operatorname{pr}_E D_{\chi_2}(e, 0), 0)$, we find that the first and fourth term add up to $\langle X, \operatorname{pr}_{T^*M} D_{\chi_1}D_{\chi_2}(e, 0)\rangle$. Similarly the second and third term add up to $-\langle X, \operatorname{pr}_{T^*M} D_{\chi_2}D_{\chi_1}(e, 0)\rangle$ and we get

$$\langle (d_{\chi_1}\phi^*_{\chi_2} - d_{\chi_2}\phi^*_{\chi_1})^*(e), X\rangle = \langle \operatorname{pr}_{T^*M}[D_{\chi_1}, D_{\chi_2}](e, 0), X\rangle.$$

The proof of the second identity is left to the reader. □

Appendix B. On the proof of Theorem 4.4

Recall that $\mathcal{D}$ has the following property:

$$\mathcal{D}_{(X,\varepsilon)}(e,\theta)=\mathcal{D}_{(X,\varepsilon)}(e,0)+(0,\pounds_X\theta) \tag{33}$$

for all $(X,\varepsilon)\in\Gamma(TM\oplus E^*)$ and $(e,\theta)\in\Gamma(E\oplus T^*M)$. (33) and the definition of δ yield together

$$\delta\circ\operatorname{pr}_E=\operatorname{pr}_E\circ\mathcal{D}. \tag{34}$$

We will use the following lemma.

Lemma B.1. $[\![\cdot\,,\cdot]\!]$ *satisfies the Jacobi identity in Leibniz form if and only if*
 (1) $[\delta_{\nu_1},\delta_{\nu_2}]=\delta_{[\![\nu_1,\nu_2]\!]}$ *and*
 (2) $\operatorname{pr}_{T^*M}[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}]\circ\iota_E=\operatorname{pr}_{T^*M}\circ\mathcal{D}_{[\![\nu_1,\nu_2]\!]}\circ\iota_E$

for all $\nu_1,\nu_2\in\Gamma(TM\oplus E^*)$.

Proof. First note that by (34), we have

$$[\delta_{\nu_1},\delta_{\nu_2}]=\operatorname{pr}_E\circ[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}]\circ\iota_E. \tag{35}$$

If $[\![\cdot\,,\cdot]\!]$ satisfies the Jacobi identity in Leibniz form, then (1) and (2) are immediate by (5).

Conversely, (1) and (2) give using (35): $[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}]\circ\iota_E=\mathcal{D}_{[\![\nu_1,\nu_2]\!]}\circ\iota_E$. We have always $[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}](0,\theta)=(0,\pounds_{X_1}\pounds_{X_2}\theta-\pounds_{X_2}\pounds_{X_1}\theta)=(0,\pounds_{[X_1,X_2]}\theta)=\mathcal{D}_{[\![\nu_1,\nu_2]\!]}(0,\theta)$ for all $\theta\in\Omega^1(M)$. This shows that (1), (2) are equivalent to $[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}]=\mathcal{D}_{[\![\nu_1,\nu_2]\!]}$, which dualises to the Jacobi identity in Leibniz form for $[\![\cdot\,,\cdot]\!]$. □

Now we can prove Theorem 4.2.

Proof of Theorem 4.2. We write $\tau=(e,\theta)$, $\tau_i=(e_i,\theta_i)$ and $\nu=(X,\varepsilon)$, $\nu_i=(X_i,\varepsilon_i)$ for $i=1,2$. By (17), we have

$$[\![\Xi(\nu_1),\Xi(\nu_2)]\!]=\left(\widehat{[\delta_{\nu_1},\delta_{\nu_2}]},\mathbf{d}\ell_{\operatorname{pr}_{E^*}\mathcal{D}^*_{\nu_1}\nu_2}-\widetilde{\operatorname{pr}_{T^*M}\circ[\mathcal{D}_{\nu_1},\mathcal{D}_{\nu_2}]\circ\iota_E}\right).$$

By Lemma B.1, this is

$$[\![\Xi(\nu_1),\Xi(\nu_2)]\!]=\left(\widehat{\delta_{[\![\nu_1,\nu_2]\!]}},\mathbf{d}\ell_{\operatorname{pr}_{E^*}\mathcal{D}^*_{\nu_1}\nu_2}-\widetilde{\operatorname{pr}_{T^*M}\circ\mathcal{D}_{[\![\nu_1,\nu_2]\!]}\circ\iota_E}\right)$$

if and only if $[\![\cdot\,,\cdot]\!]$ satisfies the Jacobi identity in Leibniz form. Since $\mathcal{D}^*_{\nu_1}\nu_2=[\![\nu_1,\nu_2]\!]$, we are done. □

Appendix C. A non-local Leibniz algebroid

Let $M=\mathbb{S}^1\times\mathbb{S}^1\simeq\mathbb{T}^2$ and consider the vector bundle $\bar{E}=T^*M\oplus\wedge^2T^*M$ over M. Let $\eta\in\Omega^1(\mathbb{S}^1)$ be the standard volume form on the circle and set $\eta_x=\operatorname{pr}_1^*\eta$ and $\eta_y=\operatorname{pr}_2^*\eta$, where $\operatorname{pr}_i\colon\mathbb{S}^1\times\mathbb{S}^1\to\mathbb{S}^1$ are the projections, $i=1,2$. Then $\eta_x\wedge\eta_y$ is a volume form on M and $\eta_x,\eta_y\in\Omega^1(M)$ form a basis of one-forms such that the pullback of η_x along any $\iota_q\colon\mathbb{S}^1\hookrightarrow\mathbb{S}^1\times\{q\}$ and the pullback of η_y to any $\{p\}\times\mathbb{S}^1$ are the standard volume form on the circle. Define the following operations for integration along the first fibre. For $f,g,h\in C^\infty(M)$: $\int_{\mathbb{S}^1}f\eta_x+g\eta_y\in C^\infty(M)$,

$$\left(\int_{\mathbb{S}^1}f\eta_x+g\eta_y\right)(p,q):=\int_{\mathbb{S}^1}\iota_q^*(f)\eta$$

and $\int_{\mathbb{S}^1} h\,\eta_x \wedge \eta_y \in \Omega^1(M)$,

$$\left(\int_{\mathbb{S}^1} h\,\eta_x \wedge \eta_y\right)(p,q) := \left(\int_{\mathbb{S}^1} \iota_q^*(h)\eta\right)\eta_y(p,q).$$

Clearly, the resulting function $\int_{\mathbb{S}^1} f\eta_x + g\eta_y \in C^\infty(M)$ is constant along the first $\mathbb{S}^1$, i.e. only a function of q in the notation above. In the same manner, the one-form $\int_{\mathbb{S}^1} h\,\eta_x \wedge \eta_y$ is constant along the first $\mathbb{S}^1$ and only has a η_y component. That is, the obtained functions and 1-forms are invariant along the fibers of pr_2.

Now we define a bracket on $\bar{E} = T^*M \oplus \wedge^2 T^*M$ as follows:

$$[\![(\alpha_1,\alpha_2),(\beta_1,\beta_2)]\!] = \left(0, \left(\int_{\mathbb{S}^1} \alpha_1\right)\beta_2 + \left(\int_{\mathbb{S}^1} \alpha_2\right)\wedge \beta_1\right)$$

and we prove that $(\bar{E} = T^*M \oplus \wedge^2 T^*M, [\![\cdot,\cdot]\!], 0\colon \bar{E} \to TM$ is a Leibniz algebroid. Since the bracket is clearly C^∞-linear in the second component and thus satisfies the Leibniz rule for functions with the zero-anchor, it suffices to check the Jacobi identity in Leibniz form. For simplicity, we just write $\int$ for $\int_{\mathbb{S}^1}$, and this is always the integration along the first $\mathbb{S}^1$. We have

$$[\![(\alpha_1,\alpha_2),[\![(\beta_1,\beta_2),(\gamma_1,\gamma_2)]\!]]\!] = \left[\!\!\left[(\alpha_1,\alpha_2),\left(0,\int \beta_1\,\gamma_2 + \int \beta_2\wedge\gamma_1\right)\right]\!\!\right]$$
$$= \left(0, \int \alpha_1 \int \beta_1\,\gamma_2 + \int \alpha_1 \int \beta_2 \wedge \gamma_1\right)$$

and in a similar manner

$$[\![[\![(\alpha_1,\alpha_2),(\beta_1,\beta_2)]\!],(\gamma_1,\gamma_2)]\!] = \left[\!\!\left[\left(0,\int \alpha_1\,\beta_2 + \int \alpha_2\wedge\beta_1\right),(\gamma_1,\gamma_2)\right]\!\!\right]$$
$$= \left(0, \int\left(\int \alpha_1\,\beta_2 + \int \alpha_2 \wedge \beta_1\right)\wedge\gamma_1\right)$$
$$= \left(0, \int \alpha_1 \int \beta_2\wedge\gamma_1 - \int \beta_1 \int \alpha_2 \wedge \gamma_1\right).$$

Therefore we get

$$[\![(\alpha_1,\alpha_2),[\![(\beta_1,\beta_2),(\gamma_1,\gamma_2)]\!]]\!] - [\![(\beta_1,\beta_2),[\![(\alpha_1,\alpha_2),(\gamma_1,\gamma_2)]\!]]\!]$$
$$-[\![[\![(\alpha_1,\alpha_2),(\beta_1,\beta_2)]\!],(\gamma_1,\gamma_2)]\!]$$
$$= \left(0, \int\alpha_1\int\beta_1\gamma_2 + \int\alpha_1\int\beta_2\wedge\gamma_1 - \int\beta_1\int\alpha_1\gamma_2 - \int\beta_1\int\alpha_2\wedge\gamma_1\right.$$
$$\left. - \int\alpha_1\int\beta_2\wedge\gamma_1 + \int\beta_1\int\alpha_2\wedge\gamma_1\right) = 0$$

This Leibniz algebroid is *non-local*, i.e. its bracket *not* given by a bilinear differential operator of any order.

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Mathematisches Institut, Georg-August Universität Göttingen.
E-mail address: `madeleine.jotz-lean@mathematik.uni-goettingen.de`

DAMTP, University of Cambridge
E-mail address: `csk34@cam.ac.uk`